Scarf's Lemma & Stable Matching

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Stable Matching (Gale & Shapley)

D = set of single doctors

H = set of hospitals each capacity $k_h = 1$

Each $s \in D$ has a strict preference ordering $\succ_s$ over $H \cup \{\emptyset\}$

Each h has a strict preference ordering $\succ_h$ over $D \cup \{\emptyset\}$

Set of Matchings

$x_{dh} = 1$ if doctor d is matched to hospital h and zero otherwise.

$$\sum_{h \in H \cup \{\emptyset\}} x_{dh} \leq 1 \ \forall d \in D$$

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Each row 'd' has a strict ordering $\succ_d$ over variables x_{dh}

Each row 'h' has a strict ordering $\succ_h$ over variables x_{dh}

Blocking

Matching x is blocked by a pair (d, h) where $x_{dh} = 0$, i.e., (d, h) is not matched and

1. $x_{dh'} = 1$ and $x_{d'h} = 1$.
2. $h \succ_d h'$.
3. $d \succ_h d'$.

A matching x not blocked by any doctor-hospital pair is called stable.

Scarf's Lemma (1967)

$A = m \times n$ nonnegative matrix, at least one non-zero entry in each row and $b \in \mathbb{R}^m_+$ with $b >> 0$.

$\mathcal{P} = \{x \in \mathbb{R}^n_+ : Ax \leq b\}$.

Each row $i \in [m]$ of A has a strict order $\succ_i$ over the columns $\{j : a_{ij} > 0\}$.

Example

$$\begin{array}{c} h_1 \\ h_2 \\ d_1 \\ d_2 \end{array} \overset{\begin{array}{cccc}(d_1,h_1) & (d_1,h_2) & (d_2,h_1) & (d_2,h_2)\end{array}}{\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}} \cdot x \leq \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}; \quad order: \begin{array}{l} column_1 \succ column_3 \\ column_2 \succ column_4 \\ column_2 \succ column_1 \\ column_3 \succ column_4. \end{array}$$

Scarf's Lemma (1967)

A vector $x \in \mathcal{P}$ **dominates** column r if there exists a row i such that

1. $a_{ir} > 0$, $\sum_j a_{ij}x_j = b_i$ and
2. $k \succeq_i r$ for all $k \in [n]$ such that $a_{ik} > 0$ AND $x_k > 0$.

$\mathcal{P}$ has an extreme point that dominates every column of A.

Example

$$
\begin{array}{c} \\ h_1 \\ h_2 \\ d_1 \\ d_2 \end{array}
\begin{array}{c}
\begin{array}{cccc} (d_1,h_1) & (d_1,h_2) & (d_2,h_1) & (d_2,h_2) \end{array} \\
\left[\begin{array}{cccc} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{array}\right]
\end{array}
\cdot x \leq \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}; \quad order: \begin{array}{l} column_1 \succ column_3 \\ column_2 \succ column_4 \\ column_2 \succ column_1 \\ column_3 \succ column_4. \end{array}
$$

$A = m \times n$ nonnegative matrix and $b \in \mathbb{R}^m_+$ with $b >> 0$.

$$\sum_{h \in H} x_{dh} \leq 1 \ \forall d \in D$$

$$\sum_{d \in D} x_{dh} \leq 1 \ \forall h \in H$$

Each row $i \in [m]$ of A has a strict order $\succ_i$ over the set of columns j for which $a_{ij} > 0$.

$$\succ_d, \ \succ_h$$

$x \in \mathcal{P}$ **dominates** column r if $\exists i$ such that $\sum_j a_{ij} x_j = b_i$ and $k \succeq_i r$ for all $k \in [n]$ such that $a_{ik} > 0$ and $x_k > 0$.

Consider $x_{dh} = 0$. There is a $d \in D$ or $h \in H$, say d, such that $x_{dh'} = 1$ and $h' \succ_d h$.

D = set of single doctors

C = set of couples, each couple $c \in C$ is denoted $c = (f_c, m_c)$

$D^* = D \cup \{m_c | c \in C\} \cup \{f_c | c \in C\}$.

H = set of hospitals

Each $s \in D$ has a strict preference relation $\succ_s$ over $H \cup \{\emptyset\}$

Each $c \in C$ has a strict preference relation $\succ_c$ over $H \cup \{\emptyset\} \times H \cup \{\emptyset\}$

k_h = capacity of hospital $h \in H$

Preference of hospital h over subsets of D^* is modeled by choice function $ch_h(.) : 2^{D^*} \to 2^{D^*}$.

$ch_h(.)$ is responsive

h has a strict priority ordering $\succ_h$ over elements of $D^* \cup \{\emptyset\}$.

$ch_h(R)$ consists of the (upto) $\min\{|R|, k_h\}$ highest priority doctors among the feasible doctors in R.

Matching with Couples

Stable matchings need not exist.

Given an instance, even determining if it has a stable matching is NP-hard.

1. Restrict preferences
2. Modify definition of stability
3. Non-existence is rare

Every 0-1 solution to the following system is a feasible matching and vice-versa.

$$\sum_{h \in H} x(d,h) \leq 1 \ \forall d \in D \tag{1}$$

$$\sum_{h,h' \in H} x(c,h,h') \leq 1 \ \forall c \in D \tag{2}$$

$$\sum_{d \in D} x(d,h) + \sum_{c \in C} \sum_{h' \neq h} x(c,h,h') + \sum_{c \in C} \sum_{h' \neq h} x(c,h',h) + \sum_{c \in C} 2x(c,h,h) \leq k_h \ \forall h \in H \tag{3}$$

Matching with Couples

Constraint matrix and RHS satisfy conditions of Scarf's lemma.

Each row associated with single doctor or couple has an ordering over the variables that 'include' them from their preference ordering.

Row associated with each hospital does not have a natural ordering over the variables that 'include' them.

Round fractional dominating solution into a stable integer solution.

Matching with Couples

Given any instance of a matching problem with couples, there is a 'nearby' instance that is guaranteed to have a stable matching.

For any capacity vector k, there exists a k' and a stable matching with respect to k' (found using IRM), such that

1. $|k_h - k'_h| \leq 2 \ \forall h \in H$
2. $\sum_{h \in H} k_h \leq \sum_{h \in H} k'_h \leq \sum_{h \in H} k_h + 4.$

Matching with Proportionality Constraints

White Plains School District (1989): same proportions of blacks, Hispanics and ‘others’ with dicrepancy of no more than 5%.

Cambridge, MA & Chicago: % of students at each school from each SES category must lie within a certain range.

Matching with Proportionality Constraints

Reserve seats for each group- assumes schools are fully allocated

Numerical upper and lower bounds for # of students in each group- stable matchings need not exist

Ignore constraints but modify how schools prioritize students- no ex-post guarantees on final distribution

Proportionality Constraints

Each $h \in H$ partitions D into types $D_1^h, D_2^h, \ldots, D_{t_h}^h$.

$$\sum_{h \in H} x_{dh} \leq 1 \; \forall d \in H$$

$$\sum_{d \in D} x_{dh} \leq k_h \; \forall h \in H$$

$$\alpha_t^h [\sum_{d \in D} x_{dh}] \leq \sum_{d \in D_h^t} x_{dh} \; \forall t \;\; h \in H,$$

Proportionality & Stability

Need to modify definition of stability.

Blocking coalitions cannot violate proportionality constraints.

Pairwise implies coalitionally stable

Proportionality Constraints

Find integer x^* that is stable such that

$$\sum_{h \in H} x^*_{dh} \leq 1 \ \forall d \in H$$

$$\sum_{d \in D} x^*_{dh} \leq k_h \ \forall h \in H$$

$$\bar{\alpha}^h_t [\sum_{d \in D} x^*_{dh}] \leq \sum_{d \in D^t_h} x^*_{dh} \ \forall t \ \ h \in H,$$

Such that

$$|\alpha^h_t - \bar{\alpha}^h_t| \leq \frac{2}{\sum_{d \in D^t_h} x^*(d,h)}.$$

Scarf's Lemma (1967)

$A = m \times n$ nonnegative matrix and $b \in \mathbb{R}^m_+$ with $b >> 0$.

$\mathcal{P} = \{x \in \mathbb{R}^n_+ : Ax \leq b\}$.

Each row $i \in [m]$ of A has a strict order $\succ_i$ over the set of columns j for which $a_{ij} > 0$.

Add side constraints to $\mathcal{P}$.

Coeff of side constraints must be non-negative.

Side constraint must have an ordering. Ordering must be chosen to so that dominating solution $=$ stable solution.

Scarf's Lemma (1967)

A vector $x \in \mathcal{P}$ **dominates** column r if there exists a row i such that

1. $a_{ir} > 0$, $\sum_j a_{ij} x_j = b_i$ and
2. $k \succeq_i r$ for all $k \in [n]$ such that $a_{ik} x_k > 0$.

$\mathcal{P}$ has an extreme point that dominates every column of A.

$$\sum_{h \in H} x_{dh} \leq 1 \ \forall d \in H$$

$$\sum_{d \in D} x_{dh} \leq k_h \ \forall h \in H$$

$$\alpha_t^h [\sum_{d \in D} x_{dh}] - \sum_{d \in D_h^t} x_{dh} \leq 0 \ \forall t \ \ h \in H,$$

Conic Version of Scarf's Lemma

Resource Constraints:

$$\sum_{h \in H} x_{dh} \leq 1 \ \forall d \in H$$

$$\sum_{d \in D} x_{dh} \leq k_h \ \forall h \in H$$

Denote this $\mathcal{A}x \leq b$.

Side Constraints:

$$\alpha_t^h [\sum_{d \in D} x_{dh}] \leq \sum_{d \in D_h^t} x_{dh} \ \forall t \ \ h \in H,$$

Denote this $\mathcal{M}x \geq 0$.

Conic Version of Scarf's Lemma

$\{x \in \mathbb{R}^n_+ | \mathcal{M}x \geq 0\}$ is a polyhedral cone and can be rewritten as $\{\mathcal{V}z | z \geq 0\}$, where $\mathcal{V}$ is a finite non-negative matrix.

Columns of $\mathcal{V}$ correspond to the **generators** of the cone $\{x \in \mathbb{R}^n_+ | \mathcal{M}x \geq 0\}$.

Apply Scarf's lemma to $\mathcal{P}' = \{z \geq 0 : \mathcal{A}\mathcal{V}z \leq b\}$.

Proportionality constraints are

$$\alpha_t^h \cdot \sum_{d \in D} x_{dh} \leq \sum_{d \in D_t^h} x_{dh} \;\; t = 1, \ldots, T_h. \tag{4}$$

The generators can be described in this way:

1. Select one doctor from each D_t^h and call it d_t.
2. Select an extreme point of the system

$$\sum_{t=1}^{T_h} v(d_t, h) = 1, \;\; \alpha_t^h \leq v(d_t, h) \;\forall t = 1, \ldots, T_h.$$