

Excursions in Computing Science: Week 1. Polarized Light

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I. Prefatory Notes

1. Here is a polarizing filter on a source of light, showing the direction of polarization.

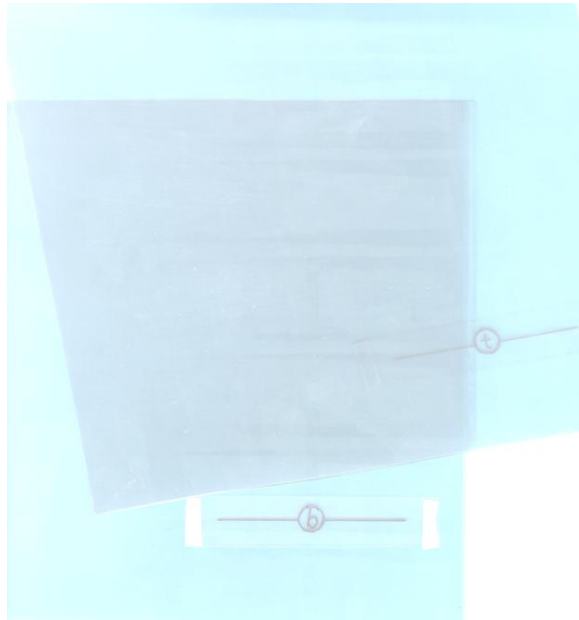


The Bottom Polarizing Filter

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The amount of light getting through does not depend on the angle of this direction of polarization.

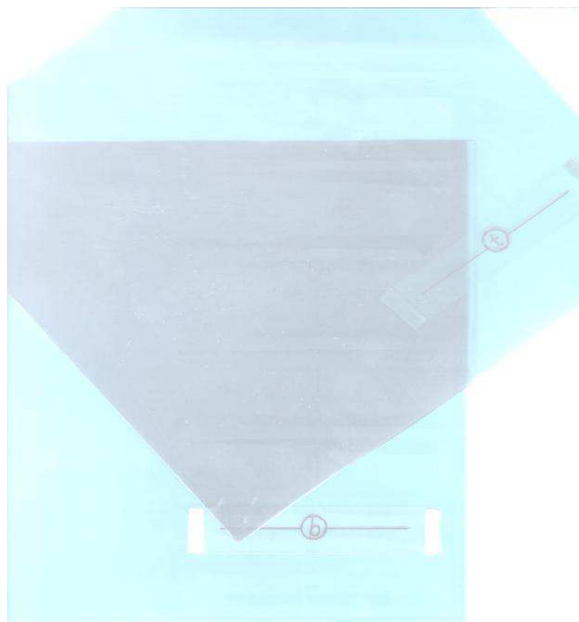
2. Here are two polarizing filters, with the top one, t, at an angle of 0.17 radians to the original, now bottom, one, b.



Top and Bottom Polarizing Filters at 10°

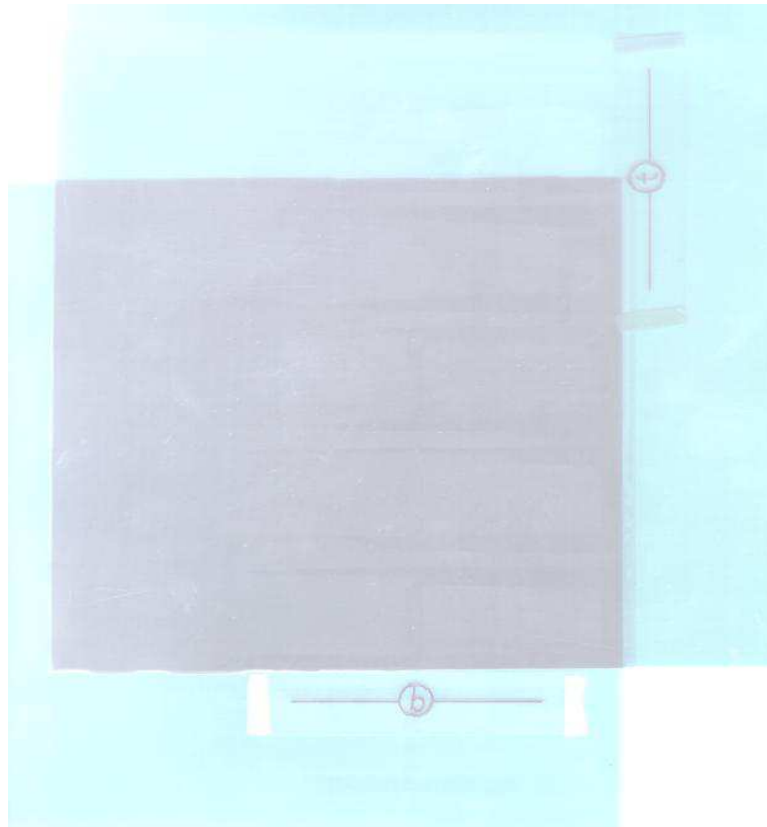
The amount of light getting through decreases as the angle between the two increases:

Here are two polarizing filters, with the top one, t, at an angle of 0.60 radians to the bottom, one, b. Less light gets through than at 0.17 radians.



Top and Bottom Polarizing Filters at 35°

Finally, here they are with t at angle $\pi/2$ to b: no light gets through.



Top and Bottom Polarizing Filters at 90

3. Can we make a theory for this?

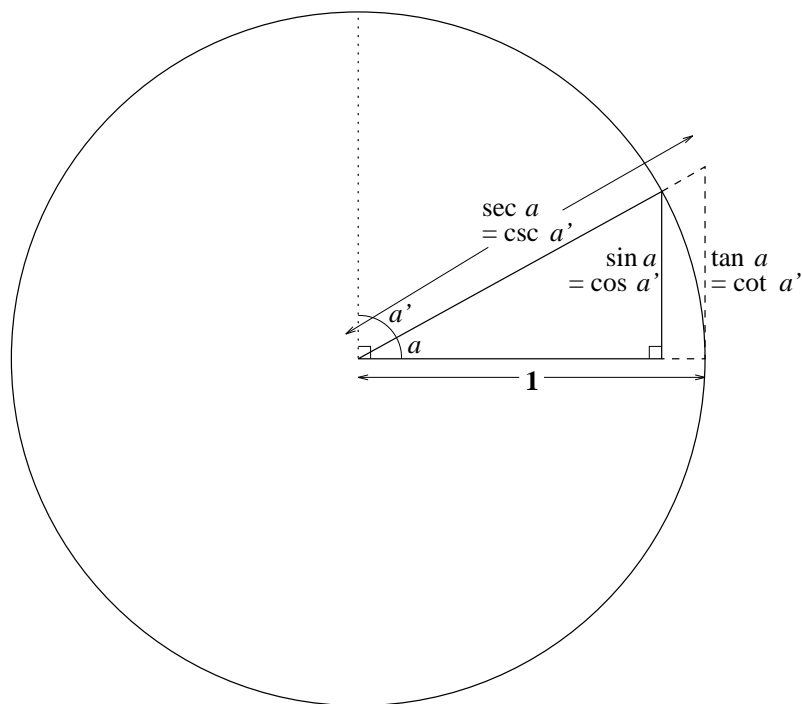
A theory should:

- be falsifiable;
- enable calculation.

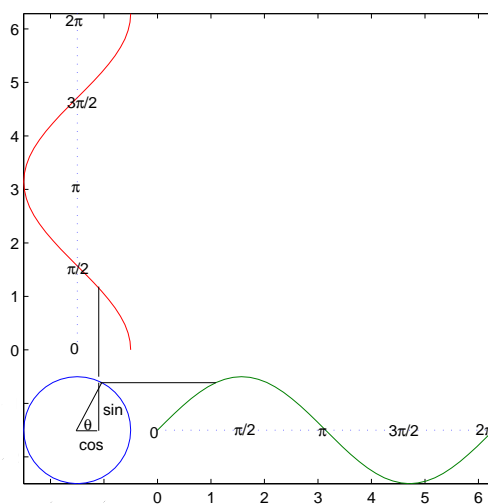
These are related: a theory which leads to deductions in the form of precise numbers can easily be proved wrong by measurements which give different numbers. A theory which survives determined attempts to prove it wrong is good science.

Since we are concerned with angles between directions of polarization, we should revisit trigonometry.

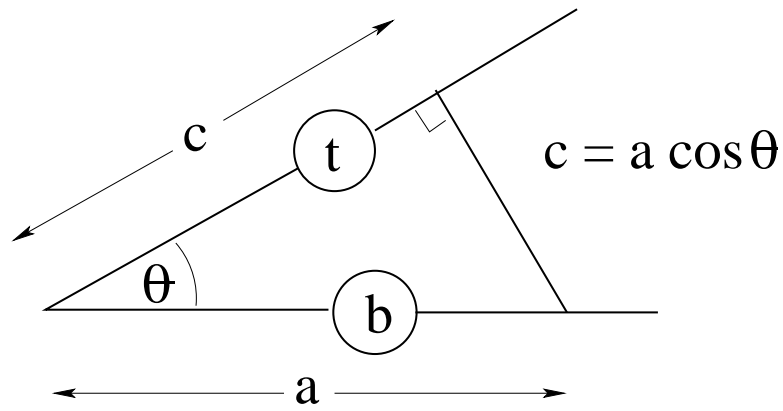
All of trig:



Let's focus on sine and cosine.

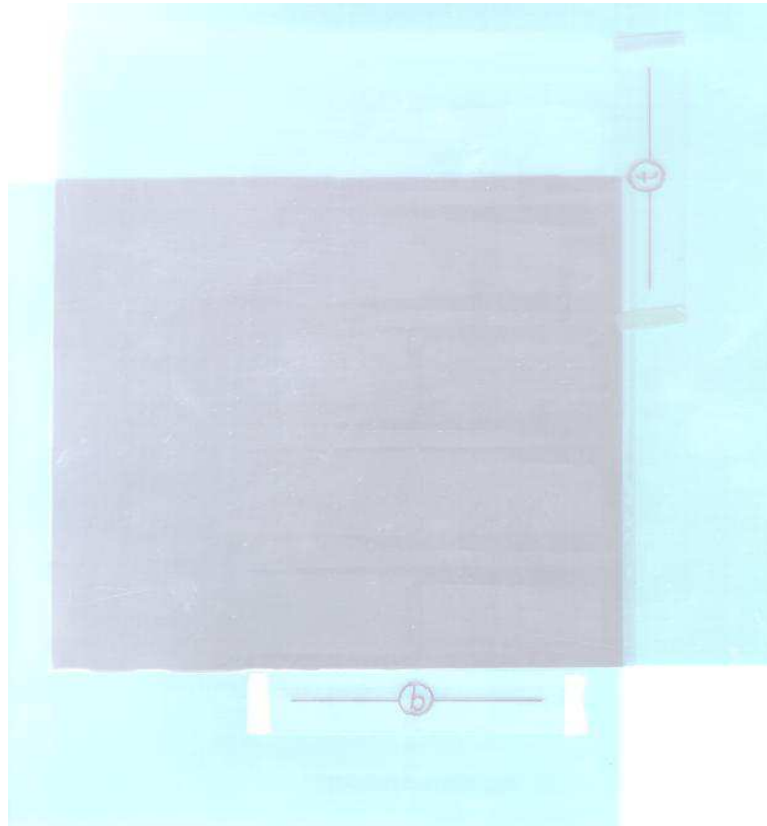


4. Which trig function diminishes to 0 as the angle increases to $\pi/2$? Let's try cosine as a theory.



(a is the *amplitude* of the light through the bottom polarizing filter.)

5. Theories should predict (this is part of being falsifiable). Let's predict what will happen with a third polarizing filter. Here are the top and bottom filters at right angles.

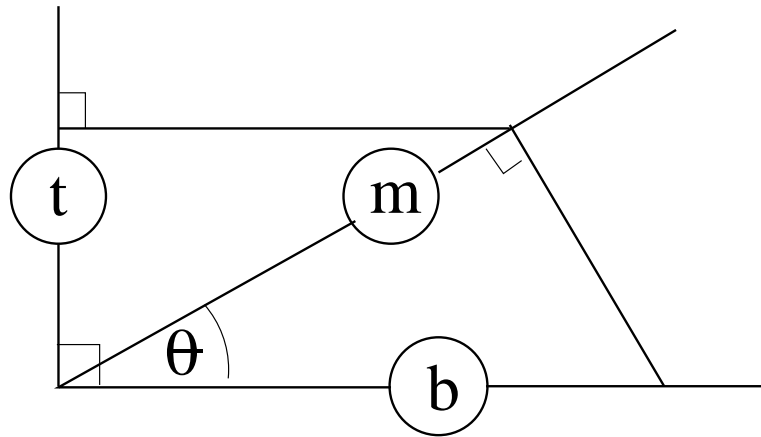


Top and Bottom Polarizing Filters at 90°

What will happen if a third polarizing filter is put on top of them?

What will happen if the third polarizing filter is put in between them?

6. Here's what our theory says about putting it in between.



$$\begin{aligned}
 a_m &= a_b \cos \theta \\
 a_t &= a_m \cos(\pi/2 - \theta) \\
 &= a_b \cos \theta \cos(\pi/2 - \theta) \\
 &= a_b \cos \theta \sin \theta
 \end{aligned}$$

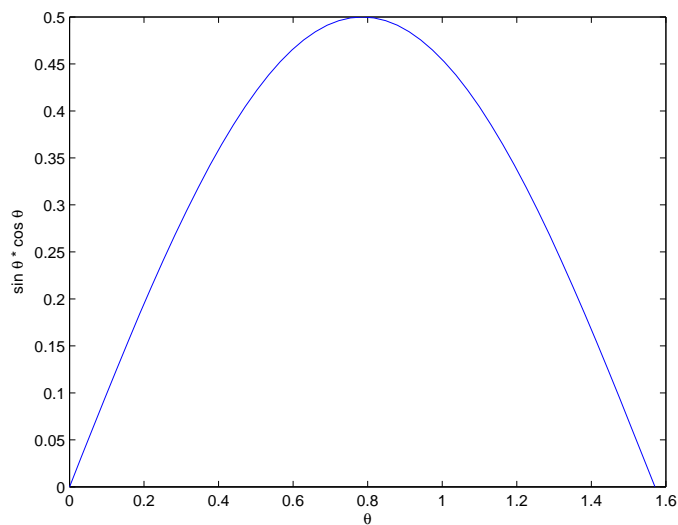
MATLAB can tell us what this is for, say $\theta = \pi/4$:

```
cos(pi/4) * sin(pi/4)
```

```
ans =
```

```
0.5000
```

This is non-zero: light gets through! Does it? Let's try the three polarizing filters at $\pi/4$..
 MATLAB can give us a plot.



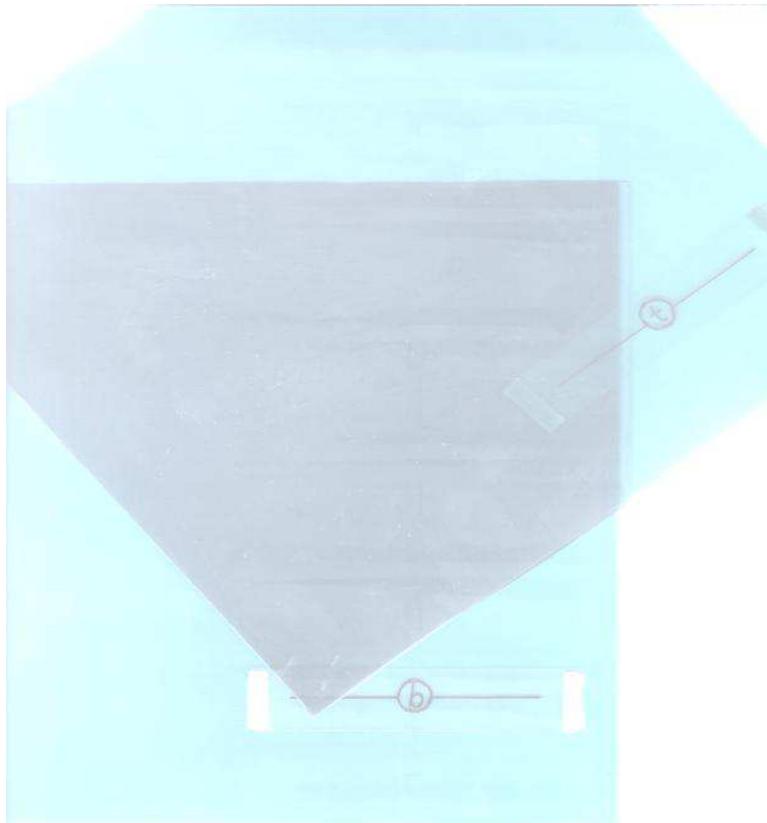
```
theta = [0:pi/100:pi/2];  
cs = cos(theta) .* sin(theta);  
plot(theta,cs)  
xlabel('\theta'), ylabel('sin \theta * cos \theta')
```

This predicts the transmitted light will go from 0 at angle 0 through a maximum of 0.5 at $\pi/4$ and back down to zero at $\pi/2$. Does it? Let's rotate the middle polarizing filter, m.

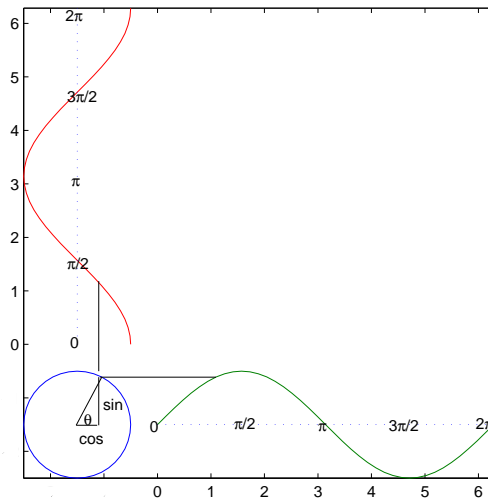
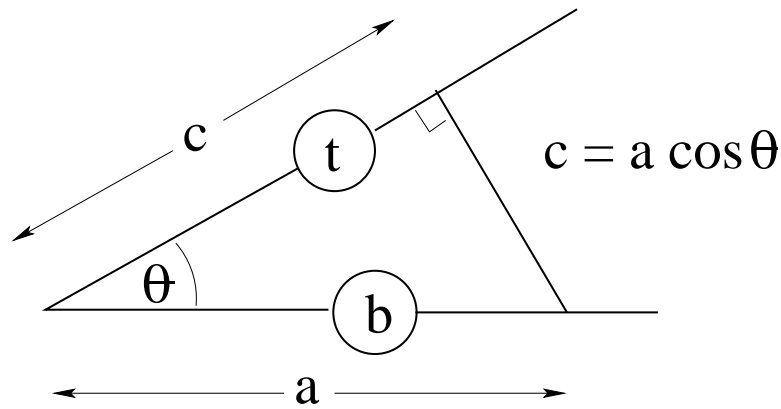
7. As far as we can tell—qualitatively, not quantitatively: without measuring the light strengths—the theory makes the right predictions.

Does it make any wrong predictions?

8. Let's go back to two polarizing filters.



Top and Bottom Polarizing Filters at 35°



Cosine goes negative!

What does negative light strength mean?

How would we force it to be positive?

9. To test it properly, we really need to make the measurements. But this is not a course on how to measure light strength, so let's go to history.

In 1810, Etienne Louis Malus, in Paris, found

$$I_t = I_b \cos^2(\theta)$$

I is the *intensity*.

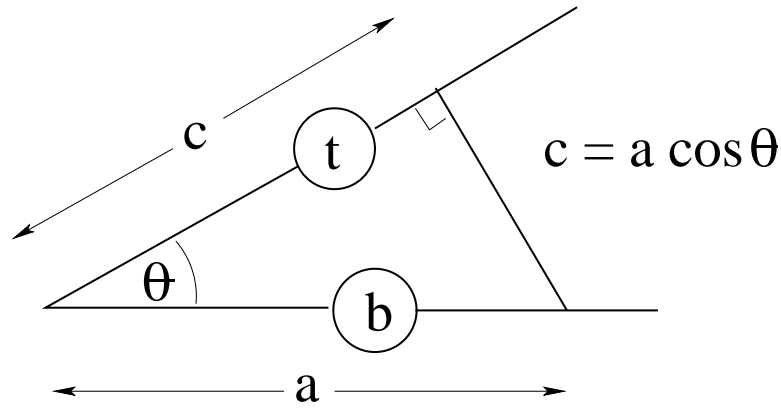
So we take the square.

Do we do this before we work out the three-filter result, or afterwards?

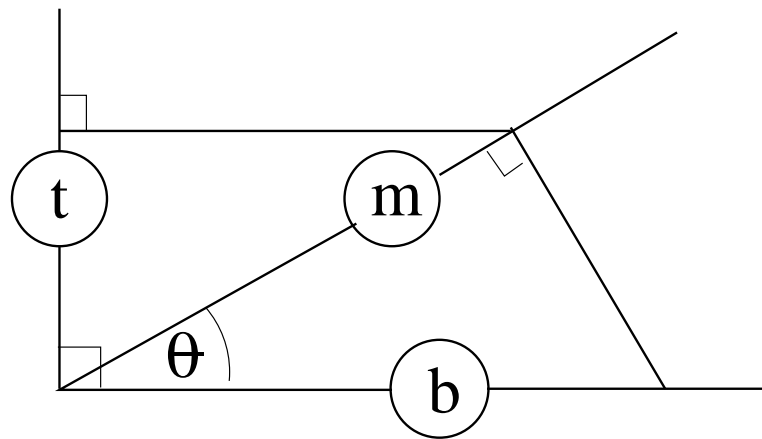
When we look at quantum theory, we'll find out why we do the calculations before squaring, then square to compare with the measurements.

10. So we'll just square our results.

Two polaroids: $I \times (\cos(\theta))^2$



Three polaroids: $I \times (\cos(\theta) \sin(\theta))^2$



11. Algorithm for scientific theories.

1. Invent a theory (imagination)
2. Make a prediction (deduction/calculation)
3. Test prediction against observation (measurement/experiment)
4. If it's wrong, goto 1; else goto 2 (falsification)

(The longer we loop from (2) the better the theory.)

12. Summary

(These notes show the trees. Try to see the forest!)

We have

- explained two polarizing filters and three polarizing filters
 - light has a direction of polarization (angle of line)
 - light has a brightness (“amplitude/intensity”) (length of line)
 - polarization in one direction has components in any other direction, except the direction at right angles to it

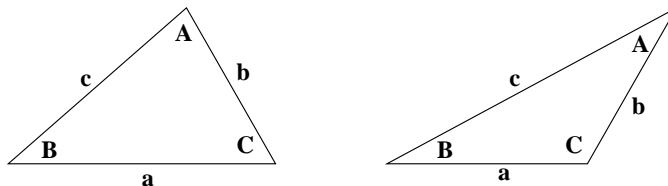
- reviewed trigonometry
- introduced MATLAB
- begun an understanding of scientific theories

II. The Excursions

You've seen lots of ideas. Now *do* something with them!

1. **Quantitative thinking is important.** (Eugene Lehman) Eugene wrapped a belt all the way around the Earth's equator, floating it across the surfaces of the oceans and tunnelling through hills and mountains at sea level, for all 40075 *kilometers*. In the dark of night, Tim cut the belt and inserted an extra length of π *meters*.
 - a) Guess whether, on a sea-level beach: an ant could now squeeze under the extended belt; a toy car could be pushed under the belt; Tim could ride a bike under the belt; Tim could drive a car under the belt; or a building could be built under the belt.
 - b) Use the relationship between diameter and circumference of a circle to see how much higher than sea level the extended belt would be, if kept a circle centered at the centre of the Earth.
 - c) How much could this be increased by moving the belt but keeping it circular?
 - d) How much could it be increased by allowing the belt to be flexible?
2. Get hold of three pairs of polaroid glasses (or two pairs, if you're allowed to break one in half) and repeat all the experiments.
3. **Measuring angles.** Angles can be measured using various units, and we pick one convention throughout. We use "radians" as the unit. A radian is the angle we must turn through to draw a circular arc whose length equals the radius. Since the ratio of circumference to radius is 2π , a full cycle is 2π radians. There are some important reasons for choosing radians, which we will see later, notably in Book 8c. Draw a hexagon inside a circle of radius 1 to see that a radian is just less than the angle inside an equilateral triangle. Other units could be "cycles" or "degrees". Angles measured in cycles would be some fraction of 1, with 1 being the angle of a full cycle. Degrees provide a way of writing the most frequently used angles as integers. 360 degrees make a full cycle, and 360 has many divisors. We must also adopt a convention for the direction of an angle. This convention is that counterclockwise angles are positive and clockwise angles are negative. Unless stated, or negative, angles are assumed counterclockwise. Write in radians, cycles and degrees the following angles: a right angle, a clockwise right angle (don't use a minus sign for this one), the angle that reverses direction, half a right angle, one fifth, one sixth and one twelfth of a full cycle.
4. Use MATLAB to plot both \cos and \cos^2 in the same plot for a full period, $0..2\pi$ What does this mean for two polarizing filters at relative angle between 0 and $\pi/2$? Why do we use $\pi(\pi/2, \pi, 2\pi, ..)$ and not degrees (90, 180, 360, ..)?
5. Use MATLAB to plot both $\sin(t)$ and $\cos(\pi/2 - t)$ in the same plot for a full period, $0..2\pi$ Give a simple argument why they should be the same.

6. (Trig. and triangles) a) On MATLAB plots, show that $\cos(-t) = \cos(t)$ and $\sin(-t) = -\sin(t)$. What are the relationships between $\cos(\pi - t)$ and $\cos(t)$? $\sin(\pi - t)$ and $\sin(t)$?
 b) In both of the triangles shown, why is $b \sin(C) = c \sin(B)$?

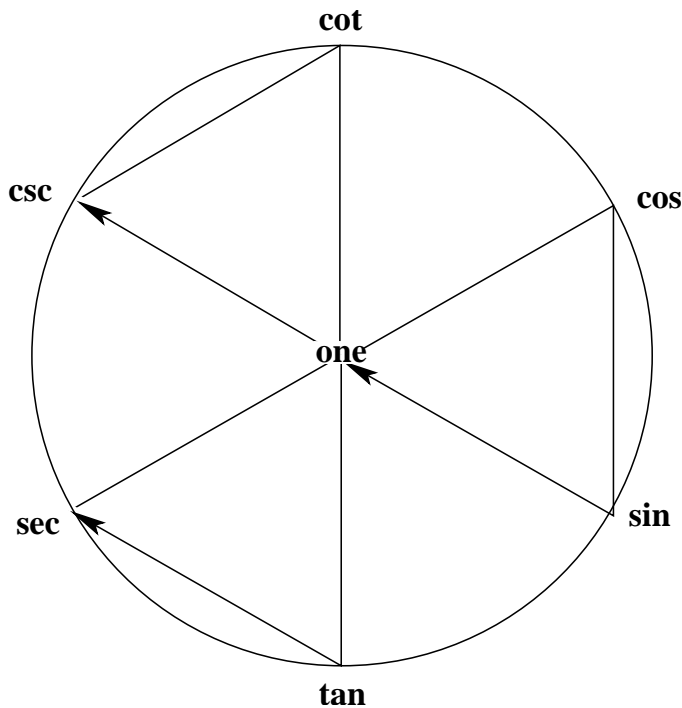


- c) Use symmetry to show that $a \sin(C) = c \sin(A)$ and $b \sin(A) = a \sin(B)$, and hence

$$\frac{a}{\sin(A)} = \frac{b}{\sin(B)} = \frac{c}{\sin(C)}.$$

- d) Going back to the triangles and the relationship of (b), and Pythagoras' theorem, show that $2ab \cos(C) = a^2 + b^2 - c^2$. By symmetry, show that $2bc \cos(A) = b^2 + c^2 - a^2$ and $2ca \cos(B) = c^2 + a^2 - b^2$.

7. Eugene's Clock (Eugene Lehman) is constructed by writing "cos, sin, tan" clockwise in alphabetical order on one side of a circle, then writing their inverses, "sec, csc, cot" diagonally opposite. Construct directed triangles, as in the radioactivity symbol, for each pair and the centre, starting cos-sin-centre, with the arrow showing the end of each loop always pointing in the same direction.



- a) Show that the sum of the squares on the first two vertices of each triangle equals the square on the third vertex, with the centre vertex being 1. (Is it the same whether

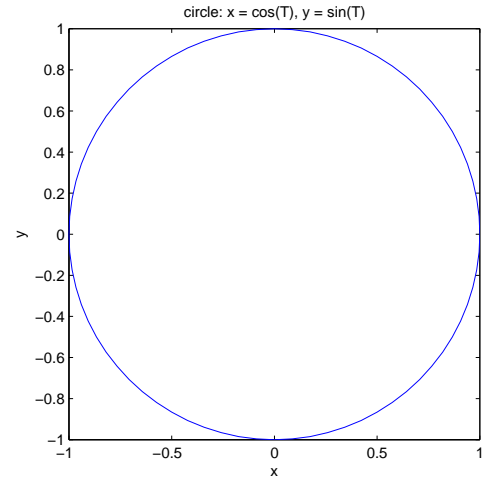
you go in or against the direction of the arrow?)

b) Show that each of the six triples, a, b, c , as you go clockwise around the circumference of the circle, satisfies $a = b/c$. Can you make a pattern for multiplying instead of dividing?

c) What happens in (b) as you go counterclockwise? What about across any diagonal?

8. **Celtic knots and other fun with cos and sin.** What happens if we plot $\cos()$ and $\sin()$ against each other?

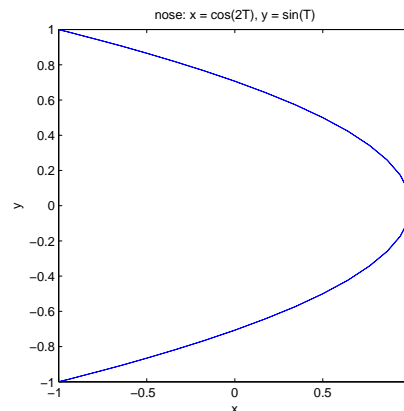
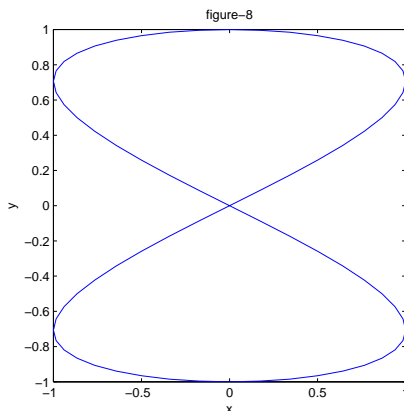
```
% function circle      THM      :
function circle
T = 0:5:360;          % angle parameter in degrees
ang = T*pi/180;      % angle parameter in radians
plot(cos(ang),sin(ang))
title('circle: x = cos(T), y = sin(T)')
xlabel('x')
ylabel('y')
axis square
```



How does this work? Let's calculate x ($\cos(T)$) and y ($\sin(T)$) for a few angles T (in degrees, for familiarity). Note that .7 is really $1/\sqrt{2}$: how does this arise?

T	0	45	90	135	180	225	270	315	360
$x=\cos(T)$	1	.7	0	-.7	-1	-.7	0	.7	1
$y=\sin(T)$	0	.7	1	.7	0	-.7	-1	-.7	0

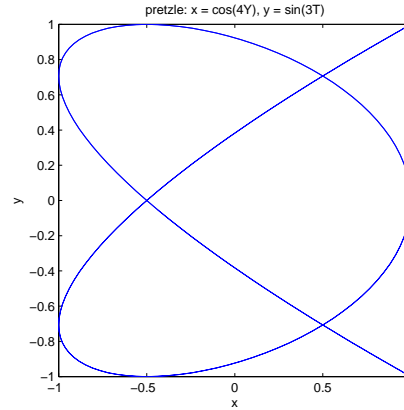
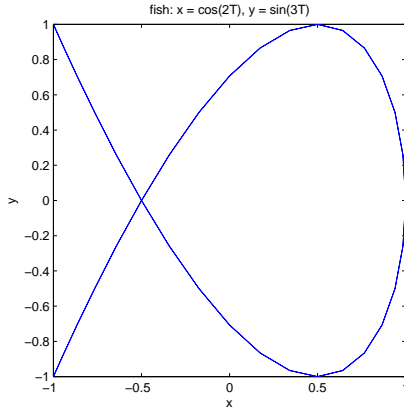
a) Trace the positions of each of these points on the circle. What would happen if we swapped the cos and the sin: $x = \sin(T)$, $y = \cos(T)$?



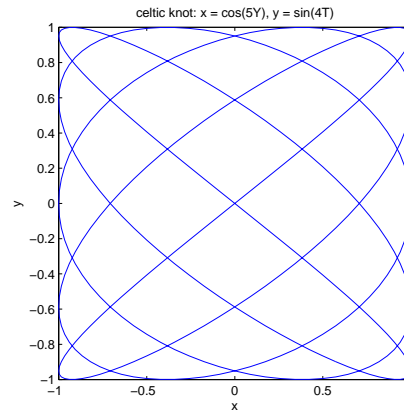
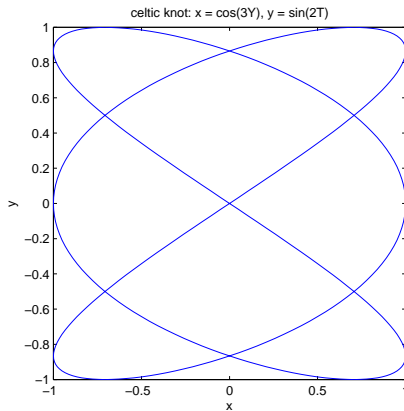
b) How would you make a figure-8? Hint: the drawing starts at the top; write down a sequence of points such as $(x, y) = (0,1), (1,.7), (0,0), ..$ and see if you can figure out

how to generate these with $\sin()$ and $\cos()$. After you've figured it out, then modify the program and try it.

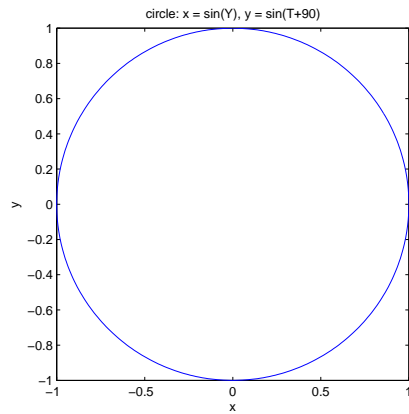
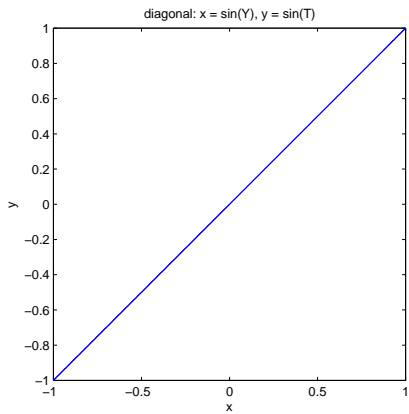
c) How would you turn the figure-8 into a bow-tie? Turning the figure-8 into a bow-tie incorrectly might result in a “nose” (above). Notice the “free ends”. What is the condition that generates such free ends? Here are some more.



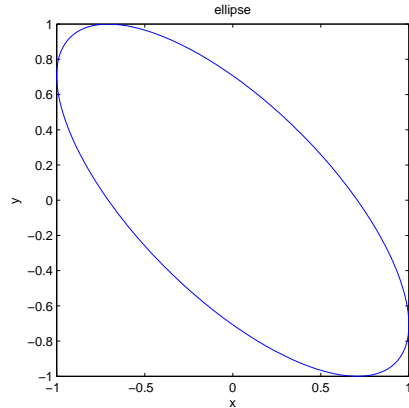
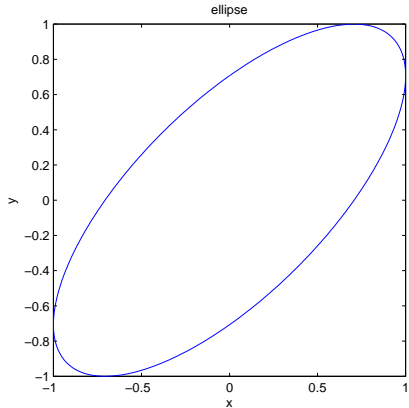
d) The prettiest pictures you can probably get are “Celtic knots”. Here are two. Find some more.



e) What can you say about the common denominator of m and n in a figure drawn by, say $x = \cos(mT)$ and $y = \sin(nT)$? What happens if you use only $\sin()$? Here are a diagonal line and a circle.



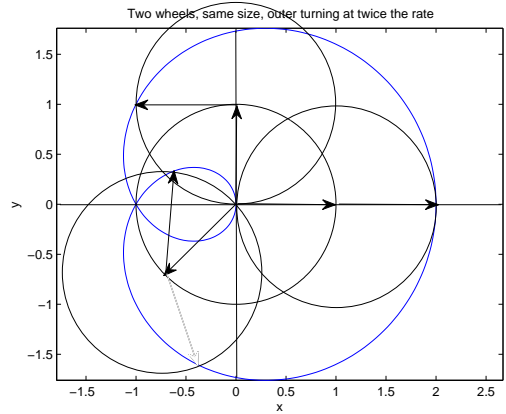
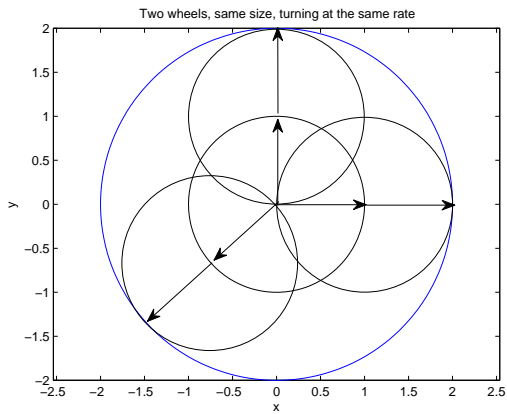
f) How would you make an ellipse? (Hint: it's “in between” a diagonal line and a circle.) How would you make the ellipse tilt the other way? How many different ways can you do this?



g) How would you make a skinnier ellipse? A really skinny ellipse? A fatter ellipse?

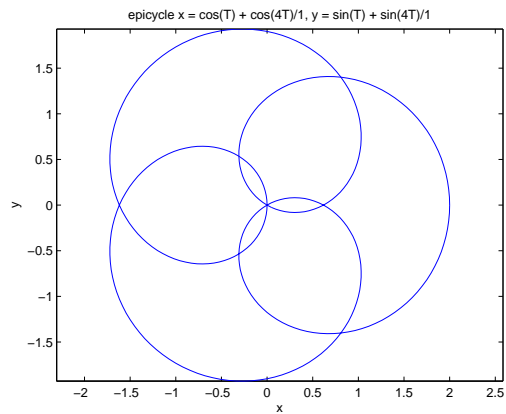
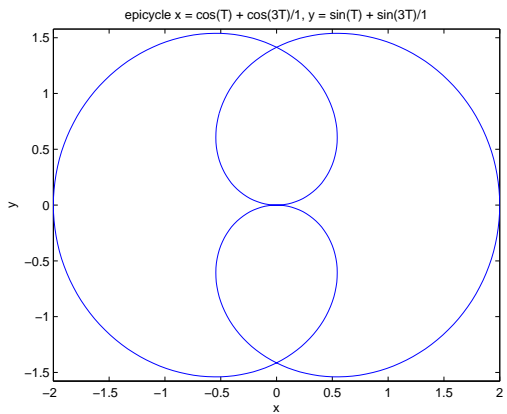
Since we can draw Celtic knots, how about a shamrock? This requires something new: *two* circles, one revolving around the other.

Let's start simply. Imagine a wheel with a pen on its rim, attached by its axle to another wheel. If both wheels are rotating at the same rate the pen will draw a circle. But if the outer wheel is turning twice as fast as the one it's attached to, the pen will do a loop.

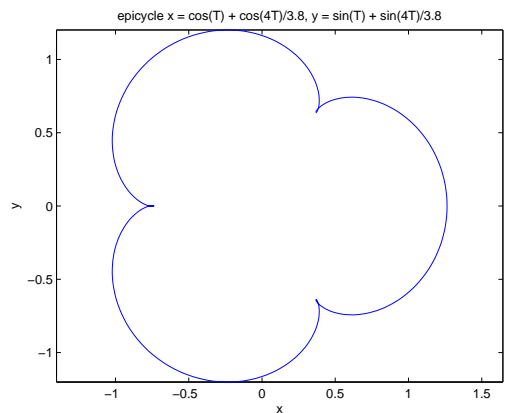


h) How would you change the program to draw (the blue part of) these two pictures? Hint: x will involve two cosines, $\cos(T)$ being one, and since when $T = 0$ the pen is at $x = 2$, these cosines must be added.

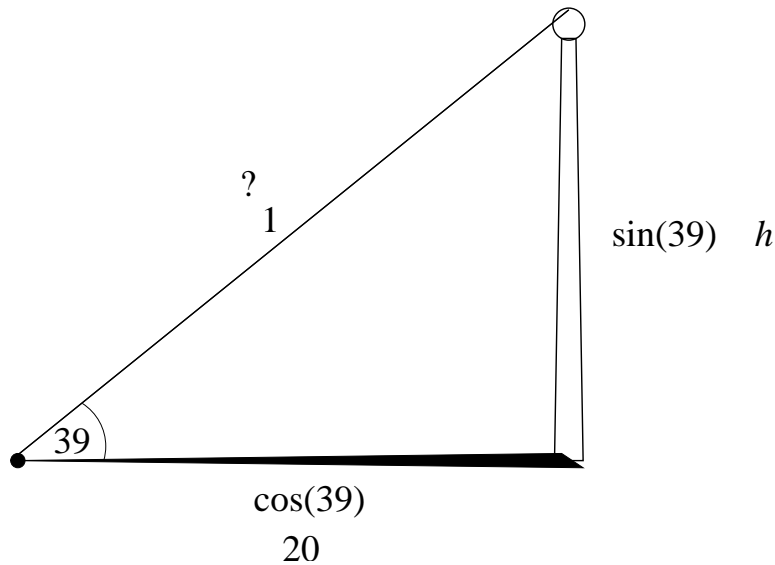
i) Now we observe that since *one* loop is made by rotating the outer wheel *twice* as fast as the inner, we can get more loops by speeding up the outer rotation. Reproduce the following.



j) Now you should be able to adjust to get the following sort-of shamrock. How would you improve it?



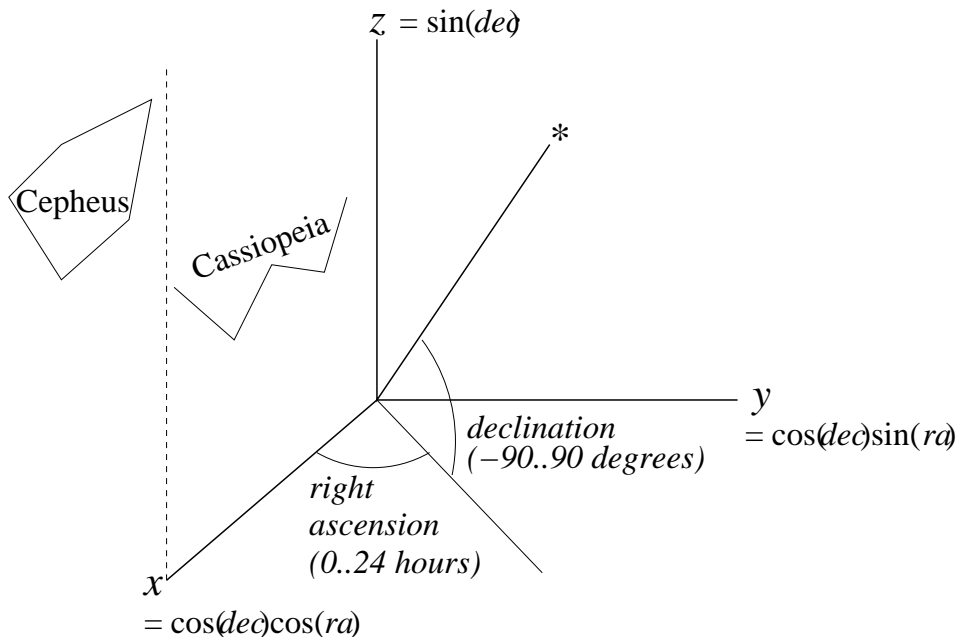
9. Use MATLAB to draw $\cos(t)\sin(t)$ for a full period, $0..2\pi$. Give a simple argument why it is positive in the first quadrant, except for two places where it is zero. What else is needed to explain the behaviour of three polarizing filters, two at $\pi/2$ from each other and the third in between at relative angle between 0 and $\pi/2$?
10. Use MATLAB to draw $(\cos(t))^2$ and $\text{abs}(\cos(t))$ on the same plot, for comparison. Do the same for $(\cos(t)*\sin(t))^2$ and $\text{abs}(\cos(t)*\sin(t))$. Why should we not take the absolute value of the cosine to repair our theory, instead of squaring it?
11. Look up the uses of “polarizing filters”. How can you tell if the “polaroid” sunglasses at a store really are polarizing filters? Look through polarizing glasses, at various angles, at: daylight reflected off a mirror, off a puddle and off a sunny highway. Which direction do you think the light is polarized in that is passed by polarizing glasses worn in the normal way? If you are walking along the bottom of a smooth and shiny wet cliff, wearing polarizing sunglasses, and suddenly the angle of the sun produces a tremendous glare off the cliff, what should you do?
12. Take a pair of polarizing sunglasses outside on a sunny day and see if you can see that sunlight has a strong component of polarization in directions that form a circle around the sun. *Do not look directly at, or near, the sun.* Look up Karl von Frisch and the waggle dance of bees (1947) and discuss the bees’ sense of direction in terms of this polarization of sunlight.
13. Hold, at the middle of two opposite sides, a rectangular piece of cardboard horizontally under a lamp so its shadow is on the floor or on a desk. Rotate the cardboard about the line across the middle. Make a theory about how the shape and area of the shadow changes. Can you find a way to make some measurements to see if your theory is wrong?
14. **Useful cos and sin.** Suppose we have an unclimbable flagpole whose height we must measure. It is perpendicular and standing in a level, horizontal plaza. At a certain time of day, the shadow of the flagpole is 20 meters long (say), which we can measure along the ground. The sun casting this shadow is 39 degrees up in the sky (say). If the triangle made by pole and shadow were a trig triangle—namely with hypotenuse 1—then its base (corresponding to the shadow) would be $\cos(39)$ and its height (corresponding to the pole) would be $\sin(39)$. We don’t know the hypotenuse of the flagpole triangle, and don’t want to know it, so we call it “?”. We do know the base: 20. And we don’t know, but would like to know, the height, which we call h . Since the trig triangle and the flagpole triangle have the common angle of 39 degrees they are similar and we can draw them on top of each other but with two sets of lengths.



The inner set of numbers are for the trig triangle. The outer set of numbers and symbols give the flagpole triangle.

Use a calculator to find $\cos(39)$ and $\sin(39)$, remembering that the 39 is *degrees*. From this, use the diagram to figure out how to find the height h .

15. Trigonometry can help us locate the stars. Using $\sin()$ and $\cos()$ we can convert to x , y and z from the astronomical coordinates of *right ascension* (measured eastwards in hours, minutes and seconds) and *declination* (measured northwards in degrees, minutes and seconds, and southward with the negative of these) as shown. (The zero of right ascension is the “vernal equinox”, part of the line where the plane that the earth’s equator lies in intersects the plane of the Earth’s orbit around the Sun. This puts it between the easily identifiable constellations of Cassiopeia and Cepheus. (Look up the mythology of these constellations!))
 (Look up “spherical coordinates”: how do they differ from astronomical coordinates?)



a) Inspect the following code, which is a slightly roundabout MATLAB way to extract data from a file, in this case the Yale Bright Star Catalog of the 9110 brightest stars, yale.txt, which is stored on the SOCS teaching computers at the location indicated. Run it with $n = 5$ and print the resulting data for the brightest five and the closest five stars. Run it, without printing, for $n = 300$, in preparation for part (b).

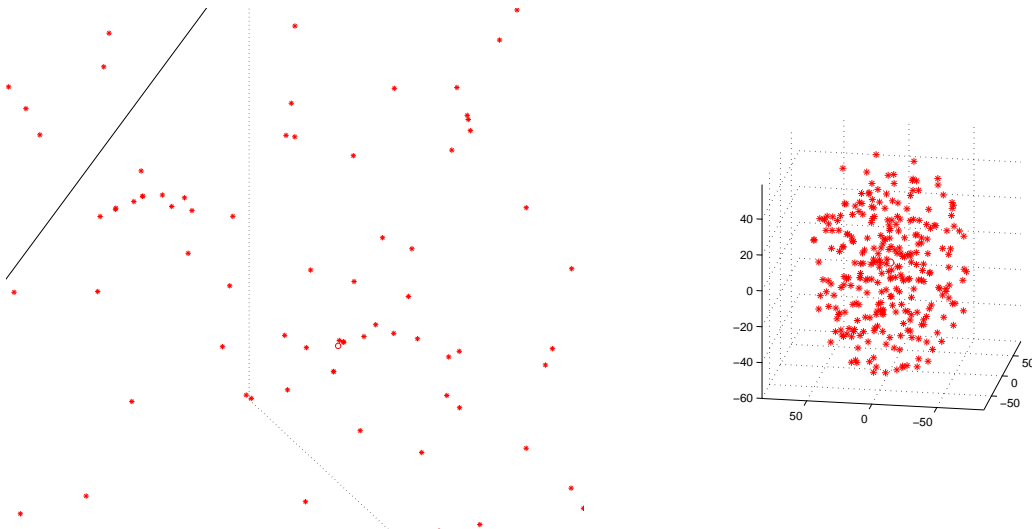
```
% function [brightest,closest] = selectStars(n)
% THM 070502 in file selectStars.m
function [brightest,closest] = selectStars(n)
yaleBSC = textread('/home/course/cs199/yale.txt','%s','delimiter','\n','whitespace',' ');
for k = 1:size(yaleBSC)
    StarNo(k) = str2double(yaleBSC{k}(1:4)); % number in the Yale BSC
    RA19h(k) = str2double(yaleBSC{k}(53:54)); % right ascension, 1900: h, m, s
    RA19m(k) = str2double(yaleBSC{k}(56:57));
    RA19s(k) = str2double(yaleBSC{k}(59:62));
    DEC19sgn(k) = yaleBSC{k}(64); % declination, 1900: +-, d, m, s
    DEC19d(k) = str2double(yaleBSC{k}(65:66));
    DEC19m(k) = str2double(yaleBSC{k}(68:69));
    DEC19s(k) = str2double(yaleBSC{k}(71:72));
    RA20h(k) = str2double(yaleBSC{k}(74:75)); % right ascension, 2000: h, m, s
    RA20m(k) = str2double(yaleBSC{k}(77:78));
    RA20s(k) = str2double(yaleBSC{k}(80:83));
    DEC20sgn(k) = yaleBSC{k}(85); % declination, 1900: +-, d, m, s
    DEC20d(k) = str2double(yaleBSC{k}(86:87));
    DEC20m(k) = str2double(yaleBSC{k}(89:90));
    DEC20s(k) = str2double(yaleBSC{k}(92:93));
    vMag(k) = str2double(yaleBSC{k}(109:113)); % visual magnitude("brightness")
    annMoRA(k) = str2double(yaleBSC{k}(156:161));% annual motion of rt ascension
    annMoDEC(k) = str2double(yaleBSC{k}(163:168));% annual motion of declination
    parx(k) = str2double(yaleBSC{k}(171:174)); % parallax (seconds)
    rankB(k) = k; % rank for sorting brightness
    rankC(k) = k;
end
magrank = sortrows([vMag',rankB']); % ascending by visual magnitude
nearrank = sortrows([parx',rankC'],-1); % descending by parallax
for k = 1:n % n brightest, closest
    bk = magrank(k,2); % select in order of brightness
    brightest(k,1) = StarNo(bk);
    brightest(k,2) = RA19h(bk);
    brightest(k,3) = RA19m(bk);
    brightest(k,4) = RA19s(bk);
    brightest(k,5) = DEC19sgn(bk);
    brightest(k,6) = DEC19d(bk);
    brightest(k,7) = DEC19m(bk);
    brightest(k,8) = DEC19s(bk);
    brightest(k,9) = RA20h(bk);
    brightest(k,10) = RA20m(bk);
    brightest(k,11) = RA20s(bk);
    brightest(k,12) = DEC20sgn(bk);
    brightest(k,13) = DEC20d(bk);
```

```

brightest(k,14) = DEC20m(bk);
brightest(k,15) = DEC20s(bk);
brightest(k,16) = vMag(bk);
brightest(k,17) = annMoRA(bk);
brightest(k,18) = annMoDEC(bk);
brightest(k,19) = parx(bk);
ck = nearank(k,2); % select in order of closeness
closest(k,1) = StarNo(ck);
closest(k,2) = RA19h(ck);
closest(k,3) = RA19m(ck);
closest(k,4) = RA19s(ck);
closest(k,5) = DEC19sgn(ck);
closest(k,6) = DEC19d(ck);
closest(k,7) = DEC19m(ck);
closest(k,8) = DEC19s(ck);
closest(k,9) = RA20h(ck);
closest(k,10) = RA20m(ck);
closest(k,11) = RA20s(ck);
closest(k,12) = DEC20sgn(ck);
closest(k,13) = DEC20d(ck);
closest(k,14) = DEC20m(ck);
closest(k,15) = DEC20s(ck);
closest(k,16) = vMag(ck);
closest(k,17) = annMoRA(ck);
closest(k,18) = annMoDEC(ck);
closest(k,19) = parx(ck);
end

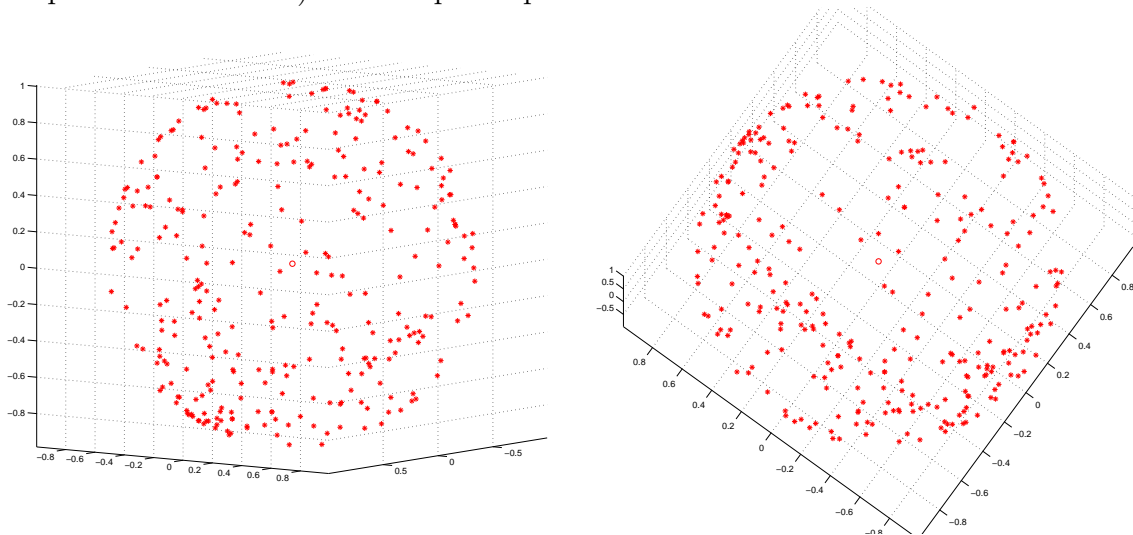
```

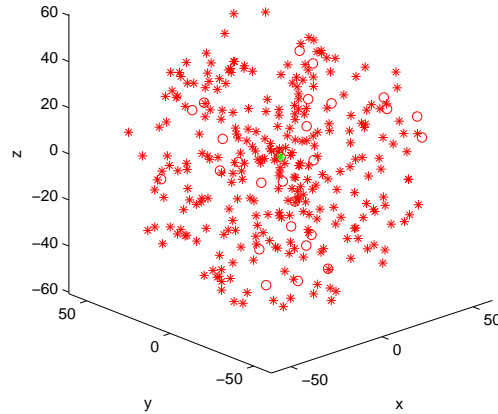
b) Write a MATLAB program to display stars represented in arrays such as **brightest** and **closest** in 3D. (Distance in light years is 3.26 times the distance in parsecs, and distance in parsecs is $1/\text{parallax (sec)}$.) (Use **quiver3()**, and note that the arguments **U**, **V** and **W** of **quiver3()** can in this case just be arrays of zeros. The seventh argument of **quiver3()** can be 0 and the eighth could be **'r*'** to show the stars as red asterisks.) How would you also display our Sun, in the middle of all these stars, as a red circle? Here are snapshots of the pictures you might get. First, the brightest 300 and the closest 300 stars. (The axis units are light years.)



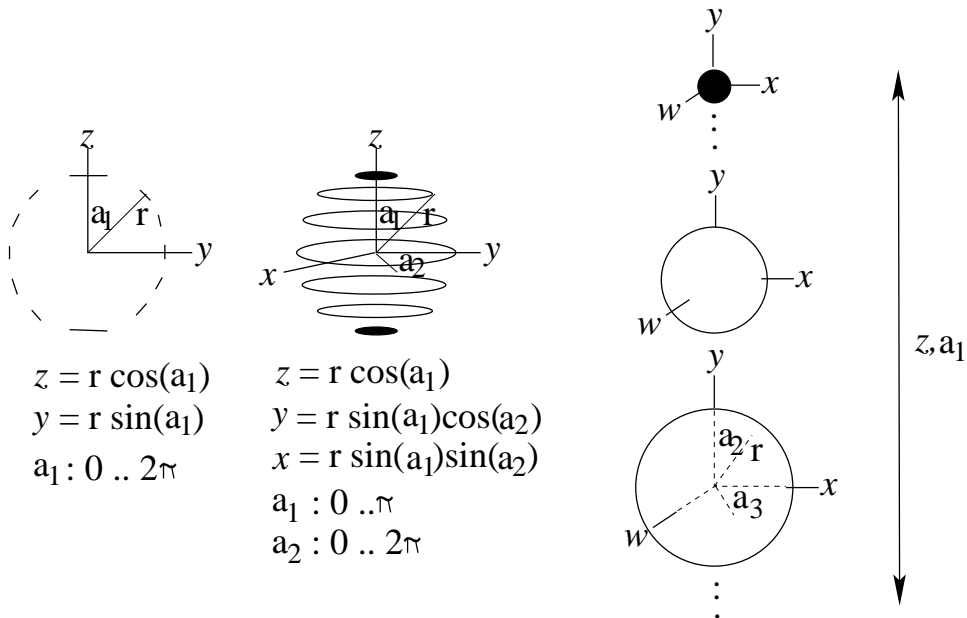
And here are three more snapshots. The first two are of the brightest 300 stars with distances all set to 1 light-year, so that we can see them as on a sphere—the “celestial sphere” seen from outside!. Can you make out the belt and sword of Orion, and the shape of the Little Dipper?

The third shows the 300 closest stars, augmented (red circles) by the closest 27 of the 168 stars so far known to have planets of their own. These stars are stored in the file `/home/course/cs199/exoplanetStars`, with entries for RA 2000 (starting at position 28), Dec 2000 (starting at position 44) and distance in parsecs (positions 76–80). If you write and run programs to select and display these together with the closest 300 stars, you can use MATLAB’s Camera Toolbar to Orbit the camera and to move the camera Forward/Back for better views. How many of the 27 closest stars known to have “exoplanets” coincide with any of the 300 closest stars from the Bright Star Catalog? What is the probability that a star has observable planets (at least from this sample of closest stars)? Look up “exoplanets”!





16. **Hyperspheres.** If you are systematic, you can begin to think about coordinates for spheres in higher dimensions—*hyperspheres*. Note how a circle can be built up from point-pairs at an angle a_1 from the z -axis, with solid “caps” at top and bottom; how a sphere can be built up from circles at an angle a_1 from the z -axis, with solid “caps” at top and bottom; how a 4-dimensional hypersphere can be built up from spheres at an angle a_1 from the z -axis, with solid “caps” at top and bottom; etc.



a) Convince yourself that the general coordinates for a d -dimensional hypersphere are

$$z_1 = r \cos(a_1)$$

$$z_2 = r \sin(a_1) \cos(a_2)$$

$$z_3 = r \sin(a_1) \sin(a_2) \cos(a_3)$$

$$\begin{aligned}
 & \vdots \\
 z_{d-1} &= r \sin(a_1) \sin(a_2) \dots \cos(a_{d-1}) \\
 z_d &= r \sin(a_1) \sin(a_2) \dots \sin(a_{d-1})
 \end{aligned}$$

where all the angles range from 0 to π except the last which is from 0 to 2π .

b) Work out expressions for r and the angles $a_i, i = 1..d - 1$ supposing the Cartesian coordinates z_1, \dots, z_d are given.

17. Look up Etienne Louis Malus, 1775-1812: why did he get interested in polarized light? what was then known about polarized light? what did he do in his career?
18. Look up “photometer” to see if you can devise the experiment needed to measure the light intensity as a top filter, t , is rotated above a bottom filter, b , or as a middle filter, m , is rotated between t and b placed at right angles to each other.
19. Look up Karl Popper’s “The Logic of Scientific Discovery”: is there a summary on the Web? What is “the logic of scientific discovery”? How can the statement “The sun rises every morning” be proved? Disproved? How does science differ from technology, mathematics or philosophy?
20. Wolfgang Pauli (1900–1958) was known to have dismissed poor work by saying “It is not even wrong”. Comment, in the light of Popper’s characterization of science as falsification.
21. Why is astrology not a science? What change in approach might make astrology scientific?
22. Any part of the Preliminary Notes that needs working through.