Note on the combinations with repetition

Suppose we are trying to solve the following two problems.

Problem 1. (Rosen, p.371 example 2)
In how many different ways can we pick 7 fruits out of a basket of apples, oranges and pears. The order in which the fruits are chosen does not matter.

Problem 2. (Rosen, p.373 example 5)
How many solutions are there to

$$x_1 + x_2 + x_3 \leq 7$$

if all variables (must) take non-negative integer values.

First, note that the answer to both question is the same. To see this, we let x_1 be the number of apples chosen, x_2 be the number of oranges chosen and x_3 be the number of pears chosen.

Second, note that it is possible to write each choice of 7 fruits by first writing down the apples followed by the oranges and finally the pears. For example, “aaaaoooppp”.

If we can now put the different kinds of fruits into separate “boxes”. The above example becomes “aaaa—ooo—ppp”.

Now if we forget which fruits we had while keeping these “separators” between boxes, we obtain a string from which we can still recover the original fruits. For example, from “????—???—???”, we can recover “aaaaoooppp” since everything before the first “—” is “a”, everything between the first and second “—” are “o” and everything after the last “—” is “p”.

In fact, every string of length 9 with exactly 2 “—” and 7 “?” corresponds to a choice of fruits. So it is only a matter of counting these strings.

We can obtain every such string exactly once by choosing the 7 positions for the “?” and making the rest “—”. There are thus $\binom{9}{7}$ such strings and therefore $\binom{9}{7}$ choices of fruits.

More generally,

Theorem 1. *The number of ways of choosing k objects out of n if repetition is allowed is* $\binom{n+k-1}{n-1} = \binom{n+k-1}{k}$.

The second problem of solving $x_1 + x_2 + \ldots + x_k \leq n$ can be solved in a “different” way. We can explicitly define a bijection f from the solutions $(x_1, x_2, \ldots, x_k)$ to a subset $\{y_1, y_2, \ldots, y_{n-1}\}$ of $\{0, 1, \ldots, n+k\}$ as follows.

$$y_i = \sum_{j=1}^{i} x_i$$

I.e., we take the partial sum of the x_i's.

This is not really a “different” solution since the y_i's are actually the position of all “—” in the string with $k-1$ “—”s and n “?”s.