Excursions in Computing Science: Week i. Rules and Calculations

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I. Prefatory Notes

1. Rules and sums. Teacher, invite your grade scholar to check some of the following rules and continue them up to, say, 13. The "rules" are given in the leftmost column and calculations up to 5 follow. Explain the meaning of Σ and let your pupil write in heir notebook the terms integer, positive integer, triangle number, odd number, square number, tetrahedral number, pyramidal number and cubic number, depending on how far hey gets. (There is even a 4-dimensional simplex of five points, each connected to each other, in the second-last line.)

Your grade scholar should get to know at least the triangle, square, tetrahedral and cubic numbers as well as the positive integers. A keen pupil will be eager to calculate, and will be interested in the relationships among these kinds of number. By all means offer a calculator. (I've been using a 1994 Texas Instruments TI81, which has variables, graphics and is programmable, but lots of others are even better. The main material later in *Excursions in Computing Science* is developed in the MATLAB® language, so that is also an alternative, albeit more sophisticated.)

(Be prepared for digressions if you offer your grade scholar a calculator as powerful as the programmable graphics calculators I am suggesting. Hey will want to explore all the other buttons, too.)

You and your grade scholar can take this material at a number of different levels. The simplest, for example, would be just to explore the calculations started below, without attempting to understand the patterns. This could provide experience to be amplified later by second and third passes through the material.

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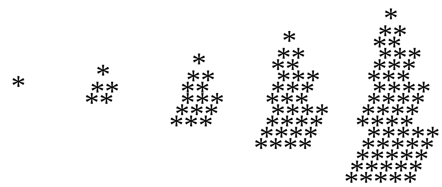
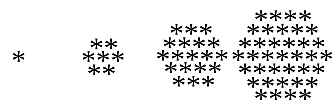
n	1	2	3	4	5	··
Σ_n	1	3	6	10	15	··
$\triangle_n$	*	* **	* ** ***	* ** *** ****	* ** *** **** *****	··
$n(n+1)/2$	1	3	6	10	15	··
$2n-1$	1	3	5	7	9	··

odd	*	**
		 *
			***
			  *
			  *
				****
				   *
				   *
				   *
					*****
					    *
					    *
					    *
					    *
						··

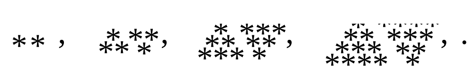
$\Sigma 2n-1$	1	4	9	16	25	··
$\square_n$	*	** **	*** *** ***	**** **** **** ****	***** ***** ***** ***** *****	··
n^2	1	4	9	16	25	··
$n+2\triangle_{n-1}$	1	4	9	16	25	··
$\Sigma\triangle_n$	1	4	10	20	35	··
 n	*	* * **	* * ** * ** ***	* * ** * ** *** * ** *** ****	* * ** * ** *** * ** *** **** * ** *** **** *****	··
$n(n+1)(n+2)/6$	1	4	10	20	35	··

n	1	2	3	4	5	..
tetrahedron$_n$						..
$\sum \square_n$	1	5	14	30	55	..
$\triangle_n$+2▱$_{n-1}$	1	5	14	30	55	..
$n(n+1/2)(n+1)/3$	1	5	14	30	55	..
⬡$_n$					..	
	1	7	19	37	61	..
$\sum$ ⬡$_n$	1	8	27	64	125	..
⬢$_n$, cube$_n$, n^3	1	8	27	64	125	..
$n+6\triangle_{n-1}+6$▱$_{n-2}$	1	8	27	64	125	..
$\sum$ cube$_n$	1	9	36	100	225	..
$\triangle_n+6$▱$_{n-1}+6$ ⬠$_{n-2}$	1	9	36	100	225	..
$\triangle_n^2$	1	9	36	100	225	..

2. Some visualizations. Here are some visual intuitions behind the sameness of some of the rows in the sum tables.

Why is the nth triangle number $n(n+1)/2$? Here is a n by $n+1$ "rectangle" (parallelogram) made from two triangles.

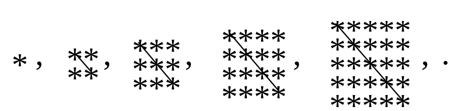
$$\sum n = (\triangle_n + \triangledown_n)/2:$$



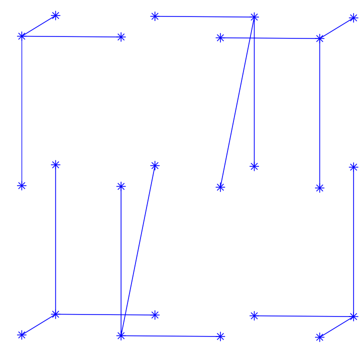
$$= n(n+1)/2$$

Why is $n^2 = n + 2\Delta_{n-1}$?

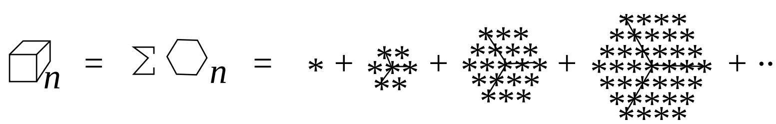
$$\square_n = n + 2\triangle_{n-1}:$$



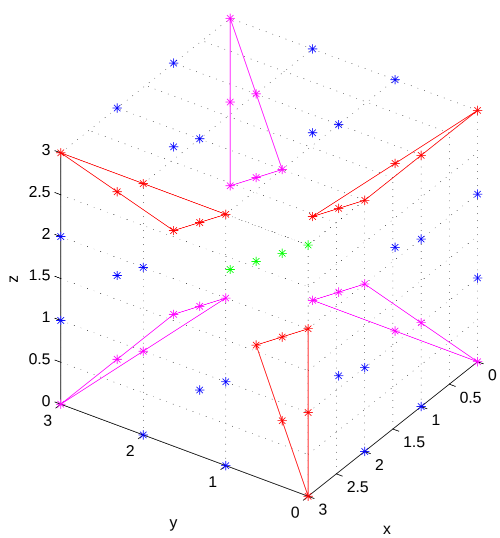
Why is the nth tetrahedral number $n(n+1)(n+2)/6$? Here is a n by $n+1$ by $n+2$ hexahedron (rectangular "cube") made from six tetrahedra.



Why do hexagonal numbers sum to cubes?



Why is $n^3 = n + 6\Delta_{n-1} + 6\diamondplus_{n-2}$? (The symbol for tetrahedral numbers is slightly modified to $\diamondplus$.)



3. Rules and differences

n	1	2	3	4	5	··
$(n+1)-n$	1	1	1	1		··
n^2	1	4	9	16	25	··
$\text{diff}_2(n) = (n+1)^2 - n^2$	3	5	7	9		··
$\text{diff}_2^2(n) = \text{diff}_2(n+1) - \text{diff}_2(n)$	2	2	2			··
n^3	1	8	27	64	125	··
$\text{diff}_3(n) = (n+1)^3 - n^3$	7	19	37	61		··
$\text{diff}_3^2(n) = \text{diff}_3(n+1) - \text{diff}_3(n)$	12	18	24			··
$\text{diff}_3^3(n) = \text{diff}_3^2(n+1) - \text{diff}_3^2(n)$	6	6				··
$\triangle_n^2 - \triangle_{n-1}^2$		8	27	64	125	··
$\square_n - \square_{n-1}$		7	19	37	61	··

4. Rules and programming. We can use a programmable calculator to find the numbers in the sum and difference tables. Here are two programs for the TI81, followed by a brief explanation of how to enter and run the first.

```
Prgm1:  TRIANGLE
Input N
N*(N+1)/2 →T
Disp T
```

```
Prgm2:  TETRAHED
Input N
N*(N+1)*(N+2)/6 →T
Disp T
```

Creating `Prgm1`:
`PRGM` button; select `EDIT`; `ENTER` button.
Type `T R I A N G L E`; `ENTER` button.
`PRGM` button; select `I/O`; select `Input`; `ENTER`.
`ALPHA` button; type `N`; `ENTER`.

ALPHA button; type N; type * (
ALPHA button; type N; type + 1) / 2
STO button; ALPHA button; type T
PRGM button; select I/O; select Disp; ENTER.
ALPHA button; type T
QUIT button (2nd QUIT).

Running Prgm1:
PRGM button; select EDIT; ENTER button.
ENTER button.
After the prompt (?) enter a number (e.g., 5) and ENTER
The answer appears (e.g., 15).

Here are the same two programs written in MATLAB.

```
% function t = triangle(n)      THM      080708
% (In file triangle.m)
% find triangle number n
function t = triangle(n)
t = n*(n+1)/2;
```

```
% function t = tetrahed(n)      THM      080708
% (In file tetrahed.m)
% find tetrahedral number n
function t = tetrahed(n)
t = n*(n+1)*(n+2)/6;
```

To run the first, in MATLAB's Command Window write, say
triangle(3)
and the response is

```
ans =
                      6
```

These programs should not need explanation. Here is a fancier one.

On the TI81

```
Prgm3:  SUMN
Input N
N→S
Lbl L
N − 1→N
If N = 0
Goto F
S + N→S
Goto L
Lbl F
Disp S
```

where to enter the label Lbl L from the EDIT PRGM state:
PRGM button; select CTL; select Lbl; ENTER
ALPHA button; type L

and to enter the condition If N = 0 from the EDIT PRGM state:
PRGM button; select CTL; select If; ENTER

`ALPHA` button; type `N`
select `TEST` button; select =; `ENTER`; type `0`

What this does is: if the condition (`N = 0`) is true, execute the next line (`Goto F`), otherwise skip that line and execute the following line.

Finally, to enter the control transfer, `Goto F`:
`PRGM` button; select `CTL`; select `Goto`; `ENTER`
`ALPHA` button; type `F`

which will jump the running program to the label `F` at the end (the program was written to let the label `F` suggest *finish* and the label `L` suggest *loop*) so that execution escapes the loop `Lbl L .. Goto L` and displays the result of the sum.

In MATLAB

```
% function s = sumN(n)      THM      080709
% (In file sumN.m)
% Sum integers from 1 to n
function s = sumN(n)
s = 0;
for k = 1:n
  s = s + k;
end
```

The nice thing about the MATLAB program is that we can easily change it to sum consecutive triangle numbers instead of just consecutive integers:

```
% function s = sumTriangle(n)      THM      080709
% (In file sumTriangle.m)
% Sum triangle numbers from 1st to nth
function s = sumTriangle(n)
s = 0;
for k = 1:n
  s = s + triangle(k);
end
```

This is called *invoking* the *function* `triangle()` (or "calling" the function). It is very handy to be able to write the code for `triangle` independently and then just use it in another program.

To do this with the TI81 program, we must rewrite `Prgm1:  TRIANGLE` so that it has no input or output (`Input, Disp`) but just accepts and returns values via variables. We cannot use `N`, however, in both `SUMTRIAN` and `TRIANGLE` because we gave different meanings to `N` in the original programs, and they will interfere with each other.

The `sums` programs have a single loop. A program to find the sums of all pairs of squares or cubes has a double loop, one nested inside the other. It also needs a place to store all its results so that they can be displayed. Here is an example calculation, for squares of the first three positive integers, to show what we need.

$j\backslash k$	1	2	3
1	2	5	10
2		8	13
3			18

The storage space needed for up to n squares is clearly the triangular number $n(n+1)/2$, which is 6 in this case for $n = 3$.

The "data structure" provided by programming languages, and some calculators, for such a storage requirement is an *array* or *matrix* of numbers. Such an array might have the name A and hold $n \times n = n^2$ numbers, each "addressed" by values of the *indices* (singular: *index*) j and k, as shown above for 3×3.

Element $j = 2$ and $k = 3$ of A, $A(j, k) = A(2, 3)$ is 13 in the above example.

Here is the MATLAB program to calculate the sums of pairs of up to n squares.

```
% function SSD = squareSumDiff(n)      THM      080720
% (Stored in file squareSumDiff.m)
% Generate matrix containing sums and differences of first n squares.
function SSD = squareSumDiff(n)
for j = 1:n
  for k = 1:n
    if j>k SSD(j,k) = j^2 - k^2;
    else   SSD(j,k) = j^2 + k^2;
    end % if j>k
  end % for k = 1:n
end % for j = 1:n
```

(Note that instead of wasting the space below the diagonal of this matrix, which we don't need (why?), the program stores the differences as well as the sums of the squares.)

The TI81 program is limited to 6×6 matrices.

```
Prgm4:  SQUARESD
1→J
Lbl A
1→K
Lbl B
if J>K
Goto C
J∧2 + K∧2→[A](J,K)
Goto D
Lbl C
J∧2 − K∧2→[A](J,K)
Lbl D
K + 1→K
If K>6
Goto E
Goto B
Lbl E
J + 1→J
If J>6
Goto F
Goto A
Lbl F
Disp [A]
```

The array name `[A]` is entered by pushing the `2nd` button then the `1` button.

Note the nesting of the loops in both programs, although it is much easier to see in the MATLAB program.

5. Reasoning with rules. Now that we have the idea, from the rules in the first three Notes and from the programs in Note 4, that a letter such as `N` or n can be used simply to stand for any number, we can start doing calculations with letters.

We can only make statements this way that are true for *any* number. So we cannot always say, for instance, $2 \times n = 3$, because that would be true only for a particular *value of* n, i.e., $n = 3/2$.

We *can* always say things such as $2 \times (n+1) = 2 \times n + 2$ because that is true for every possible number that could be represented by n. (In fact, because there cannot be any confusion, we often omit the $\times$ and write just $2(n+1) = 2n+1$. We could not do this for numbers only: 2×3 is 6, but 23 is not.)

Let's show that the sum of all positive integers up to n (any number n, but you can start by trying, say $n = 3$ and $n = 4$ to see if it works) is $n(n+1)/2$. The analog of the visualization in Note 2, in which we combined two triangles to get a rectangle, is to take half of

$$\begin{array}{llllllll} 1 & +\,2 & +\,3 & .. & +\,(n-2) & +\,(n-1) & +\,n & + \\ n & +\,(n-1) & +\,(n-2) & .. & +\,3 & +\,2 & +\,1 & \end{array}$$

Since $1+n = 2+(n-1) = 3+(n-2) = .. = (n-2)+3 = (n-1)+2 = n+1$ and since there are n of these sums, all equal, we see that twice the sum from 1 to n is $n(n+1)$, which is what we hoped to show.

Here is an argument that $n^2 = n + 2\Delta_{n-1}$. Note how it can all be written out in a single chain. It is based on the argument we just made that $\Delta_n = n(n+1)/2$. Note the modification for Δ_{n-1}.

$$\begin{aligned} n + 2\Delta_{n-1} &= n + 2(n-1)n/2 \\ &= n + (n-1)n \\ &= n + n^2 - n \\ &= n^2 \end{aligned}$$

We can make a similar, if slightly longer, argument about cubes.

$$\begin{aligned} n + 6\Delta_{n-1} + 6\,\Diamond_{n-2} &= n + 6(n-1)n/2 + 6(n-2)(n-1)n/6 \\ &= n + (3+n-2)(n-1)n \\ &= n + (n+1)(n-1)n \\ &= n + (n^2-1)n \\ &= n + n^3 - n \\ &= n^3 \end{aligned}$$

We can show that the sum of odd numbers, $2n-1$ for $n = 1, 2, 3, ..$, is a square number.

$$\begin{aligned} \Sigma(2n-1) &= \Sigma 2n - \Sigma 1 \\ &= 2\Sigma n - \Sigma 1 \\ &= 2\Delta_n - n \\ &= n(n+1) - n \\ &= n^2 \end{aligned}$$

Finally, here is the sum of squares, using $n^2 = n + 2\Delta_{n-1}$ from above.

$$\Sigma n^2 = \Sigma(n + 2\Delta_{n-1})$$

$$
\begin{aligned}
&= \Sigma n + 2\Sigma\Delta_{n-1} \\
&= \Delta_n + 2 \diamondplus_{n-1} \\
&= \frac{n(n-1)}{2} + 2\frac{(n-1)n(n+1)}{6} \\
&= n(n+1)(\frac{1}{2} + \frac{n-1}{3}) \\
&= n(n+1)\frac{3+2n-2}{6} \\
&= n(n+1)(2n+1)/6 \\
&= n(n+1/2)(n+1)/3
\end{aligned}
$$

6. Square roots and cube roots are the ways of going backwards from squares and cubes, respectively.

Thus, since $3^2 = 9$, we use the sign $\surd$ to go the other way

$$\sqrt{9} = 3$$

Since $2^3 = 8$ (so you see it is very important not the get the order mixed up) we go the other way by a modified $\surd$ sign

$$\sqrt[3]{8} = 2$$

There are alternative symbols, too. See the table:

n	1	2	3	4	5	6	7	8	..
$n^{\frac{1}{2}}$, $\sqrt{n}$, sqrt(n)	1	1.4142..	1.7321..	2	2.2361..	2.4495..	2.6458..	2.8284..	
$n^{\frac{1}{3}}$, $\sqrt[3]{n}$	1	1.2599..	1.4422..	1.5874..	1.7100..	1.8171..	1.9129..	2	

7. Primes. A *prime number* is a positive integer with exactly two different divisors, itself and 1. For example, 2, 3 and 5 are prime. 4 is not (1, 2 and 4 are its divisors). 1 is not (why?).

To check that a number is prime, we could try dividing it by every positive integer smaller than itself.

This would be wasteful in two ways. First, we need not try dividing it by any even number bigger than 2, because if such an even number divides it exactly then so do 2 and whatever times 2 equals the even number, and vice-versa. So we need only try 2 and then odd numbers between 2 and itself.

Second, we don't need to check any potential divisors bigger than the square root of the number, because all bigger integers, if they do go in exactly, will just be the result of dividing by one of the integers smaller than the square root.

For example, for 50, test 2, 3, 5 and 7 only ($7^2 = 49; 8^2, 9^2 > 50$): the test of 2 will reveal 25 as a divisor, so we need not test 25; if we went on to test 5 that would reveal 10 as a divisor so we need not test 10.

Here is a TI81 program which tries 2 then all odd numbers from the integer just below $\sqrt{N}$. (Note that the program starts by making sure N is an integer and at least 2. If not, it stops without producing any result.)

```
Prgm5:  PRIME
Input N
If N ≠ Ipart N                                  If N is not an integer.
Stop
```

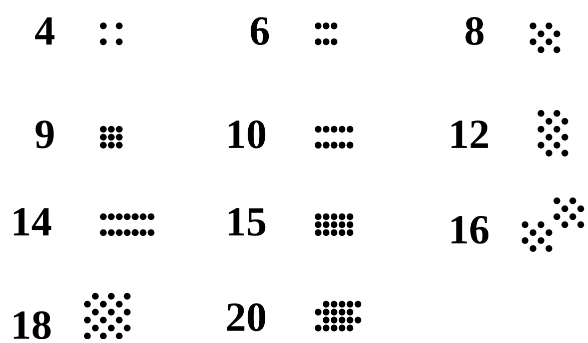
```
If N < 2
Stop
If N = 2
Goto P
If N/2 = Ipart(N/2)                                  If N is even.
Goto G
Ipart √ N → D                                        D is test divisor.
If D/2 = Ipart(D/2)
D − 1 → D                                            Make D odd.
Lbl L                                                Loop back to here.
If D = 1                                             We haven't found any divisor.
Goto P
N/D → V
If V = Ipart(V)                                      D divides N exactly.
Goto F
D − 2 → D                                            Next smaller odd number.
Goto L                                               Loop back.
Lbl P                                                N is prime.
Disp "PRIME"
Stop
Lbl F                                                N has divisor.
Disp "DIVISOR"
Disp D
Stop
Lbl G
Disp "EVEN"
```

The only new programming operations are $\surd$, and `Ipart` which gives the integer part (e.g., `Ipart(2.1)` is 2, `Ipart(2)` is 2, `Ipart(−2.2)` is −2).

Here is the (much shorter) MATLAB program. In MATLAB, `floor()` does the same for positive integers as `Ipart()` does on the TI81. The `mod()` function gives as shorthand way of writing what is in the comment following it.

```
% function p = prime(n)      THM      080709
% (In file prime.m)
% test n for primacy using 2 and odd integers up to sqrt(n)
function p = prime(n)
if n==floor(n) & n>1              % test for plural integer.
  p = true;
  if n~=2
    if mod(n,2) == 0              % if n/2 == floor(n/2)
      p = false;
    else
      for k = 3:2:sqrt(n)
        if mod(n,k) == 0           % if n/k == floor(n/k)
          p = false;
          break
        end % if mod(n,k) == 0
      end % for k = 3:2:sqrt(n)
    end % if mod(n,2) == 0
  end % if n~=2
end % if n==floor(n) and n>1
```

8. Multiplication: rectangles. We can think of multiplication as building “rectangles” in any number of dimensions. Here are some examples.

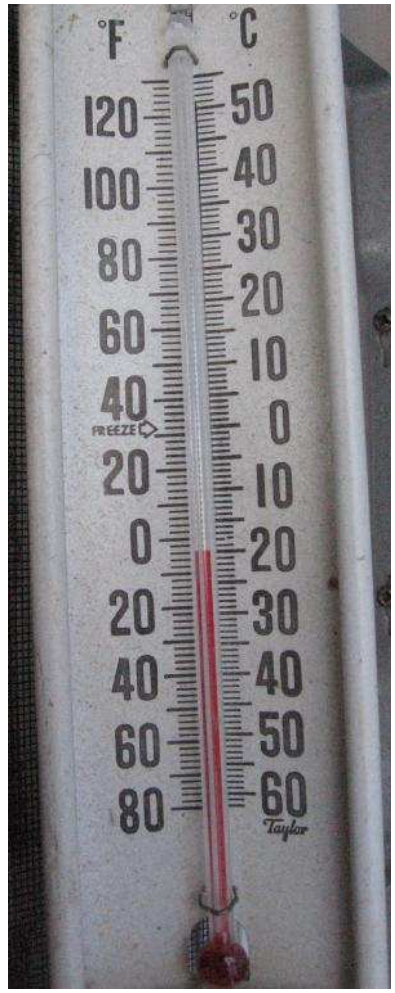


9. Division: slopes. We can think of division as generating sloping lines. Here are some examples. (The notation $\div 2$ etc. means $1/2$ etc., the *reciprocals*.)

14	14	7	14/3	7/2	14/5	7/3	2	7/4	14/9	7/5	14/11	7/6	14/13	1
13	13	13/2	13/3	13/4	13/5	13/6	13/7	13/8	13/9	13/10	13/11	13/12	1	13/14
12	12	6	4	3	12/5	2	12/7	3/2	4/3	6/5	12/11	1	12/13	6/7
11	11	11/2	11/3	11/4	11/5	11/6	11/7	11/8	11/9	11/10	1	11/12	11/13	11/14
10	10	5	10/3	5/2	2	5/3	10/7	5/4	10/9	1	10/11	10/12	10/13	5/7
9	9	9/2	3	9/4	9/5	3/2	9/7	9/8	1	9/10	9/11	3/4	9/13	9/14
8	8	4	8/3	2	8/5	4/3	8/7	1	8/9	4/5	8/11	2/3	8/13	4/7
7	7	7/2	7/3	7/4	7/5	7/6	1	7/8	7/9	7/10	7/11	7/12	7/13	÷2
6	6	3	2	3/2	6/5	1	6/7	3/4	2/3	3/5	6/11	÷2	6/13	3/7
5	5	5/2	5/3	5/4	1	5/6	5/7	5/8	5/9	÷2	5/11	5/12	5/13	5/14
4	4	2	4/3	1	4/5	2/3	4/7	÷2	4/9	2/5	4/11	÷3	4/13	2/7
3	3	3/2	1	3/4	3/5	÷2	3/7	3/8	÷3	3/10	3/11	÷4	3/13	3/14
2	2	1	2/3	÷2	2/5	÷3	2/7	÷4	2/9	÷5	2/11	÷6	2/13	÷7
1	1	÷2	÷3	÷4	÷5	÷6	÷7	÷8	÷9	÷10	÷11	÷12	÷13	÷14
	1	2	3	4	5	6	7	8	9	10	11	12	13	14

On the other hand, division is also just multiplying by the reciprocal, so $3/2$ is just $3 \times (1/2)$, and we can also imagine this as a 3-by-$(1/2)$ rectangle.

10. Negative numbers. Arithmetic with negative numbers is sometimes difficult. This is not surprising: even the idea of zero needed getting used to and did not enter western mathematics until the Muslim mathematicians passed it on from the Hindu mathematicians.



To make negative arithmetic concrete let's use the example of temperature. We can model the addition operation as a rise in temperature and the subtraction operation as a drop. Temperatures in "temperate" climates such as Canada's often go below 0°C, where they are called "minus", meaning negative, temperatures.

(It is important to distinguish the two uses of the $-$ sign. It can operate on two quantities, as in $3 - 2$, which is usually prononced "three minus two". Or it can operate on only one quantity, as in -7, which should be pronounced "negative seven". In the temperature example, we'll pronounce $3 - 2$ as "three drop two" and -7 as "minus seven".)

The arithmetic of $3 - 2$ is easy: $3 - 2 = 1$. It is $2 - 3$ that is trickier. $2 - 3 = -1$, generally prononced "two minus three equals negative one" but, for temperature, pronounced "two drop three equals minus one".
Temperature make this clearer. If it was 2°C yesterday but dropped by 3C° overnight, it must be -1°C by morning.

Negative numbers make us rethink multiplication, too. For positive numbers we have thought of multiplication as generating rectangles, so 2×3 gives a 2-by-3 rectangle of area 6. When we multiply negative numbers we must take a further step, namely multiplying by -1.

Multiplying by -1 is the operation of *changing direction* on the line of numbers. Thus, on the thermometer, if the temperature has risen 3C°, then changing direction would make it drop 3C°: we say it has risen -3C°.

If we multiply a second time by -1, we change direction again, which on a line can only mean that we are again going in the original direction: $-1 \times -1 = 1$. (Or, being fussy, $(-1) \times (-1) = 1$.)

So multiplying two negative numbers, say

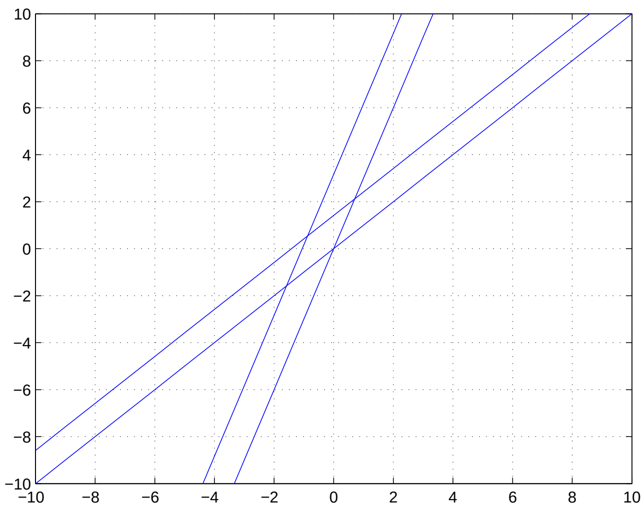
$$-2 \times -3 = (-1 \times 2) \times (-1 \times 3) = -1 \times -1 \times 2 \times 3 = 2 \times 3 = 6$$

requires us to deal with the -1s first, then to think about rectangles.

To multiply a positive by a negative number, take the same steps, but note that the result is negative

$$2 \times -3 = 2 \times (-1 \times 3) = -1 \times 2 \times 3 = -1 \times 6 = -6$$

11. Pictures of rules. Use your calculator or MATLAB to draw the following pairs of "railway tracks".



On the TI81
`RANGE` button; select `Xmin`; type `(-)10`;
(Note that `(-)`, "negative", is different on the calculator from −, "minus". `(-)`is an "adjective", modifying a single number. − is a "verb", operating between two numbers.)
Still in `RANGE`:
select `Xmax`; type `10`;
select `Xscl`; type `1`;
select `Ymin`; type `(-)10`;
select `Ymax`; type `10`;
select `Yscl`; type `1`;
select `Xres`; type `1`.
Now press the `Y=` button; select Y_1; type `X`|`T`;
`GRAPH` button.

This will plot one of the lines. The other lines may be described under Y_2, Y_3 and Y_4 after pressing `Y=` again. Let's get this far in MATLAB, too, before thinking about how to get the other lines.

In MATLAB, create a file `railwayTracks.m`

```
% railwayTracks.m                               THM                         090113
% plot pairs of straight lines: equally separated "railway tracks"
X = -10:1:10;
Y1 = X;
plot(X,Y1,'b')
axis([-10 10 -10 10])
grid on
```

and then type `railwayTracks` in the command window.

We get the parallel track by adding a "constant" to `Y = X`. You can try Y_2 `= X + 1` on the calculator. In MATLAB, this would be `Y2 = X + 1` and you must change the `plot()` line to `plot(X,Y1,'b',X,Y2,'b')`. Look carefully at the result. It is not what the figure shows. You'll see in a moment that I did not use 1 as the constant.

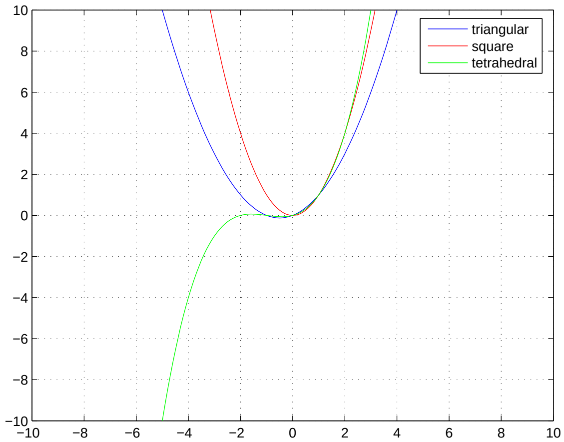
We get a line going in a different direction by multiplying `X` by a constant. I used Y_3 `= 3X` on the calculator. (`Y3 = 3*X` in MATLAB and change `plot()` to `plot(X,Y1,'b',X,Y2,'b',X,Y3,'b')`.)

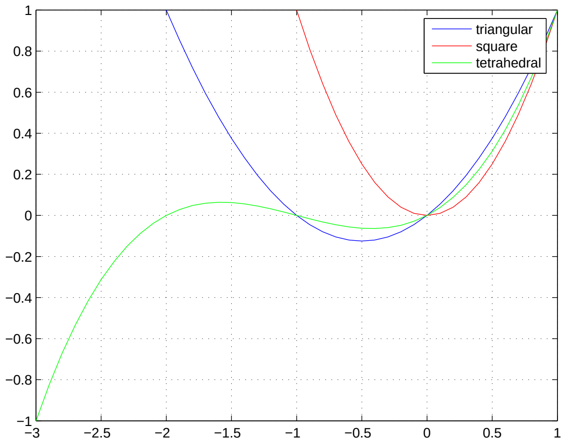
The multiplier of `X` is called the "slope" because it gives the direction of the line. For the first two lines, the slope is 1, For the third line it is 3.

The fourth line is another line of slope 3, but moved upwards by adding another constant. Try `Y4 = 3*X+1`. This is not what I used.

How did I get two pairs of lines separated by equal distances? In fact, if you look closely, those separations are both 1, going perpendicularly from one track to the next in each pair of railway tracks. Try making the constants $\sqrt{s^2+1}$, where s is the slope. That is, $\sqrt{1^2+1}$ for the first pair and $\sqrt{3^2+1}$ for the second.

12. Nonlinear plots. MATLAB and the calculator can also plot the other expressions we have seen this week. Let's try plotting the rule that gives triangular numbers (and the sum of $1+..+n$), $n(n+1)/2$, the rule that gives square numbers, n^2, and the rule that gives tetrahedral numbers (the sum of triangluar numbers), $n(n+1)(n+2)/6$.



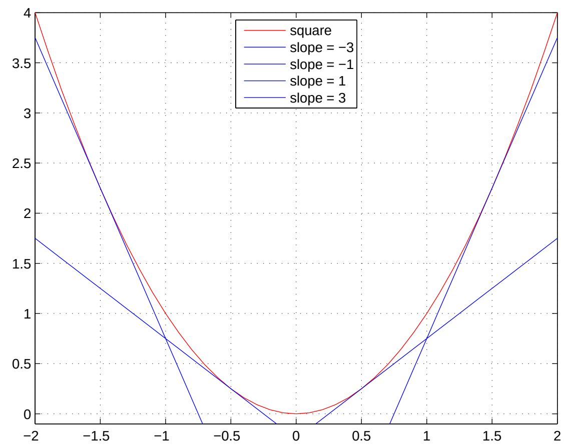


Here is the MATLAB program that gives the zoomed plot on the right.

```
% nonlinPlot.m                              THM                              090113
% plot rules for triangular, square, tetrahedral numbers
X = -3:.1:1;
Y1 = X.*(X+1)/2;
Y2 = X.^2;
Y3 = X.*(X+1).*(X+2)/6;
plot(X,Y1,'b',X,Y2,'r',X,Y3,'g')
axis([-3 1 -1 1])
grid on
legend('triangular','square','tetrahedral')
```

Note that n in the expressions has been renamed `X` for the program. This is not necessary in MATLAB, but is consistent with the `railwayTracks` program and with the caclulator.

Since the slope of a nonlinear expression in `X` changes with `X`, it cannot simply appear as s in the expression itself. Here are a picture of the slopes of X^2 at $X = -3/2, -1/2$, $1/2$ and $3/2$, and the MATLAB program that made the plot.




```
% parabSlope.m        THM          090113
% slope of the rule for square numbers
X = -2:.1:2;
Y1 = X.^2;
Y2 = -3*X - 9/4;
Y3 = -X - 1/4;
Y4 = X - 1/4;
Y5 = 3*X - 9/4;
plot(X,Y1,'r',X,Y2,'b',X,Y3,'b',X,Y4,'b',
     X,Y5,'b')
axis([-2 2 -.1 4])
grid on
legend('square','slope = -3','slope = -1',
 slope = 1','slope = 3','Location', 'Best')
```

13. Summary

(These notes show the trees. Try to see the forest!)

1. Squares and cubes and how to find them using odd and hexagonal numbers or triangle and tetrahedral numbers, and how to sum them using triangle, tetrahedral numbers and the like; pyramidal numbers.
2. Visualizing the relationships among these kinds of number.
3. Programs to calculate some of these kinds of number.
4. Differences of positive integers, their squares and cubes; how far we can go on finding differences.
5. Letters representing any number: the rules for working with letters are the same as the rules of arithmetic.
6. Square roots and distances; cube roots.
7. Prime numbers and programs to test for primacy.
8. Multiplication as rectangle, division as slope.
9. Negative numbers.
10. Plotting: the rules make pictures.

II. The Excursions

You've seen lots of ideas. Now *do* something with them!
(Some excursions are credited to the people who are not necessarily their originators but who suggested them to me.)

1. **Mathematical truth** Does 1 + 1 always = 2? Think about a drop of rain on the car window (1) encountering a nearby raindrop (+ 1): do they remain 2 raindrops or become 1 (1 + 1 = 1)? Think about lending a friend your pet female rabbit (1) while your family goes away on an assignment for a year or two, and your friend already has a pet male rabbit (+ 1): are you sure there will be exactly 2 rabbits when you return, or does, maybe, 1 + 1 = 7 in this case?
   Mathematical truths are different from scientific truths. If a counterexample is found for a

scientific statement (such as "the sun rises every morning (in Montreal)"), that statement is not a scientific truth. Mathematical statements are abstractions from important situations (most of the times we can think of, $1 + 1 = 2$) and it is being useful that makes them important: they help us reason in the situations where they do apply, or they give us useful ideas to understand such relevant situations.
Think of some situations where counting and adding are useful, and of some situations where they are not.

2. **Counting in tongues.** Find out how to count from 0 to a million−1 in as many different languages as possible. How many different words does each language need to do this? Which language has the fewest different words? How many more words does it take to count from a thousand to a million−1? From a hundred to a thousand−1? (Travellers' phrase books are usually easier to use for this research than translation dictionaries.)

3. `1 2 3 4 5 6 7 8 9 10 = 100`
Find a way to replace some of the blanks by `+` so that the above is true. If a blank is not replaced by `+` then the digits either side of it are combined to form a multi-digit number, e.g.,
`1 + 2 + 3 = 6,   1 2 + 3 = 12 + 3 = 15,   1 + 2 3 = 1 + 23 = 24,   1 2 3 = 123.`
Can you do it with only the nine digits `1 .. 9 = 100`? What if you use other operators as well as `+`: $-$, $\times$, $\div$?

4. **Squares.** Find pairs of square numbers which:
a) sum to another square (Find three examples and say which can be calculated from which others. Such sets are called *Pythagorean triples*. Look up Pythagoras of Samos 580–572 BCE to 500–490 BCE);
b) sum to a prime number (A *prime* is a positive integer which has exactly two different divisors, itself and 1. Why is 1 not a prime under this definition?);
c) sum to a number, not necessarily square or prime, which is also the sum of two other squares.
d) Find a number whose square and cube sum to the square of twice itself ($n^2 + n^3 = (2n)^2$ Try $(2n)^2 - n^2 = n^3$).

5. **Cubes.** Find pairs of cubic numbers which:
a) sum to a prime number (Think laterally!);
b) sum to a number which is also the sum of two other cubes (The smallest such is called Ramanujan's number: look up Srinivasan Raman*u*jan, 1887–1920.).
c) A *Fermat triple* is three numbers, k, m, n, such that $k^p + n^p = m^p$ for any integer $p > 2$. Can you find any for $p = 3$? Look up Pierre Fermat (1601–1665) and "Fermat's Last Theorem".
d) Find three adjacent positive integers whose cubes sum to the cube of the next integer.
e) Show that 1, 3+5, 7+9+11, 13+15+17+19, 21+23+25+27+29 and so on are cubes. Use results in the sum table and the difference table to argue that this is generally true. (Note the average value and the number of entries in each of these groups.)

6. Write n^2 as the sum of two triangles of different sizes and nothing else.

7. Write n^3 as the sum of six tetrahedra of different sizes and nothing else.

8. (Ken Murata) How would you display ten nectarines for a party?

9. The connection between powers and triangle numbers, tetrahedral numbers and so on is given by *Stirling numbers*, which take relationships such as

$$n^2 = n + 2\Delta_{n-1}$$

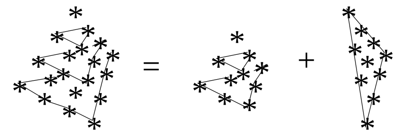
and

$$n^3 = n + 6\Delta_{n-1} + 6 \diamondplus_{n-2}$$

to any number of dimensions. Look up Stirling numbers.

10. Find a way to write the nth pyramidal number as two tetrahedral numbers, and a way to write the nth square number as three pyramidal numbers.

11. Supposing that $\diamondplus_n = n(n+1)(n+2)/6$, show that $\diamondplus_{n+1}$ has the corresponding form $(n+1)((n+1)+1)((n+1)+2)/6$.



12. Compare the constants we got at the end of the differencing processes for n^2 and n^3 with the denominators of the formulas giving the triangular numbers and the tetrahedral numbers, respectively.

13. Rewrite the TI81 `Prgm1:  TRIANGLE` and invoke it from `Prgm3:  SUMN` modified to sum triangle numbers instead of integers.

14. “Martian arithmetic” (Eugene Lehman.) Martian arithmetic uses $+$, $-$, $\times$, / and ^ in the proper way, but scrambles the meanings of the ten digits 0,1,2,3,4,5,6,7,8,9. The Marinaris Stone, discovered on the Moon and identified as an ejectum from a Martian asteroid strike, contains the only clues:
a) $8\times7 = 8$
e) 7^4 = 6
c) $7 - 33 = -1$
b) $4\times9 = 39$
d) $51/2 = 2$
f) $3+65 = 69$
What are ? (in Martian **and** Earthian) in
h) 7+? = 9
g) 40+94 = ?

15. Choose two specific successive values of n to work through the argument in Note 5 about the sum of the first n positive integers. (For instance, 100 and 101.)

16. For each step in the arguments in Note 5, explain which property of arithmetic on “any number n” allows the step to be made.

17. (Eugene Lehman.) Eugene usually takes the 17:00 train home after a day of teaching unnecessarily advanced mathematics in the school, but last week a class was cancelled and he took the 16:00 train. Shirley always drives to the station arriving just in time to meet the 17:00 train. Since she was not there yet when Eugene arrived on the 16:00 train last week, he started walking along her route. When they met she picked him up and took him straight home, arriving there half an hour earlier than usual. How long did Eugene walk?
Use letters, t for the usual arrival of the 17:00 train, d for Shirley’s usual *round-trip* drive, variants t' and d' for the corresponding times and durations last week, and w for how long Eugene walked last week, make an equation relating the usual commute with last week’s, and see what arithmetic on the letters tells you.

18. (Eugene Lehman.) The bank teller made a mistake and instead of withdrawing and giving you D dollars and C cents, which you asked for, gave you instead C dollars and D cents. You had spent 5 cents in a chewing gum machine before you counted up and found you now had exactly twice what you had asked for. How much was that?
a) Show that $98C - 5 = 199D$. The process of arriving at this equation from the equation you should start with is called ”The Balance”, or, to the Arabs who invented it, “al jabr”: if you add anything on one side of the “=” you must also add it on the other side, or else the equation will get out of balance; if you subtract anything from one side of the “=” you must also subtract it from the other. This is true whether you are doing arithmetic on letters or

just on numbers.
b) (a) gave you one equation in two unknowns. Think of a second equation in C and D which is approximately true, from the statement of the problem, and see what the two equations give you. You may have to modify your approximation, but after just a little trial and error you should find a solution.
c) Write a calculator program instead of (b), which takes, say, C through all possible values (what are they?) and uses (a) to find, say, D for each one. It can stop the first time you find a D which is an integer (whole number).
d) Integer problems such as this are called Diophantine. Look up Diophantus ($\sim$ 200–$\sim$ 284).

19. (Wofgang Rasmussen.) A jug contains exactly one litre of white wine. A second jug contains exactly one litre of red wine.
a) Suppose you take a ladleful of white wine from the first jug and pour it into the second. Then take exactly the same amount of the mixture from the second jug and pour it back into the first. Show that, after this, the amount of red wine in the first equals the amount of white wine in the second. Hint: let W litres be the amount of white now in the second jug. How much red is in the second jug? How much white is in the first jug? How much red is in the first jug?
b) What if you had used a teaspoon instead of a ladle? A half-litre cup? The whole litre? What if you had done several transfers (an even number) with exactly the same amounts going each way?
c) Another way to look at (a) is to show, assuming red and white molecules are the same size, that the number of molecules of red in the ladle going from jug 2 back to jug 1 is the same as the number of molecules of white left in jug 2 after the mixture has been transferred to jug 1. Hint: let m be the number of molecules in a ladleful and w be the number of white molecules left in jug 2 at the end.
d) You are led to a table in a darkened room, on which you are told that there are a couple of decks of cards, all face-down except for 10 that have been turned face-up. You are instructed to make a second pile from among the cards in that pile, such that the number of face-up cards is exactly the same in the two piles, old and new. You are allowed to do anything you like with the cards, but not damage them or turn on a light. How do you do it?

20. (Ramanujan's address.) Ramanujan lived at address R on a one-sided street of N houses (houses are numbered consecutively from 1 to N, including R); he could never remember his own address (unlikely for Ramanujan, who was friends with every number) but he could remember that the sum of the addresses below but not including his equalled the sum of the addresses above but not including his; how many houses were on the street and where did he live?
a) Why does this problem involve finding triangular numbers which are also square numbers? Try writing three columns, n, Δ_n and $\Box_n$, and rows for $n = 1..10$ or 12. Can you find two solutions among these numbers?
b) Can you think of a shortcut? Why must either both $N/2$ and $N+1$ or both N and $(N+1)/2$ be squares?
c) If Ramanujan's street contains between 50 and 500 houses how long is it and where does he live?
d) Write a program to find subsequent pairs (R, N). Look for patterns in the results.

21. Check the roots of Note 6 and confirm in particular that $(\sqrt{n})^2 = n$ and $(\sqrt[3]{n})^3 = n$.

22. What are $\sqrt{3^2+4^2}$, $\sqrt{5^2+12^2}$, $\sqrt{9^2+12^2}$, $\sqrt{8^2+15^2}$, $\sqrt{6^2+8^2}$?

23. On a new sheet of paper (which will have right angles at each corner) carefully measure 4cm down from the top left corner and 3cm rightwards from the same corner. Carefully measure the length of the line connecting the two points you marked at the edges.

Measure 5cm rightwards from the bottom left corner and 12cm upwards. How long is the connecting line?
Measure 12cm leftwards from the bottom right corner and 9 cm upwards, and complete and measure the triangle. Measure 8cm left and 15cm down from the top right corner.
Cut out the four triangles you have just drawn and see what kinds of patterns you can make with them. Make extra copies of the triangles and more patterns. Find a fifth triangle with the same property you noticed in the first four and make still more patterns.

24. What is the pattern that finds Pythagorean triples which include two adjacent integers?

25. a) Explore the sequence

$$\frac{1}{1}, \frac{3}{2}, \frac{7}{5}, \frac{17}{12}, ..$$

in which each fraction is made from the preceding fraction $\frac{n}{d}$ by the rule

$$\frac{n}{d} \rightarrow \frac{n+2d}{n+d}$$

starting with $\frac{1}{1}$.
Look for a pattern in the sequence of squares of these fractions.
b) Explore the sequence made by the rule

$$\frac{n}{d} \rightarrow \frac{n^2+2d^2}{2nd}$$

starting with $\frac{1}{1}$.
Is there any overlap with the previous sequence?
c) Now try

$$\frac{n}{d} \rightarrow \frac{n+sd}{n+d}$$

for any given number s, say $s = 2$, $s = 3$, $s = 3/2$, .. Compare these new sequences with

$$\frac{n}{d} \rightarrow \frac{n^2+sd^2}{2nd}$$

for the corresponding s.

26. Find the twenty-five prime numbers less than 100. Hint: it is fastest to eliminate as many non-primes (*composites*) as possible, instead of checking many numbers for being prime, even if you have available the prime-finding programs. Write down 2 and odd numbers > 1 unless they are divisible by 3, by 5 or by 7.
(Numbers are divisible by 3 if the sum of their digits is divisible by 3: e.g., the number 1023. Numbers are divisible by 5 if the last digit is 0 or 5, and 0 will not happen for odd numbers. All that is left is to check divisibility by 7.)
Why do we not need to go beyond 7?

27. The *Sieve of Eratosthenes* is a more sophisticated method to test if a positive integer is prime. Instead of checking 2 and all odd numbers above 2 as divisors, the "sieve" checks only primes up to the square root of the number. (The first argument in Note 7 applies not only to even divisors but to all composite divisors.)
a) Look up the Sieve of Eratosthenes (E$\rho\alpha\tau o\sigma\theta\epsilon\nu\eta\varsigma$, Eratosthenes of Cyrene, 276–195 BCE). To program this you will need a table of prime numbers, which you add to every time you discover a new prime. Think about how you would use an array to do this.
b) Leonhard Paul Euler (1707–83) discovered (or invented) an application of the Sieve which

converts an infinite sum over all numbers into an infinite product over all primes. Show that the "harmonic series"

$$\zeta = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \frac{1}{10} + \frac{1}{11} + \frac{1}{12} + \frac{1}{13} + ..$$

satisfies

$$(1 - \frac{1}{2})\zeta = 1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{9} + \frac{1}{11} + \frac{1}{13} + ..$$

and then

$$(1 - \frac{1}{3})(1 - \frac{1}{2})\zeta = 1 + \frac{1}{5} + \frac{1}{7} + \frac{1}{11} + \frac{1}{13} + ..$$

and

$$(1 - \frac{1}{5})(1 - \frac{1}{3})(1 - \frac{1}{2})\zeta = 1 + \frac{1}{7} + \frac{1}{11} + \frac{1}{13} + ..$$

So, eventually,

$$\zeta = 1/\Pi_p(1 - \frac{1}{p})$$

where Π_p means take the product over all primes, p.
(Actually the harmonic series becomes arbitrarily large, and Euler's "product formula" is more interesting for the generalization

$$\begin{aligned}\zeta(s) &= 1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \frac{1}{5^s} + \frac{1}{6^s} + \frac{1}{7^s} + \frac{1}{8^s} + \frac{1}{9^s} + .. \\ &= 1/((1 - \frac{1}{2^s})(1 - \frac{1}{3^s})(1 - \frac{1}{5^s})(1 - \frac{1}{7^s})..)\end{aligned}$$

which does have a finite value for every $s \neq 1$. Georg Friedrich Bernhard Riemann's (1826–66) "zeta function" $\zeta(s)$ is thoroughly chronicled by [Der03], who discusses its relationship with the primes: knowing the zeros of the zeta function will tell us how the primes are distributed—not the zeros at every even negative integer value of s but those in the 2-number plane off the 1-number line.)
c) How do we know that there are infinitely many primes? Ευκλειδης, Euclid of Alexandria ∼300 BCE, argued that if N is the last prime then $2 \times 3 \times 5 \times .. \times N - 1$ is either prime or has a prime factor larger than N, so there can be no last prime in either case.

28. **Rectangular numbers: multiplication** a) Make "rectangles" in two or three or more dimensions in as many different ways as you can for as many different integers as you like, building on the table in Note 8. For each "rectangle" write in numbers the multiplication it represents.
b) Use the two-dimensional rectangular numbers to show that multiplication is *commutative*

$$a \times b = b \times a$$

Use the three-dimensional rectangular numbers to show that multiplication is *associative*

$$(a \times b) \times c = a \times (b \times c)$$

What other arithmetic operations are commutative and associative?

29. (Eugene Lehman: How much is your name worth?)
a) Calculate the value of your name by multiplying the letters coded `a` = 1\$, `b` = 2\$, .., `z` = 26\$. For example, `Eugene` $= 5 \times 21 \times 7 \times 5 \times 14 \times 5\$ = 2 \times 3 \times 5^2 \times 7^3\$ = 51450\$$; `Lehman` $= 12 \times 5 \times 8 \times 13 \times 1 \times 14\$ = 2^6 \times 3 \times 5 \times 7 \times 13\$ = 87360\$$
b) Now find a word or short phrase that will sell for exactly a million dollars. (Hint: which letters are divisors of 1 000 000?)

30. (Eugene Lehman) Two of the guests at a birthday party turn out to have the same birthday as the celebrant, even though all three are not necessarily the sme age. Figure out all their ages from the following facts.
i) The three ages multiply to 36.
ii) The three ages sum to the number of people at the party.
iii) The eldest of the three is from the U.S.
Also say why (iii) is needed.

31. **Al jabr** The Arabic term for “the balance” describes the process of keeping equations balanced: we can do any arithmetic operation on one side of the equation and as long as we match it with the same operation on the other, the two sides are still equal.
a) Suppose we happen to know that $\phi = 1 + 1/\phi$. How can we find out that $\phi^2 - \phi - 1 = 0$ by (i) multiplying both sides by the same thing and (ii) subtracting the same thing from both sides?
b) *Diophantine* equations are restricted to integer solutions. Al jabr applies for addition, subtraction and multiplication but not always for division. Inspect Eugene’s bank teller excursion, above, from this point of view.
c) **The mystery of cancellation.** *Al jabr* also applies to fractions, in an even more limited way. Only multiplication and division can be done to both the numerator and the denominator if we want to reach a fraction having the same value. Thus in Note 10, 8/12 is shown as 2/3 because we can divide both the top and the bottom by 4.
But if we *add* the same number to both numerator and denominator we do not get a fraction having the same value: try adding 3 to both top and bottom of 1/2.

32. **Slopes: division** a) Extend the division diagram in Note 9 to as many divisions as you like. Explain the appearance of more than one fraction on some of the lines. In the full diagram, which would go on forever upwards and rightwards, what would be the smallest number of fractions on any one line?
b) Figure out the following diagram and add some more fractions in their approximate positions on the horizontal line of numbers from 0 to 1.

$\mathbf{0}$ $\frac{1}{9}\frac{1}{8}\frac{1}{7}$ $\frac{1}{6}$ $\frac{1}{5}\frac{2}{9}$ $\frac{1}{4}$ $\frac{2}{7}$ $\frac{1}{3}$ $\frac{3}{8}$ $\frac{2}{5}$ $\frac{3}{7}\frac{4}{9}$ $\frac{1}{2}$ $\frac{5}{9}\frac{4}{7}$ $\frac{3}{5}$ $\frac{5}{8}$ $\frac{2}{3}$ $\frac{5}{7}$ $\frac{3}{4}$ $\frac{7}{9}$ $\frac{4}{5}$ $\frac{5}{6}$ $\frac{6}{7}\frac{7}{8}\frac{8}{9}$ $\mathbf{1}$ $\frac{9}{8}\frac{8}{7}$ $\frac{7}{6}$ $\frac{6}{5}$ $\frac{5}{4}$ $\frac{9}{7}$ $\frac{4}{3}$ $\frac{10}{8}$ $\frac{7}{5}$ $\frac{10}{7}$ $\frac{3}{2}$

1/3 = 2/6 < 3/6 = 1/2

3/7 = 27/63 < 28/63 = 4/9

n/d < m/c or n/d = m/c or n/d > m/c

n/d ⩽ m/c or n/d > m/c

n/d ⩽ m/c or n/d ⋠ m/c

if a ⩽ b and b ⩽ c then a ⩽ c

if a ⩽ b and b ⩽ a then a = b

if a = b then a ⩽ b and b ⩽ a

a ⩽ b and b ⩽ a iff a = b

c) *Al jabr* works for inequalities as well as equalities. Just above we see that $3/7 < 4/9$ because we can multiply top and bottom of the first by 9, to give a fraction of the same value, and we can multiply top and bottom of the second by 7, to give a fraction of the same value as 4/9. Since 27<28, so 27/63<28/63.
Show that if $\frac{n_1}{d_1} \leq \frac{n_2}{d2}$ then

$$\frac{n_1}{d_1} \leq \frac{n_1 + n_2}{d_1 + d_2} \leq \frac{n_2}{d2}$$

d) Show that division is *palindromic*

$$a = b/c$$
$$\Updownarrow$$

$$c = b/a$$

What other arithmetic operations are palindromic?
e) **Approximate arithmetic** An example of division is speed

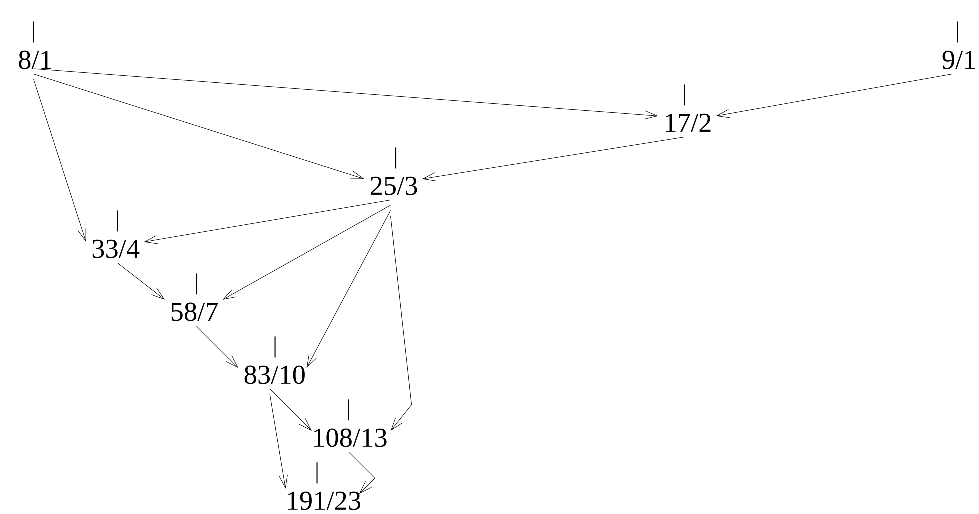
$$s = d/t$$

The speed s is the distance d travelled divided by the time t taken.
Palindromically we can calculate how long it will take to make a trip if we know the distance and the speed (or, at least, the average speed)

$$t = d/s$$

If we are driving 100Km on a quiet highway at the speed limit of 100Km/hour, we know it will take us an hour. But if the highway gets busy and we are only making 90Km/hour we can calculate 100Km/(90Km/hour) = 1.11 hours or about 67 minutes.
However it is not legal in many jursidictions to use a calculator or a pencil and paper while driving, even at 10Km/hour under the speed limit. Nor is it safe, especially on a busy highway. So we would like to do this kind of calculation in our head.
Here is an approximation. Since we are driving 10% slower than the speed that would get us there in an hour, it should take us 10% longer than the 60 minutes, i.e., 66 minutes.
This is pretty close. If we were driving 50% slower, though, the time would be double, not just 50% longer.
Make a table of speeds, both under and over 100Km/hour by 1%, 2%, 5%, 10% 20% and 50%, and put into it the exact and the approximate calulations of how many minutes the trip will take. When is the approximation close enough to make no difference to the nearest minute?
(By the way, how much time do you actually save by driving 100Km at 20% over the speed limit? Since it will cost you more than 20% more fuel, and possibly a heavy fine to boot, is it worth it?)

33. (Brocot fractional approximation) The Brocot algorithm [Hay08, Ch.7] finds fractional approximations to given numbers, motivated by the mid-19th century need to design gear ratios. Here it finds successive approximations to 191/23, starting at the underestimate 8/1 and the overestimate 9/1. 17/2 = (8+9)/(1+1) will be closer than either because it averages both numerator and denominator. 9/1 had a larger error than 8/1, so it gets replaced and the next "average" is of 8/1 and 17/2: 25/3. Eventually we get the exact answer, 191/23, since this was itself a fraction all along. But we see that 108/13 and 83/10 are approximations to it, if we could not make a gear with 191 teeth, for example.



The diagram shows the successive approximations positioned to the left and to the right of 191/23 according to their error. Be careful: this error is not measured $G - N/D$, where the goal G is the number sought (191/23 in this case) and N and D are the numerator and denominator, respectively, of the approximation. Rather, the "error" is $GD - N$, otherwise the algorithm does not give the final 191/23. (The final result also depends on the choice of the initial straddling approximations, in this case 8/1 and 9/1.)

Write a program for your calculator or in MATLAB to implement Brocot's algorithm. Let it input both the number to be approximated and the tolerance acceptable, and read the initial straddling approximations from an array of twice three elements: the numerator, the denominator, and the error which your program will calculate at each step. Replace the old approximation that has the greater "error" $GD - N$ by the "averaged" approximation.

The tolerance should be compared with the true error, $G - N/D$.

Experiment with your program. For instance 191 and 0 can be input as goal and tolerance, and the array [8,1,0;9,1,0] as the straddling approximations. Or π and 0.01, straddled by [3,1,0;4,1,0], gives the sequence of fractions 3/1, 4/1, 7/2, 10/3, 13/4, 16/5, 19/6 and 22/7, the last being the classical schoolchild approximation to π. Or $(1+\sqrt{5})/2$ and 0.001, straddled by 1/1 and 2/1, gives the first dozen terms of the Fibonacci sequence (Week ii Note 2).

34. (Eugene Lehman) How many times and exactly when do the hour and minute hands coincide on an analog clock, starting at one second after midnight and ending the following noon?
a) Try an approximate solution to start with, by saying to the nearest minute what time the first coincidence will occur. (Not at midnight, since we are starting just after the midnight coincidence.) Then step this forward, stopping before you go past noon.
b) By how much does this approximation get wrong the last coincidence before noon? What corrections to all the other coincidence timings will fix this error? Confirm that the final coincidence is exactly at noon. Work with fractions for this part.
c) Repeat (b) but this time work with decimals. You will need a way of writing infinitely repeating decimals. A convention is the bar: $0.232323.. = 0.\overline{23}$. Be careful when adding two infinitely repeating decimals written this way.
d) Check your work in (c) by calculator, which should show enough decimal places to give the right idea.

35. a) Invent and work through a number of examples of temperature drops and rises, across 0°C, above 0°C and below 0°C, until you are comfortable with working with negative arithmetic. How would you interpret and calculate a drop of -8C°? a rise of -8C°?
b) Martin Gardner imagined a room full of good and bad people. He supposed that adding is sending people into the room and subtracting is calling people out of the room. He supposed that good people are positive and bad people are negative. Thus adding $+5$ to the room sends in 5 good people, while adding -5 sends in 5 bad people. Subtracting $+5$ calls out 5 good people and subtracting -5 calls out 5 bad people.
Multiplying by $+3$ is adding 3 times. Multiplying by -3 is subtracting 3 times. So $(-3)\times(-5)$ successively subtracts -5 3 times, thereby *in*creasing the goodness of the room by 15 people. Work through several examples of this "model" of arithmetic.
c) Find and practice with some other "model"s of negative arithmetic—ones, for instance, which might be suitable for people who do not live in climates where the temperature can drop below freezing. Can you find any persuasive ones which involve placing the numbers along a straight line and reversing direction for multiplication by -1? (Hint. Look up Margaret Atwood's CBC Massey Lectures, 2008, Toronto, House of Anansi Press.)
d) What is $2 \times -4 \times 3 \times -6 \times -7$? Why does an odd number of negative values in a product give a negative result? Why does an even number of negative values in a product give a positive result?

36. a) Modify the `railwayTracks` program, or the corresponding plot on your calculator, to explore a variety of slopes and separations of straight-line ("linear") equations. Can you

work out a rule for the value of `X` that makes `Y = 0`? (If you use the `TRACE` button and the arrow keys on your calculator, you can display values of `X` and `Y` to test your rule.)
b) What constant must be added to a line of slope s to give a second line, also of slope s but a distance a apart from it, measuring the distance perpendicular to the two lines, as in Note 11?

37. a) Write and run a calculator program to plot the nonlinear expressions of Note 12.
b) What are the rules for finding the values of `X` that make `Y = 0` in the nonlinear plots of Note 12?
c) The general form of a "quadratic" expression is aX^2+bX+c, and of a "cubic" expression is AX^3+BX^2+CX+D. Work out the values of a, b and c for the triangular and square number expressions. Work out the values of A, B, C and D for the tetrahedral number expression.
d) Using the expression `2X + c` in a plotting program, find c by trial and error so that this straight line just touches the $\texttt{X}^2$ curve at `X = 1`. (The straight line is then said to be "tangent" to the curve, and its slope, 2, is the slope of the curve at the point of tangency.

38. At most and at least how many times do linear, quadratic and cubic expressions cross the horizontal $y = 0$ line? *Any* horizontal line?

39. Ramanujan called the integers his friends. Now we know enough about the integers to begin to give each one a "personality". We can do this by noting whether or not the integer is prime, is a triangular number, is square, is a sum of squares, and so on.
Make a table such as the following, with all the possible properties of integers discussed in these Notes, and fill it out for as many integers as you like. Can you find two integers with identical "personalities", i.e., the same columns have $\surd$s in them?

	...	odd	prime	$\triangle$	$\square$	cube	$\sqrt[2]{}$	$\sqrt[3]{}$	$\square+\square$	hexag	tetrahed	..
1		$\surd$		$\surd$	$\surd$	$\surd$	$\surd$	$\surd$		$\surd$	$\surd$	..
:		:	:	:	;	:	:	:	:	:	:	

40. Any part of the lecture that needs working through.

References

[Der03] John Derbyshire. *Prime Obsession: Bernhard Riemann and the Greatest Unsolved Problem in Mathematics.* Penguin Group (USA) Inc, New York, 2003.

[Hay08] Brian Hayes. *Group Theory in the Bedroom, and Other Mathematical Diversions.* Hill and Wang (Farrar, Straus and Giroux), New York, 2008.