Excursions in Computing Science: Week 7a. $E = mc^2$

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I. Prefatory Notes

1. Overview.

1. Frequency and wavenumber transform the same way as time and space.

2. Energy and momentum are proportional to frequency and wavenumber.

3. *Energentum* components (energy and momentum components) are conserved in physical processes.

2. (1) As a particle, $e^{-i\omega t}$, travels, what does its phase do?

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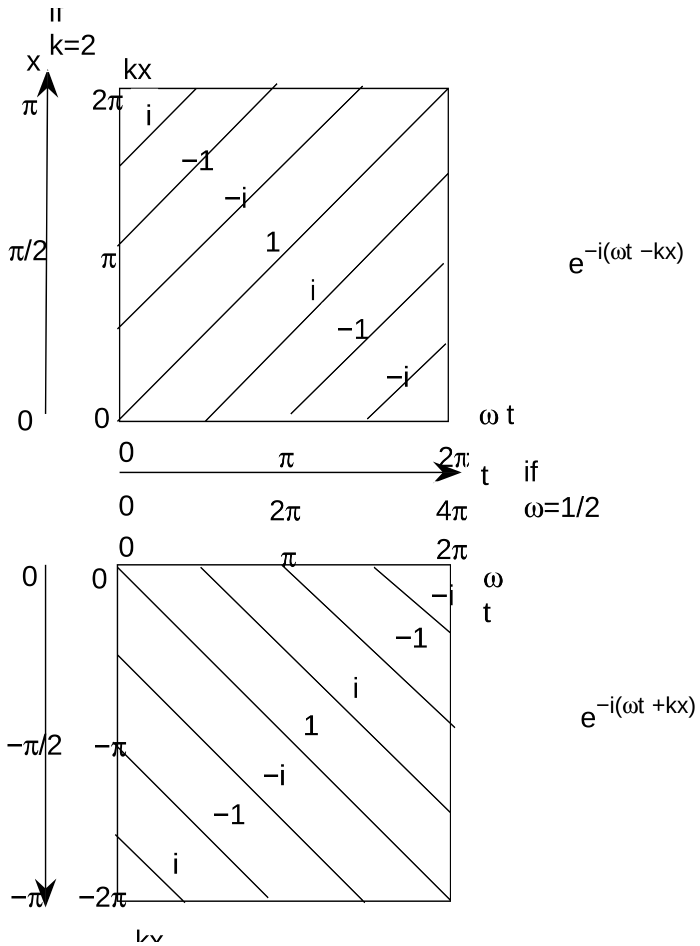


ω: angular frequency (radians/sec); $\omega/(2\pi) = f$ cycles/sec $= 1/T$, period T sec.

k: (angular) wavenumber; $k/(2\pi) = \nu$ waves/light-sec $= 1/\lambda$, wavelength λ light-sec.

3. Phase should be independent of any observer: $\omega t - kx = \omega' t' - k' x'$

Since $\begin{pmatrix} t \\ x \end{pmatrix} = \gamma \begin{pmatrix} 1 & v \\ v & 1 \end{pmatrix} \begin{pmatrix} t' \\ x' \end{pmatrix}$,

$$\begin{aligned} \omega t - kx &= \gamma\omega(t' + vx') - \gamma k(vt' + x') \\ &= \gamma(\omega - kv)t' - \gamma(k - \omega v)x' \end{aligned}$$

So $\begin{pmatrix} \omega' \\ k' \end{pmatrix} = \gamma \begin{pmatrix} 1 & -v \\ -v & 1 \end{pmatrix} \begin{pmatrix} \omega \\ k \end{pmatrix}$.

QED frequency and wavenumber transform the same way as time and space.

"timespace", "freqnum"

NB $\omega^2 - k^2 = \omega'^2 - k'^2$

4. (2) Energy and momentum are proportional to frequency and wavenumber.

$$\begin{aligned} E &\stackrel{\text{def}}{=} \hbar\omega = hf; \\ p &\stackrel{\text{def}}{=} \hbar k = h\nu \end{aligned}$$

$\hbar$ is Planck's constant ($\hbar \stackrel{\text{def}}{=} h/2\pi$). (Note that the 2π comes in because we measure angles in radians (Week 1). The versions using f and ν measure angles in cycles.)

This is *observed*—we can't prove it mathematically.

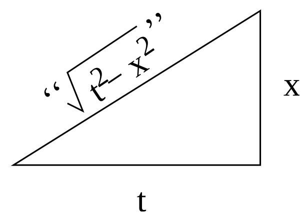
Einstein proposed it in 1905 to explain photoelectricity.

Now we have "energentum"

NB $E^2 - p^2 = E'^2 - p'^2$

5. Anti-Pythagoras: review

$$\begin{aligned}
\begin{pmatrix} t' \\ x' \end{pmatrix} &= \gamma \begin{pmatrix} 1 & -v \\ -v & 1 \end{pmatrix} \begin{pmatrix} t \\ x \end{pmatrix} \\
t'^2 - x'^2 &= \gamma^2((t-vx)^2 - (x-vt)^2) \\
&= \gamma^2(t^2 - 2vtx + v^2x^2) \\
&\quad -\gamma^2(v^2t^2 - 2vtx + x^2) \\
&= \frac{1}{1-v^2}((1-v^2)t^2 - (1-v^2)x^2) \\
&= t^2 - x^2
\end{aligned}$$

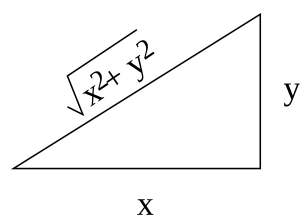


$$\begin{aligned}
\text{Similarly } \omega'^2 - k'^2 &= \omega^2 - k^2 \\
\text{and } E'^2 - p'^2 &= E^2 - p^2
\end{aligned}$$

These differences are "invariants" of timespace axis transformations (the Lorentz transformation). We can give them meanings, but first we compare with length, which is invariant under ordinary space transformations such as rotation.

It's a new kind of "length":

$$\begin{aligned}
\begin{pmatrix} x' \\ y' \end{pmatrix} &= \begin{pmatrix} c & s \\ -s & c \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \\
x'^2 + y'^2 &= (cx+sy)^2 + (-sx+cy)^2 \\
&= c^2x^2 + 2csxy + s^2y^2 \\
&\quad s^2x^2 - 2csxy + c^2y^2 \\
&= x^2 + y^2
\end{aligned}$$



6. The *length* $x^2 + y^2$ is invariant under axis transformations such as rotation.

$t^2 - x^2$ is the *interval*, invariant under axis transformations such as Lorentz: it is the "proper time", τ, which is the time experienced by the traveller ($x' = 0 : \begin{pmatrix} t \\ x \end{pmatrix} = \gamma \begin{pmatrix} 1 & v \\ v & 1 \end{pmatrix} \begin{pmatrix} \tau \\ 0 \end{pmatrix}$, so for the nontraveller $t = \gamma\tau$ and $x = \gamma v\tau$).

$E^2 - p^2$ is—what? It is another invariant, the *mass*, m of the particle. (We'll only mean the "rest mass" when we use the word "mass", so this is indeed invariant under axis transformations. It is the mass observed by somebody travelling with the particle.) $E^2 - p^2 = m^2$.

7. Dimensional analysis and pre-timespace limits.

("Pre-timespace limit" is usually called "classical limit" but there are two "classical" limits and we will distinguish pre-timespace from pre-quantum limits.)

quantity	t	x	ω	k
"dimensions"	T	L	T^{-1}	L^{-1}

T *time*, L *length*

Objective of Notes 7, 8, 9: putting c (lightspeed) back into the equations.

So timespace

$$\begin{pmatrix} t' \\ x' \end{pmatrix} = \gamma \begin{pmatrix} 1 & -v \\ -v & 1 \end{pmatrix} \begin{pmatrix} t \\ x \end{pmatrix}$$

1. $x' = \gamma(x - vt)$
   x: L, t: T, thus v: L/T, e.g., m/sec, light-sec/sec. OK
2. $t' = \gamma(t - vx)$
   t: T, x: L, thus v: T/L? No, must fix.

   Try v/c^2: $(\mathrm{L/T})/(\mathrm{L^2/T^2}) = \mathrm{T/L}$

   So it really is $x' = \gamma(x - vt/c^2)$
3. $1/\gamma^2 = 1 - v^2$
   Is v dimensionless? No, must fix.
   Try v/c: $(\mathrm{L/T})/(\mathrm{L/T})$

Finally $\begin{pmatrix} t' \\ x' \end{pmatrix} = \frac{1}{\sqrt{1-(v/c)^2}} \begin{pmatrix} 1 & -v/c^2 \\ -v & 1 \end{pmatrix} \begin{pmatrix} t \\ x \end{pmatrix}$

The pre-timespace limit is $c \to \infty$

$$\begin{pmatrix} t' \\ x' \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -v & 1 \end{pmatrix} \begin{pmatrix} t \\ x \end{pmatrix}$$

This is the Galilean transformation. It also gives Newton's absolute time.

8. The pre-timespace limit for freqnum.

Some more dimensional analysis.

quantity	t	x	ω	k	m	$\hbar$ ("action")	E	p
"dimensions"	T	L	T^{-1}	L^{-1}	M	$\mathrm{ML^2T^{-3}}$	$\mathrm{ML^2T^{-2}}$	$\mathrm{MLT^{-1}}$

M *mass*

$$\begin{pmatrix} \omega' \\ k' \end{pmatrix} = \gamma \begin{pmatrix} 1 & -v \\ -v & 1 \end{pmatrix} \begin{pmatrix} \omega \\ k \end{pmatrix}$$

As before, $\omega' = \omega - vt$: $\mathrm{T}^{-1}-$ $(\mathrm{L/T})\mathrm{L}^{-1}$ OK

But $k' = k - v\omega$: $\mathrm{L}^{-1}-$ $(\mathrm{L/T})\mathrm{T}^{-1}$. No,must fix

Try $(v/c)^2$

$$\begin{aligned} \begin{pmatrix} \omega' \\ k' \end{pmatrix} &= \gamma \begin{pmatrix} 1 & -v \\ -v/c^2 & 1 \end{pmatrix} \begin{pmatrix} \omega \\ k \end{pmatrix} \\ &\to \begin{pmatrix} 1 & -v \\ 0 & 1 \end{pmatrix} \end{aligned}$$

Thus, in pre-timespace terms, $k' = k$ and $\omega' = \omega - vk$.

9. Similarly $\begin{pmatrix} E' \\ p' \end{pmatrix} = \gamma \begin{pmatrix} 1 & -v \\ -v/c^2 & 1 \end{pmatrix} \begin{pmatrix} E \\ p \end{pmatrix}$

But there is no pre-timespace limit this way because the pre-timespace limit of quantum theory requires $\hbar \to 0$.

But the "dimensional" approach gives the explicit form of $E^2 - p^2 = m^2$:

$$E^2 - c^2p^2 = m^2c^4$$

(and when the particle is stationary, $p = 0$ and $E = mc^2$).

A form of energentum which *can* be taken to the pre-timespace limit is

$$\begin{aligned} p &= m\frac{\Delta x}{\Delta \tau} \\ E &= m\frac{\Delta t}{\Delta \tau} \end{aligned}$$

(This does not follow from the arguments we've seen this week, and I won't prove it.)

Let's see what energy looks like in pre-timespace terms

$$E = mc^2\gamma = \frac{mc^2}{\sqrt{1-(v/c)^2}} = mc^2(1 + \frac{1}{2}(v/c)^2 + \frac{3}{8}(v/c)^4 + ..)$$

by an expansion which I also will not prove. We see it gives mc^2 for the "rest energy" and $\frac{1}{2}mv^2$ for the kinetic energy if $v \ll c$.

Because E and p transform the same way as t and x, respectively, and because $v = x/t$, so $v = p/E$, or, fixing this up dimensionally,

$$v = \frac{pc^2}{E} = \frac{pc^2}{mc^2} = \frac{p}{m}$$

Since in the second step we have replaced E by the rest energy only, mc^2, this gives the pre-timespace limit for momentum, $p = mv$.

10. (3) Energentum is conserved.

In any coordinate system, each component of energentum is the same before and after a physical process.

$$\begin{aligned} E_a &= E_b \\ p_a &= p_b \end{aligned}$$

(The latter equation is true for each component, p_x, p_y and p_z, of the momentum if we were working in three dimensional space.)

This is also an observation and so can only be accepted—carefully.

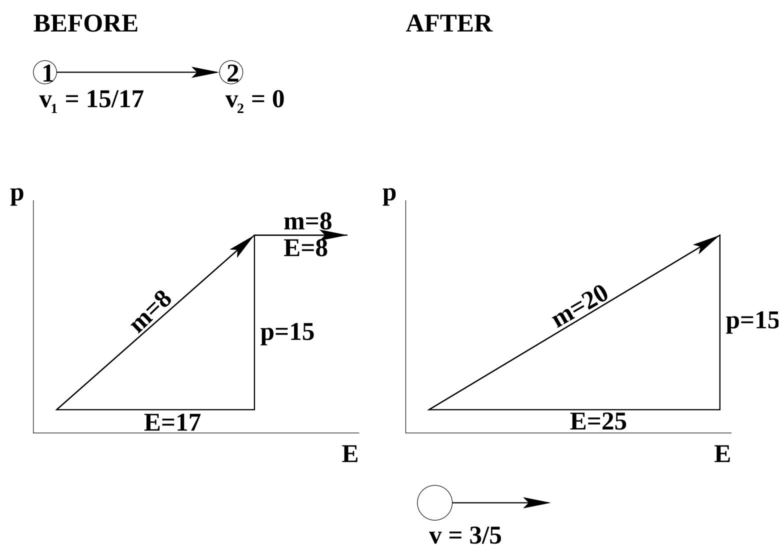
Conservation of E and p and invariance of $E^2 - p^2$ together make it easy to analyze many physical processes.

(Note the difference between *conservation* (same before and after any *physical* process) and *invariance* (unchanged by axis transformations). The second is a property of the transformation: all linear transformations have invariants, so this is a mathematical result—except that it is the physics which tells us which transformation to use and so which invariant to expect. The first is a directly physical property.)

We can now calculate the consequences of collisions, nuclear fusion and nuclear fission.

11. Inelastic collisions (the bodies stick.)

Here we see that energentum is conserved but mass is not.



$$\left.\begin{array}{rcl} E & = & E_1 + E_2 \\ p & = & p_1 + p_2 \end{array}\right\} \text{conservation}$$

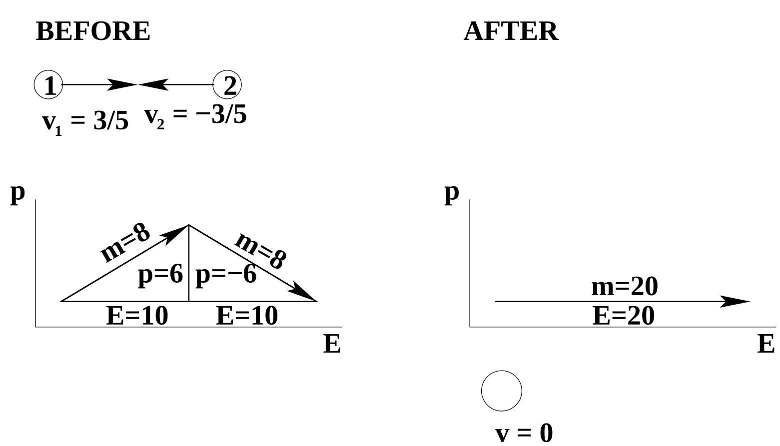
$$\left.\begin{array}{rcl} m_1^2 & = & E_1^2 - p_1^2 \\ m_2^2 & = & E_2^2 - p_2^2 \\ m^2 & = & E^2 - p^2 \end{array}\right\} \text{invariance}$$

12. Or, to stop the final object, choose reference frame moving at $u = 3/5$.

So if the old v_1, which we'll now call v, is $15/17$, then the new v_1 is

$$v_1 = \frac{v - u}{1 - uv} = \frac{3}{5}$$

and the new $v_2 = -3/5$ (Why?)



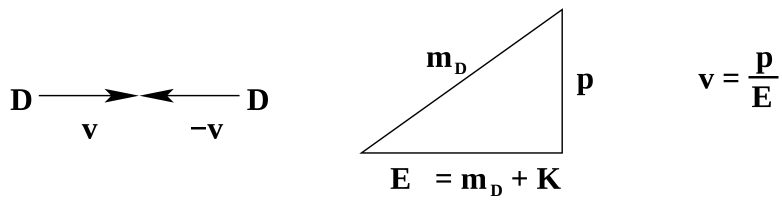
This could be nuclear fusion; the inverse could be nuclear fission.

13. Fusion

$^{2}_{1}\mathrm{D} + {}^{2}_{1}\mathrm{D} \rightarrow {}^{4}_{2}\mathrm{He}$

$$\begin{aligned} m_{\mathrm{D}} &= 2.0141 \\ m_{\mathrm{He}} &= 4.0026 \\ 2m_{\mathrm{D}} - m_{\mathrm{He}} &= 0.0256 \\ &= 0.00636 \times (2m_{\mathrm{D}}) \end{aligned}$$

Because the helium energy is less than the sum of the deuterium energies, we can get energy out of this fusion.



The two deuterons have been fired at each other at velocities v and $-v$ (in the centre-of-mass reference frame). It costs energy to give them these velocities. So if we are to get net energy out of the fusion process, we must limit v. What is the maximum v so we still gain energy?

Kinetic energy $K = E - m$, and this must be less than $0.00636 m_{\mathrm{D}}$ for each deuteron.

$$\begin{aligned} p^2 &= E^2 - m_{\mathrm{D}}^2 \\ &= (m_{\mathrm{D}} + K)^2 - m_{\mathrm{D}}^2 \\ &= K(K + 2m_{\mathrm{D}}) \\ &= 0.00636 m_{\mathrm{D}}(0.00636 m_{\mathrm{D}} + 2m_{\mathrm{D}}) \\ &= 0.01276 m_{\mathrm{D}}^2 \\ p &= 0.11296 m_{\mathrm{D}} \\ E &= 1.00636 m_{\mathrm{D}} \\ \text{So } v &\leq 0.113 \end{aligned}$$

0.113 lights is 34 megameters/sec.

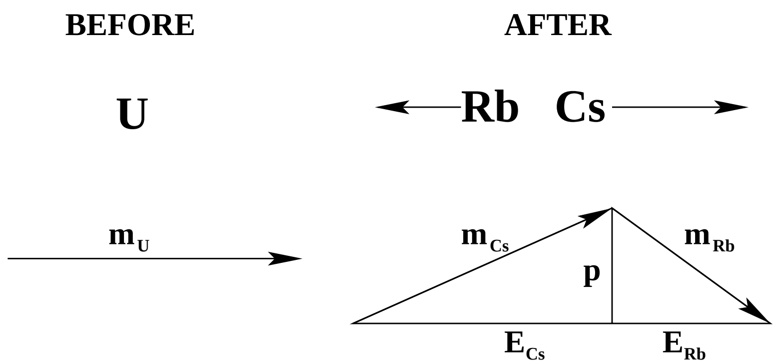
14. Fission

$^{1}_{0}\mathrm{n} + {}^{235}_{92}\mathrm{U} \rightarrow {}^{236}_{92}\mathrm{U} \rightarrow {}^{45}_{37}\mathrm{Rb} + {}^{141}_{55}\mathrm{Cs}$

$$\begin{aligned} m_{\mathrm{U}} &= 236.0455680 \\ m_{\mathrm{Rb}} &= 94.929303 \\ m_{\mathrm{Cs}} &= 140.920046 \\ \Delta &= 0.1962 = 0.00083 m_{\mathrm{U}} \end{aligned}$$

(This energy yield is about 8 times smaller proportionately than the fusion Δ.)

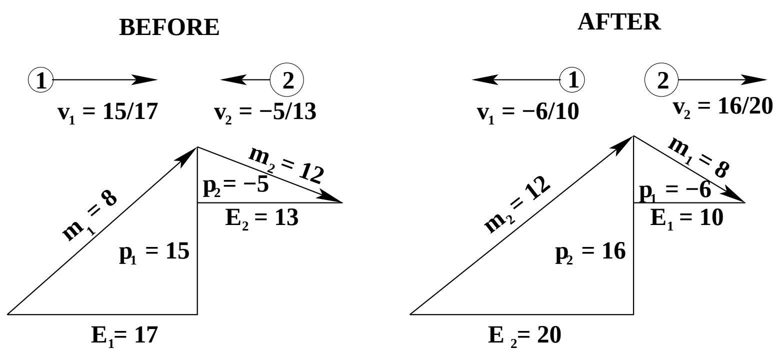
If the $^{236}_{92}\mathrm{U}$ is initially at rest, what velocities do the Rb and Cs get?

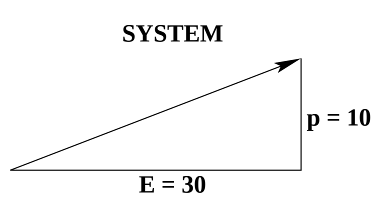


$$
\begin{aligned}
p_{\mathrm{Cs}} + p_{\mathrm{Rb}} &= 0 \qquad\qquad \text{so } p \stackrel{\mathrm{def}}{=} p_{\mathrm{Cs}} = -p_{\mathrm{Rb}} \\
E_{\mathrm{Cs}} + E_{\mathrm{Rb}} &= m_{\mathrm{U}} \\
E_{\mathrm{Cs}}^2 &= p^2 + m_{\mathrm{Cs}}^2 \\
(m_{\mathrm{U}} - E_{\mathrm{Cs}})^2 = E_{\mathrm{Rb}}^2 &= p^2 + m_{\mathrm{Rb}}^2 \\
E_{\mathrm{Cs}} &= \frac{1}{2}(m_{\mathrm{U}} + \frac{m_{\mathrm{Cs}}^2 - m_{\mathrm{Rb}}^2}{m_{\mathrm{U}}}) \\
&= 140.999 \\
E_{\mathrm{Rb}} &= 95.0466 \\
p_{\mathrm{Rb}}^2 &= p_{\mathrm{Cs}}^2 = E_{\mathrm{Cs}}^2 - m_{\mathrm{Cs}}^2 = (4.72)^2 \\
v_{\mathrm{Cs}} &= \frac{p_{\mathrm{Cs}}}{E_{\mathrm{Cs}}} = 0.033 \\
v_{\mathrm{Rb}} &= \frac{p_{\mathrm{Rb}}}{E_{\mathrm{Rb}}} = -0.050
\end{aligned}
$$

which means that the cesium flies off rightwards at 10 megameters/sec, and the rubidium moves leftwards at 15 megameters/sec.

15. Elastic collision (the bodies bounce.)





$$
\begin{aligned}
\text{System } E &= E_1 + E_2 = 30 \\
p &= p_1 + p_2 = 10
\end{aligned}
$$

$$
\begin{aligned}
\text{AFTER } E &= E_1' + E_2' \\
p &= p_1' + p_2' \\
\textbf{Show} : E_1' &= 12 + p_1'/3
\end{aligned}
$$

This last line gives the required relationship between E_1' and p_1'.
Subject to this we can pick any value for E_1'.

In the diagram, I picked $E_1' = 10$, from which p_1' must be -6.

16. Equations of relativistic quantum mechanics (A sketch)

Quantum mechanics considers measurement to be *operators* (see Week 2).

For quantities such as energy and momentum the discrete math is approximated by continuous mathematics, i.e. differentials. We haven't done the background in this course, so this sketch is a lookahead. (See Notes 34 and 38 of Week 8c part IV. $\frac{\partial}{\partial x}$, etc., has the same meaning as $\text{slope}_x()$, etc., in Note 10 of Week 8.)

$$
\begin{aligned}
E &\rightarrow i\hbar\frac{\partial}{\partial t} \\
p &\rightarrow -i\hbar\nabla \\
\nabla &= (\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z})
\end{aligned}
$$

With this, the relativistic relationship $E^2 = p^2c^2 - m^2c^4$ gives the *Klein-Gordon* equation

$$\frac{1}{c^2}\frac{\partial^2\Psi}{\partial t^2} = \nabla^2\Psi - \frac{m^2c^4}{\hbar^2}\Psi$$

where Ψ is the "wave function" giving the amplitude, varying in timespace, for the particle.

(Before Klein and Gordon, Schrödinger used the pre-timespace kinetic energy, $E = p^2/2m$ to get

$$\frac{\partial\Psi}{\partial t} = \frac{1}{2m}\nabla^2\Psi$$

Heisenberg independently derived an equivalent, discrete-math, formulation, "matrix mechanics"—with infinite-dimensional matrices.)

Dirac wanted to reduce the Klein-Gordon equation to a first-order differential equation, and found that he needed the Pauli matrices, σ_x, σ_y, and σ_z:

$$
\begin{aligned}
E\Psi &= (\vec{\alpha}.\vec{p} + \beta m)\Psi \\
\vec{\alpha} &= \begin{pmatrix} & \vec{\sigma} \\ \vec{\sigma} & \end{pmatrix} \\
\beta &= \begin{pmatrix} I & \\ & -I \end{pmatrix}
\end{aligned}
$$

Note that the four matrices $\vec{\alpha}$ and β each have four components, which turn out to describe spin up and spin down, each with either positive or negative energy. The negative energies were eventually associated with antimatter.

17. Summary

(These notes show the trees. Try to see the woods!)

- Frequency and wavenumber transform the same way as time and space.
- Energy and momentum are proportional to frequency and wavenumber.

- *Energentum* components (energy and momentum components) are conserved in physical processes.
- These physical processes include
  - inelastic collision
  - nuclear fusion
  - nuclear fission
  - elastic collision
  - Compton effect (see Excursions)

II. The Excursions

You've seen lots of ideas. Now *do* something with them!

1. **Visualizing waves** a) The plots of $e^{i(\omega t-kx)}$ and $e^{i(\omega t+kx)}$ in Note 2 show lines of constant phase, $\omega t-kx$ or $\omega t+kx$. Thus the lines labelled 1 have phase 0 radians, the lines labelled i have phase $\pi/2$ radians, and so on. Since the plots are timespace diagrams, they show phases moving through space as time increases. To figure out what velocity the phase, say, $\phi=\omega t-kx$ moves at, let's look at a small change, Δx, in x and at the related small change, Δt, in t. The change in value of the phase will be
$$\omega(t+\delta t)-k(x+\Delta x)-(\omega t-kx)=\omega\Delta t-k\Delta x$$
But since we are tracking constant phase, this difference equals $\phi-\phi=0$.
Using $\omega\Delta t-k\Delta x=0$, find the velocity of this phase.
Note that this velocity depends on the observer, unless it is lightspeed, because the measurements of $x, \delta x, t$ and Δt depend on the observer.
b) The plots of $e^{i(\omega t-kx)}$ and $e^{i(\omega t+kx)}$ in Note 2 show example axes for t if $\omega=1/2$ and for x if $k=2$. Why can these two examples not hold simultaneously if the mathematics is describing photons? If $k=2$ what must ω be for a photon?
c) How do the plots of $e^{i(\omega t-kx)}$ and $e^{i(\omega t+kx)}$ in Note 2 change if i is changed to $-i$, i.e., the phase, the direction of rotation of the arrow (week 5 Note 2), is reversed? What do the *sums*,
$$e^{i(\omega t-kx)}+e^{-i(\omega t-kx)}$$
and
$$e^{i(\omega t+kx)}+e^{-i(\omega t+kx)}$$
look like? Redraw the plots with the sums as labels and draw matching cosine waves on top of your plots: horizontally, vertically, and in a couple of other directions. What does the horizontal cosine wave mean? The vertical wave?
d) Draw the plot showing
$$e^{i(\omega t-kx)}+e^{i(\omega t+kx)}$$
the sum of two waves moving in opposite directions. Instead of lines labelled by 2-numbers, you will now get a lattice whose nodes are labelled by 2-numbers. Note that there are locations in space where the amplitude is zero at all times: this is called a "standing wave". Draw a representative cosine wave of this result plus its 2-number conjugate ("complex conjugate").
e) How would you get a "synchronous wave", with locations in time where the amplitude is simultaneously zero everywhere in space (think of marching in step)?

2. Give the dimensional argument showing that p must be multiplied by c and m by c^2 in the invariance relationship $E^2-p^2=m^2$.

3. If τ is the time experienced by a traveller, show that (t, x) for the observer hey travels past at speed v is $(\gamma\tau, \gamma v\tau)$ and so that $(t_2 - t_1)/(\tau_2 - \tau_1) = \gamma$ and $(x_2 - x_1)/(\tau_2 - \tau_1) = \gamma v$

4. Look up isotopic masses of a range of elements, from the lightest to the heaviest in the periodic table, and use MATLAB to plot the mass per nucleon against (a) the numbers of protons (atomic numbers) and (b) the numbers of nucleons (atomic weights). Show that iron (56 protons) has the least mass per nucleon: this makes it the most stable element. What does the shape of the curve tell us about elements worth considering for fusion? For fission? Be sure you look up *isotopic* masses, since atomic masses (on the other hand) are weighted averages over all isotopes with the same atomic number.

5. Look up a few different isotopes of some element having at least three isotopes; look up relative abundances of each isotope; and calculate the mean atomic weight of the element. Look up the atomic weight of that element as a check.
Find an element with only two isotopes and, from its given atomic weight, calculate the relative abundances of the two isotopes. (An element with an extremely rare third isotope is an alternative possibility to investigate.)

6. For the elastic collision of $m_1 = 8$ units and $v_1 = 15/17$ with $m_2 = 12$ units and $v_2 = -5/13$, show that, afterwards, $E_1 = 12 + p_1/3$.

7. Planck's constant, $h = 4.136$ peV-sec (peV = pico eV = 10^{-12} eV); electron mass = 0.5 MeV (MeV = mega eV); nucleon mass (proton, neutron) = 1 GeV (GeV = giga eV = 10^9 eV).
a) Show that the energy of a green photon, with wavelength 500 nm (nm = nanometers = 10^{-9} meters) and frequency 600 THz (THz = tera Hertz = 10^{12} Hertz = 10^{12} cycles/sec), has energy 2.5KeV (KeV = kilo eV).
b) Show that a photon with energy equal to the electron mass is an ultraviolet photon and give its frequency and wavelength. [www.usbyte.com/common/approximate_wavelength.htm]

8. A photon of energy 3/8 MeV (ultraviolet) collides elastically head-on with an electron moving towards it at 3/5 lightspeed. Show that either the two pass through each other, unaffected, or each reverses direction. Show the energies, momenta and mass or frequency before and after.
The collision of photons with electrons is called Compton scattering and captures the essence of relativistic quantum mechanics. Look up Arthur Holly Compton, 1892-1962. (Compton used X-rays and scattering in two space dimensions.) [TW92, p.267]

9. An infrared photon of energy 0.45 eV is absorbed by a stationary water vapour molecule (a greenhouse gas) in the atmosphere. Find the mass of the molecule in Daltons and find its conversion to eV. Draw an energentum diagram and estimate the subsequent motion of the molecule.

10. (This excursion elaborates on the "world line" excursion of Week 3. The first three parts are mathematically identical in both excursions but physically describe different aspects.)
a) Plot (by hand or somehow by computer) the following points on an energentum diagram.

$$\begin{array}{cccccccccc} & & a & b & c & d & e & f & g & h \\ \begin{pmatrix} E \\ p \end{pmatrix} & = & \begin{pmatrix} -5 \\ 4 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 4 \\ 5 \end{pmatrix} & \begin{pmatrix} 9 \\ 1 \end{pmatrix} & \begin{pmatrix} -4 \\ -4 \end{pmatrix} & \begin{pmatrix} 5 \\ -5 \end{pmatrix} & \begin{pmatrix} -1 \\ 10 \end{pmatrix} & \begin{pmatrix} 8 \\ 9 \end{pmatrix} \end{array}$$

b) On your plot, connect the dots as follows to show the "world lines" of three particles: a-b-c-d, e-b-f and g-c-h. Describe what is happening, including any special or strange aspects.
c) Calculate the new positions of these points under the Lorentz transformation of an observer travelling at 0.9 lights, plot them and the transformed world lines, and again describe all aspects of your result.

d) Confirm that masses are invariant under the Lorentz transformation.

e) Confirm that the energy and momentum components are each conserved before and after each collision. Note that you must always move forward in time, so some quantity(ies) will appear on one side of the equals sign in the original plot and on the other side after the Lorentz transformation. (Note that these energentum components are the *differences* between the (E, p) vectors at each end of the straight line segments.)

f) Supposing that one of the world lines in the original plot describes a charged particle, say an electron, what happens to the particle, and the charge, in the Lorentz transformed plot?

g) Use the two plots to make an argument showing that what one observer sees as a particle turning through 360 degrees, another observer sees as swapping two different but identical particles. (To get the 360 degree turning you might want to add a second space dimension to your plot (well, to the timespace plot from Week 3) and imagine the two collisions, each reversing the motion of the particle, replaced by a more gradual process of forcing the particle around a circle—or at least a polygon—in the two spatial dimensions.) This approach is based on [Bae06], which in turn cites Feynman's lecture in [FW87]. The gap in the argument is that so far it supposes "tachyons" are possible: see Week 9.

h) Since turning an electron through 360 degrees changes the sign of its amplitude, you now know that swapping two electrons also changes the sign of the amplitude describing the pair. Show that this means two electrons cannot be in the same state.

i) Photons, on the other hand, can rotate 360 degrees without changing sign. Look up Feynman's argument [FLS64b, Sects. 4-2,4-3] that photons and bosons in general not only are permitted to share a state but prefer it. (Bosons are particles with whole-integer spins.)

11. Look up Wolfgang Pauli, 1900–1958, again. How long after originally formulating the exclusion principle for electrons did he derive it from relativistic quantum mechanics for fermions in general? (This is called the "spin-statistics theorem". Fermions are particles with half-integer spins.) The mutual exclusion of electrons is the basis for all normal matter and in particular for atomic structure and chemistry (see Week 6).

12. Look up Albert Einstein, 1879–1955, again, and Satyendra Nath Bose, 1894–1974, and the consequences of their work showing that bosons attract each other into the same state: lasers, superconductivity, superfluidity and the recently explored fifth state of matter, "Bose-Einstein condensates". Bosons mediate all forces: for instance, the electromagnetic force is conveyed by photons. (How can exchanging a particle *attract* two particles? See [FLS64b, Sect. 10-1] for a related discussion.)

13. A photon emitted from a light source with frequency f appears to have frequency $f\sqrt{(1-v)/(1+v)}$ to a traveller moving away from the source with velocity v. Show this, using the fact that photons have zero mass. What is the apparent frequency for a traveller moving at v towards the source? This is the relativistic Doppler shift [TW92, p.263].

14. Given the pre-timespace limit that kinetic energy, $\mathcal{T} = (1/2)mv^2$, and that a uniform gravitational field (e.g., close to the Earth's surface) accelerates all masses at the same rate, g (e.g., 10 m/sec^2), you can work out that, to conserve energy, raising a mass, m, a height, h, gives it a "potential energy" mgh. (You need to show mgh $= (1/2)mv^2$, where v is the velocity the mass acquires falling the distance h under g.)

a) If you climb the Eiffel tower (300 m), what is the ratio of your potential energy to your rest energy, mc^2? Does this depend on your mass?

b) If a photon climbs the same height as you did, what is its proportional change in frequency? This is the gravitational red shift [TW92, p.258].

c) From its spectrum, the Sun was once thought to be made of iron. Look up the spectra of iron and hydrogen and see how much of a red shift separates them. Look up Cecilia Payne-Gaposchkin, 1900–1979, and her role in changing this perception.

15. We can use the energy levels of atomic electrons, from Week 6, to understand the immense range of electrical conductivity in crystalline solids; from insulators to metals, conductivity changes by a factor of 10^{24}. Since calculating atomic electron energies is beyond what we've done in this course, we'll have to be nonquantitative in this excursion.
    First, because an electron is a particle with mass, its energy varies as its momentum squared, forming a parabola if we plot $E = \hbar\omega$ against wavenumber, k: you can show this using the quantum mechanical relationships between energy and frequency (just given) and between momentum and wave number, and the pre-timespace relationships between energy, velocity and momentum (one is $E = mv^2/2$ and the other is $p = mv$).
    Second, quantum mechanically, electron amplitudes have frequency and wavenumber (amplitude $\propto \exp(i(\omega t - kx))$) and these waves get reflected by the regular array of atoms in a crystal when their wavelength approaches any multiple of the inter-atom spacing ("Bragg reflection": see Week 8c part II.). The combination of $\exp(ikx)$ and $\exp(-ikx)$ when $k = 2\pi/\lambda$ and $\lambda = a$ (for atoms in a one-dimensional "crystal" spaced a units apart) gives either $\cos(2\pi x/a)$ or $\sin(2\pi x/a)$, if we consider everything at a fixed time, say $t = 0$: show how! You can now show that the *probability*, associated with these amplitudes, of where the electron is found, is a cosine wave either with peaks (maximum probability) right on top of the atoms in the first case (cos), or with troughs (zero probability) right on top of the atoms in the second case (sin). Since atoms attract electrons, the first case has lower energy than the second case. At any rate, the smooth parabola showing the relationship between energy and wavenumber for the free electron now has *gaps* between the two different energies of the electron in the crystal when the electron wavelength is near $\pm a$, $\pm 2a$, $\pm 3a$, etc.
    Third, this broken parabola with the energy gaps has two basic relationships with the types of atom that make up the crystal. If the atom is of an element where the highest energy of the electrons of that element just reaches the bottom of one of these gaps, the electrons cannot move, because this would require them to have more energy than their "ground state" just below the gap, putting them into the gap, which is illegal. Such a crystal is an insulator. On the other hand, if the atom is of an element, typically a metal, whose ground state energies do not completely fill an "energy band" up to the gap, the electrons are allowed to move and this crystal is a conductor. (Semiconductors are insulators with such a narrow gap that it is possible for electrons to have enough energy to put them into the band *above* the gap.)
    A book that covers all this, and much more, is Peierls' *Quantum Theory of Solids* [Pei55]. Like Hertzberg's book (Week 6), this book is well beyond this course, and will also be an exercise in skimming what you don't understand and trying to string together the bits you do understand, with the help of the discussion in this excursion, into an overall picture. Feynman's *Lectures on Computation* [Fey99] uses these ideas at the beginning of his discussion on semiconductors and VLSI.

16. Look up Feynman's discussion of relativistic mass in the light of elastic collisions [FLS64a, Sect. 16-4]. We follow Wheeler [TW92, pp.250,251] and do not consider either that *mass* varies with velocity, or the contrasting concept of "rest mass".

17. Look up Einstein's derivation of the equivalence of mass and energy [TW92, pp.254ff], and elsewhere.

18. Look up John Archibald Wheeler, 1911–2008 . Who are three famous people whose Ph.D. theses he supervised?

19. Look up Ernest Rutherford, 1871–1937. What historical role was played by Rutherford scattering? What did he get his Nobel for?

20. Any part of the lecture that needs working through.

References

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