BRIEF INTRO TO GRAPH THEORY

Definition. A graph is an ordered pair $G = (V, E)$ where V is a set of vertices and E is a (multi) set of edges: 2-element subsets of V.

- Models binary relations between elements of V .
- Examples:
  - The bridges of Königsberg
  - Scheduling
  - 3 Utility problem.
- A graph is simple if it contains no loops (i.e. $\forall v \in V, vv \notin E$), no multi-edges (i.e for each edge $e \in E$ there is exactly one copy of $e \in E$.)
- A graph undirected if $uv \in E \implies vu \in E$.
- For now, we assume a graphs are simple and undirected.

Neighbourhood and Degree:

- u is adjacent to v if $uv \in E$.
- If u is adjacent to v then we say v is a neighbour of u.
- The set of all neighbours of u is the neighbourhood of u and is denoted $N(u)$.
- $|N(u)|$ is the degree of u and is denoted $\deg(u)$.

The Handshaking Lemma: The sum of the degree for each vertex is equal to twice the number of edges. i.e.

$$\sum_{v \in V} \deg(v) = 2|E|.$$

Proof. picture.

Simple Observations: The number of vertices of odd degree is even.

Paths, Cycles, and Connectedness.

Definition: A walk consists of an alternating sequence of vertices and edges consecutive elements of which are incident, that begins and ends with a vertex. A trail is a walk without repeated edges. A path is a walk without repeated vertices.

Definition: If a walk (resp. trail, path) begins at x and ends at y then it is an $x - y$ walk (resp. $x - y$ trail, resp. $x - y$ path).

Definition: A walk (trail) is closed if it begins and ends at the same vertex. A closed trail whose origin and internal vertices are distinct is a cycle.

Definition: Given a walk W_1 that ends at vertex v and another W_2 starting at v, the concatenation of W_1 and W_2 is obtained by appending the sequence obtained from W_2 by deleting the first occurrence of v, after W_1.

Observation I: The concatenation of any two walks is also a walk. Furthermore, if we concatenate two edge disjoint trails then we obtain a trail.

Observation II: If v is a vertex of a walk W then W is the concatenation of a walk that ends at v with a walk that begins at v. If W is a trail, so are the two subwalks.