Excursions in Computing Science:
Week 8 Higher Dimensions: 2D Graphics and Internet Search

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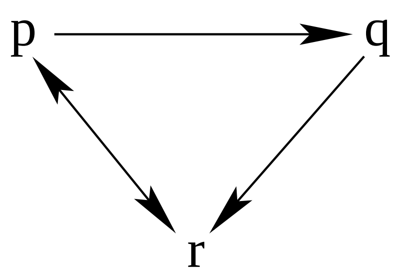
March 6, 2007

A. Eigenvectors and Google Page Ranking

1. How do we find out how important a web page is?

We can suppose, circularly, that a page is important if important pages link to it.



↓	p	q	r
p	0	0	1
q	1	0	0
r	1	1	0

Since p has two outgoing links, let's say it divides its importance between q and r.

↓	p	q	r
p	0	0	1
q	$\frac{1}{2}$	0	0
r	$\frac{1}{2}$	1	0

Let's play with this.
Suppose p has importance 1 and q and r have importance 0.

$$\begin{pmatrix} 0 & 0 & 1 \\ 1/2 & 0 & 0 \\ 1/2 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \rightarrow \begin{pmatrix} 0 \\ 1/2 \\ 1/2 \end{pmatrix} \rightarrow \begin{pmatrix} 1/2 \\ 0 \\ 1/2 \end{pmatrix} \rightarrow \begin{pmatrix} 1/2 \\ 1/4 \\ 1/4 \end{pmatrix} \rightarrow \begin{pmatrix} 1/4 \\ 1/4 \\ 1/2 \end{pmatrix} \rightarrow \begin{pmatrix} 1/2 \\ 1/8 \\ 3/8 \end{pmatrix} \rightarrow \begin{pmatrix} 3/8 \\ 1/4 \\ 3/8 \end{pmatrix} \rightarrow \cdots$$

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Is this settling down to something?

Try

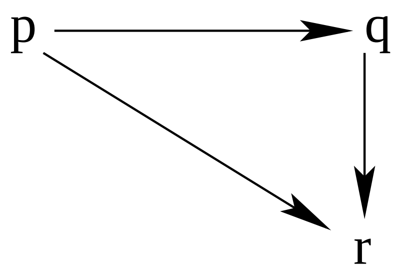
$$\begin{pmatrix} 0 & 0 & 1 \\ 1/2 & 0 & 0 \\ 1/2 & 1 & 0 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} \rightarrow \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$$

or

$$W\vec{R} = w\vec{R}$$

$\vec{R}$ is called an "eigenvector" of W; w, an ordinary number, is called an "eigenvalue".

2. But the web doesn't have many cycles.



$$\begin{array}{c|ccc} \swarrow & p & q & r \\ \hline p & 0 & 0 & 0 \\ q & \frac{1}{2} & 0 & 0 \\ r & \frac{1}{2} & 1 & 0 \end{array}$$

And now the importance just fades away.

$$\begin{pmatrix} 0 & 0 & 0 \\ 1/2 & 0 & 0 \\ 1/2 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \rightarrow \begin{pmatrix} 0 \\ 1/2 \\ 1/2 \end{pmatrix} \rightarrow \begin{pmatrix} 0 \\ 0 \\ 1/2 \end{pmatrix} \rightarrow \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

So we need a way to sustain the importance of a page.

Try $W\vec{R} + \vec{S}$ for a "source vector", $\vec{S}$, i.e.,

$$\vec{R'} = m(W\vec{R} + \vec{S}), \qquad m \stackrel{\text{def}}{=} 1/w$$

If we can get $\vec{R'} = mA\vec{R}$ for some matrix A, we get back to an eigenvector problem:

- make $\vec{S}$ a matrix, $\vec{S}\vec{1}^T = \vec{S}(1,1,1)$;
- keep $\| \vec{R} \|_1 \stackrel{\text{def}}{=} \vec{1}^T\vec{R} = R_1 + R_2 + \cdots = 1$:

$$\vec{S} = \vec{S} \parallel R \parallel_1 = \vec{S}\vec{1}^T\vec{R}$$

So $\vec{R'} = m(W\vec{R} + \vec{S}) = \vec{R'} = m(W + \vec{S}\vec{1}^T)\vec{R}$
and, to find eigenvectors, $\vec{R'} = \vec{R}$.

What should $\vec{S}$ be?

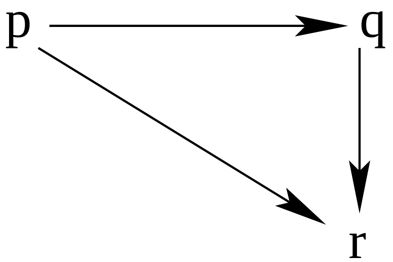
The Google page rank method uses

$$\vec{S} = \alpha\vec{1}; \quad \| \vec{S} \|_1 = 0.15$$

So, for 3D

$$\vec{S} = 0.05 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}; \quad \vec{S}\vec{1}^T = 0.05 \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

3. So now W becomes $W + S1^T$

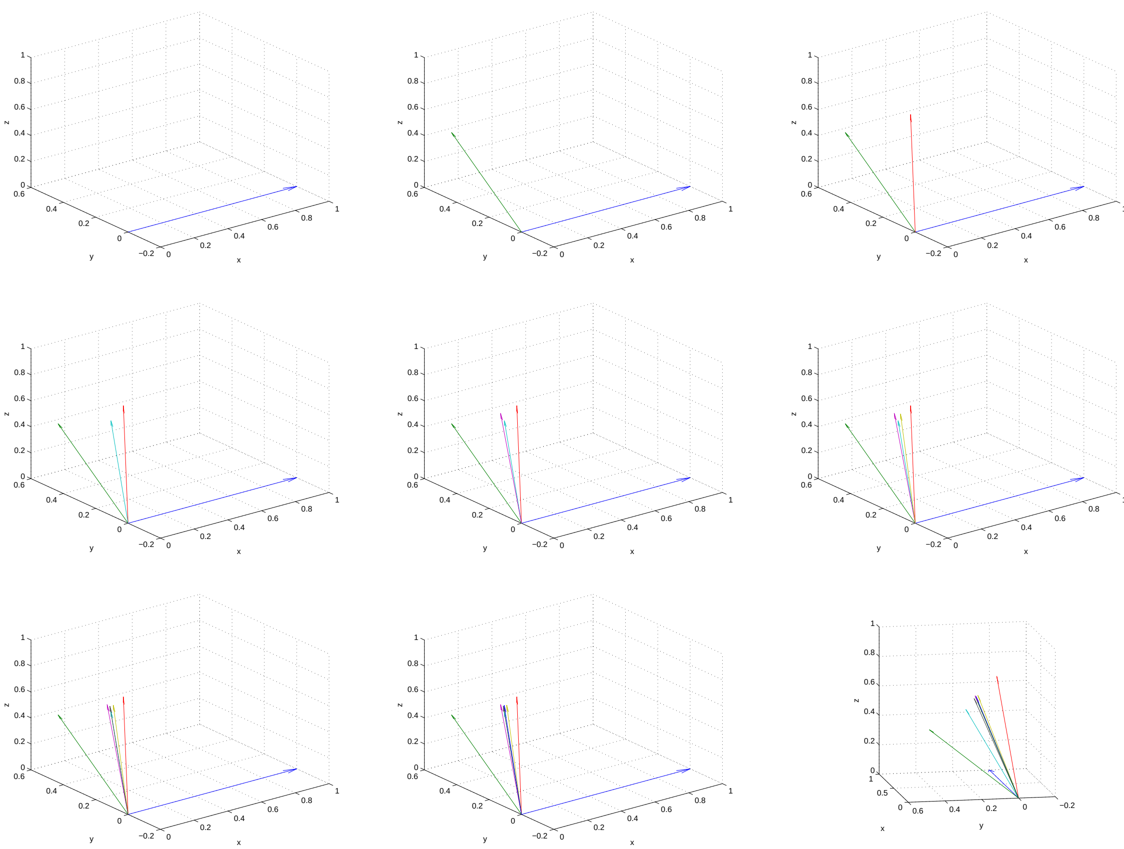


$$\downarrow \quad \begin{array}{c c} & \begin{array}{ccc} p & q & r \end{array} \\ \begin{array}{c} p \\ q \\ r \end{array} & \begin{pmatrix} 0.05 & 0.05 & 0.05 \\ 0.55 & 0.05 & 0.05 \\ 0.55 & 1.05 & 0.05 \end{pmatrix} \end{array}$$

$$\begin{pmatrix} 0.05 & 0.05 & 0.05 \\ 0.55 & 0.05 & 0.05 \\ 0.55 & 1.05 & 0.05 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \rightarrow \begin{pmatrix} 0.05 \\ 0.55 \\ 0.55 \end{pmatrix} \rightarrow 1.15 \begin{pmatrix} 0.0435 \\ 0.4783 \\ 0.4783 \end{pmatrix} \rightarrow ? \begin{pmatrix} 0.0744 \\ 0.1068 \\ 0.8188 \end{pmatrix} \rightarrow ? \begin{pmatrix} 0.1510 \\ 0.2673 \\ 0.5837 \end{pmatrix} \rightarrow \cdots$$

$$\rightarrow ? \begin{pmatrix} 0.1507 \\ 0.2174 \\ 0.6769 \end{pmatrix} \rightarrow ? \begin{pmatrix} 0.1507 \\ 0.2174 \\ 0.6769 \end{pmatrix} \rightarrow \cdots$$

Here, the ? indicates the factor that has been removed at each iteration in order that $\| R \|_1 = 1$.



The values have converged to four decimal places after a dozen iterations.
(They still keep changing at higher precisions.)

Roughly, $p : q : r = 1 : 2 : 7$.

It is certainly plausible from the web diagram that r should be more important than p or q, both of which supply it with importance, and that p should be of least importance, since nobody else links to it.

B. Linear Equations and Sketchpad

4. **Elementary row operations.**

1. Multiply row by nonzero constant.
2. Swap two rows.
3. Add multiple of one row to another row.

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} a & c \\ b & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

1. $\begin{pmatrix} 2x' \\ y' \end{pmatrix} = \begin{pmatrix} 2a & 2c \\ b & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$

2. $\begin{pmatrix} y' \\ x' \end{pmatrix} = \begin{pmatrix} b & d \\ a & c \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$

3. $\begin{pmatrix} x' + 2y \\ y' \end{pmatrix} = \begin{pmatrix} a + 2b & c + 2d \\ b & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$

Solving equations: given (x', y') find (x, y)!

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} a & c \\ b & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

Use (1) $\begin{pmatrix} x'/a \\ y' \end{pmatrix} = \begin{pmatrix} 1 & c/a \\ b & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$ "normalize row"

Use (3) $\begin{pmatrix} x'/a \\ y' - bx'/a \end{pmatrix} = \begin{pmatrix} 1 & c/a \\ 0 & d - bc/a \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$ "zeros below the 1"

Use (1) $\begin{pmatrix} x'/a \\ \frac{y' - bx'/a}{d - bc/a} \end{pmatrix} = \begin{pmatrix} 1 & c/a \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$ "normalize row"

Use (3) $\begin{pmatrix} x'/a - \frac{c}{a}\frac{y' - bx'/a}{d - bc/a} \\ \frac{y' - bx'/a}{d - bc/a} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$ ("zeros above the 1")

Note. (2) is used to avoid division by zero (or by very small numbers).
Note. This can be used to find the determinant: keep the product of the denominators,

$$a(d - bc/a) = ad - bc$$

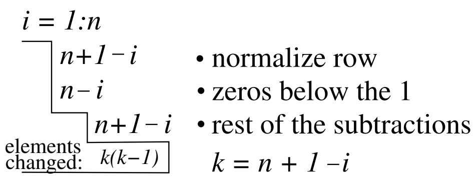
.

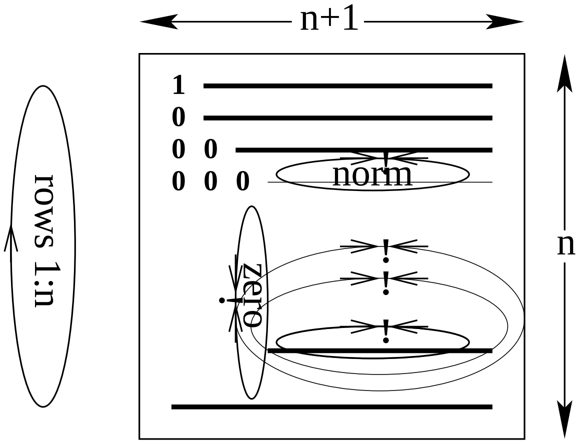
5. **Gauss elimination**

Do elementary row operations on the "augmented matrix"

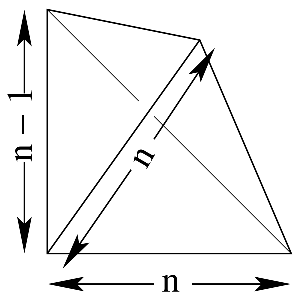
$$\begin{pmatrix} a & b & x' \\ c & d & y' \end{pmatrix}$$

1. Triangularization

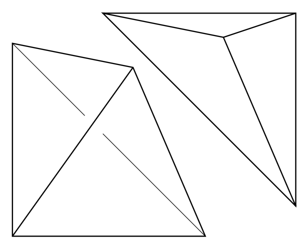




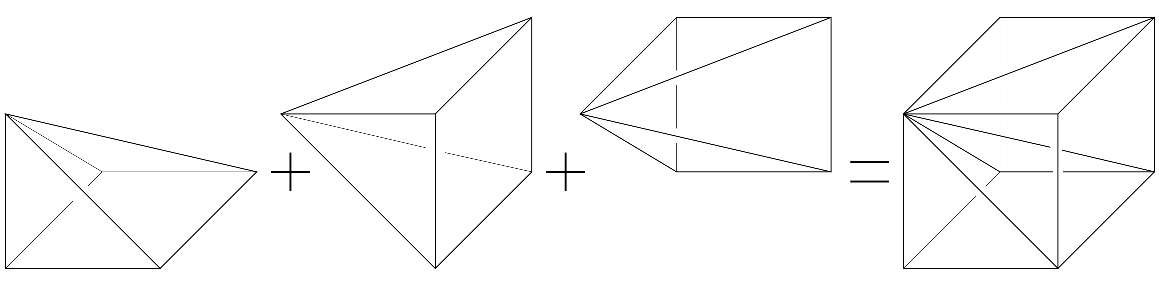
Therefore $\mathcal{O}(n^3)$ elements are changed.
Or, more precisely, $\frac{1}{3}n^3 + \mathcal{O}(n^2)$,
which is twice the volume of a tetrahedron:
$(k(k-1) = 2(k!2)$ elements are changed in the innermost loop and $\sum_1^n k!2 = (n+1)!3$, which can be presented as a tetrahedron.)



(2 tetrahedra = 1 pyramid

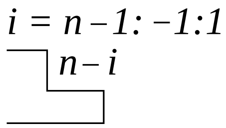


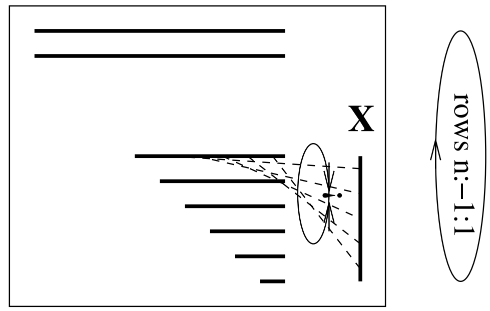
and 3 pyramids = 1 cube:



)

2. Back-substitution





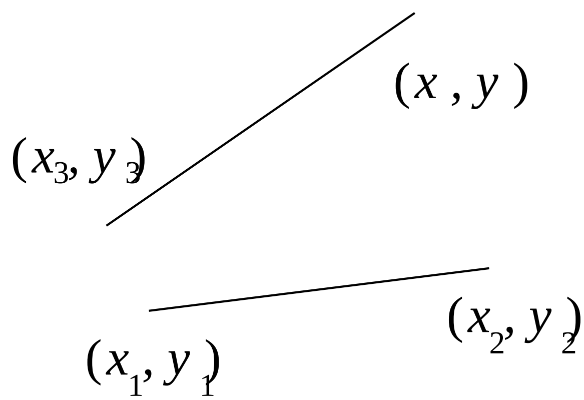
$\mathcal{O}(n^2)$, in fact, $n^2/2$.

6. **Sketchpad : 3D for 2D graphics**

Drawing program which allows users to specify *constraints*.

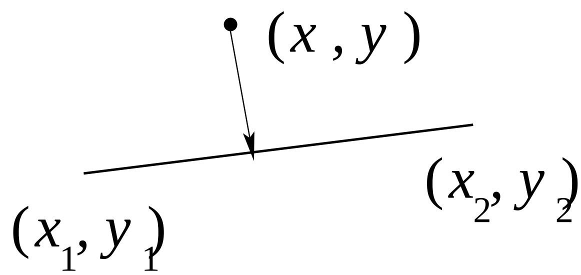
E.g., two lines must have the same length.

$$
\begin{aligned}
0 = \text{ error } &= (x - x_3)^2 + (y - y_3)^2 \\
&= (x_2 - x_1)^2 + (y_2 - y_1)^2
\end{aligned}
$$

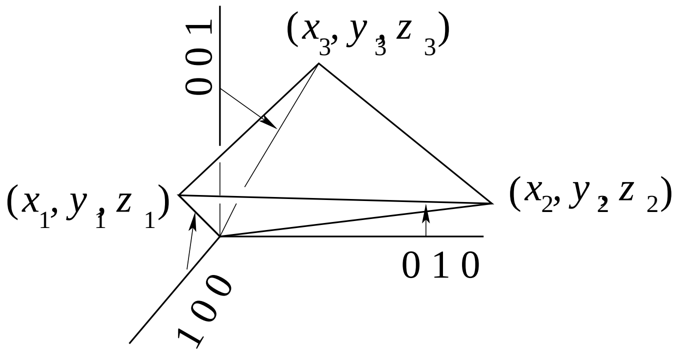


E.g., three points must be collinear.

$$
0 = \text{ error } = \begin{vmatrix} x_1 & x_2 & x \\ y_1 & y_2 & y \\ 1 & 1 & 1 \end{vmatrix}
$$

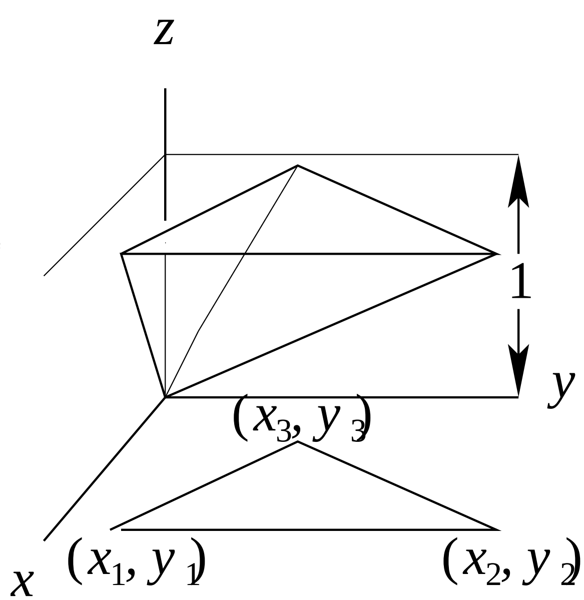


Note. Determinant of 3×3 is volume (by *definition*):



$$\begin{pmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}$$

etc. (The definition of the determinant must also include the requirement that it changes sign whenever any pair of columns is swapped. This is a property of volumes, as suggested by Clifford algebra (Week 7c).)

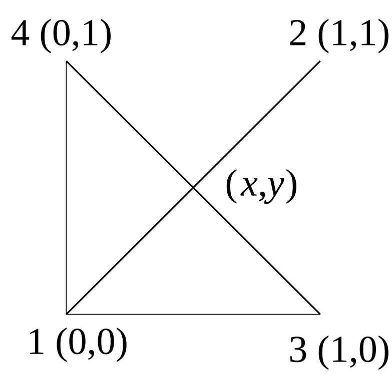


So, determinant of 3×3 with 1s is triangle area:

$$\frac{1}{2}\begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ 1 & 1 & 1 \end{vmatrix}$$

7. Sketchpad example 1 (linear constraint):

Find where two lines intersect.



$$0 = \begin{vmatrix} x_1 & x_2 & x \\ y_1 & y_2 & y \\ 1 & 1 & 1 \end{vmatrix} = \begin{vmatrix} 0 & 1 & x \\ 0 & 1 & y \\ 1 & 1 & 1 \end{vmatrix} = y - x$$

$$0 = \begin{vmatrix} x_3 & x_4 & x \\ y_3 & y_4 & y \\ 1 & 1 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 0 & x \\ 0 & 1 & y \\ 1 & 1 & 1 \end{vmatrix} = 1 - x - y$$

That is

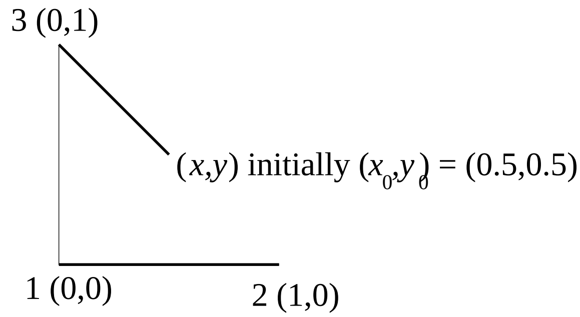
$$\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

Or, by Gaussian elimination,

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix};\ \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix};\ \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix}$$

8. Sketchpad example 2 (nonlinear constraint):

Let's constrain ((0,1),(x,y)) to have the same length as ((0,0),(1,0)), i.e., 1



$$1 = (x - x_3)^2 + (y - y_3)^2 = (x - 0)^2 + (y - 1)^2 = x^2 + (y + 1)^2$$

$\text{error}_{\text{length}} = x^2 + (y + 1)^2 - 1$

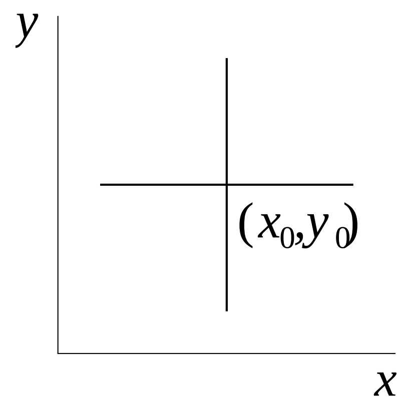
a) This is underdetermined: (x, y) could be anywhere on a circle of radius 1.

So we'll say that the ((0,1),(x,y)) line should keep its slope: $x + y = 1$:
$\text{error}_{\text{slope}} = x + y - 1$

b) The length constraint is not linear but quadratic. Because we don't always know how to solve nonlinear equations, we'd like to make a *linear approximation* to the nonlinear equation.

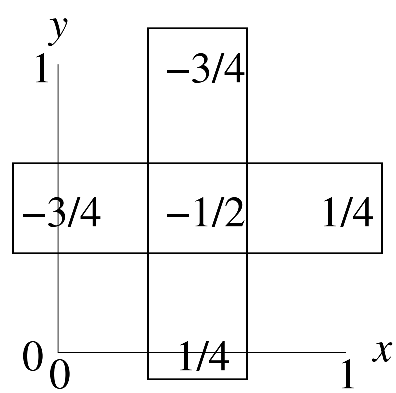
We'll call this a *slope approximation*, because we're going to say that the nonlinear function is approximately equal to its slope at a point under consideration.

What does the error look like near $(x_0, y_0) = (0.5, 0.5)$?

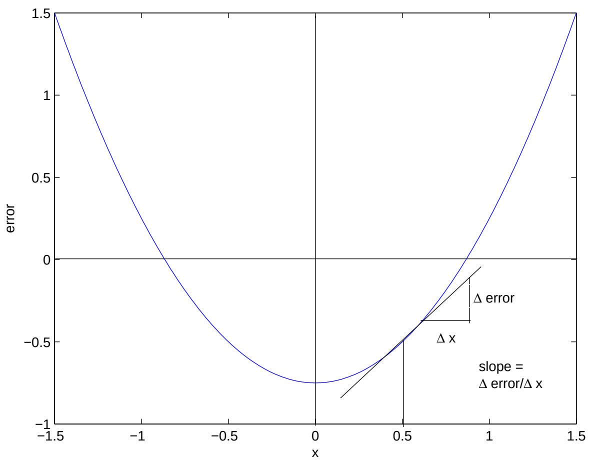


1. $y = y_0 = 0.5$; vary x:
   $\text{error}^x_{\text{length}} = x^2 + y_0^2 - 2y_0 = x^2 - 3/4$

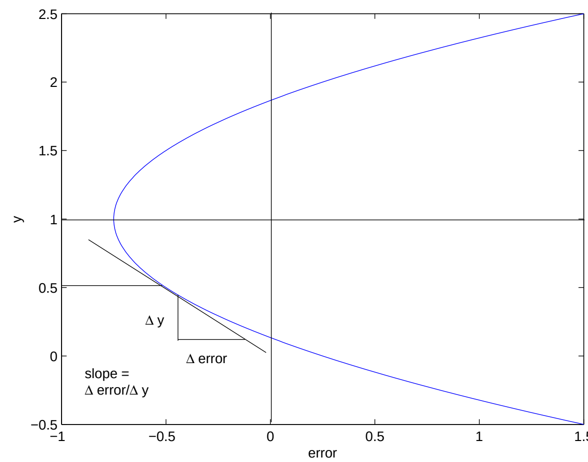
2. $x = x_0 = 0.5$; vary y:
   $\text{error}^y_{\text{length}} = x_0^2 + y^2 - 2y = y(y-2) + 1/4$



$\text{error}^x_{\text{length}} = x^2 - 3/4$



$\text{error}^y_{\text{length}} = y(y-2) + 1/4$



$$
\begin{aligned}
\text{slope}^{x}_{(}x_0, y_0) &= \frac{\Delta\text{error}}{\Delta x} \\
&= \frac{1/4 - (-3/4)}{1} \\
&= 1 \\
\text{error}^x_{\text{length}} &\simeq 1 * (x - 0.5) + \cdots
\end{aligned}
$$

$$
\begin{aligned}
\text{slope}^{y}_{(}x_0, y_0) &= \frac{\Delta\text{error}}{\Delta y} \\
&= \frac{-3/4 - 1/4}{1} \\
&= -1 \\
\text{error}^y_{\text{length}} &\simeq -1 * (y - 0.5) + \cdots
\end{aligned}
$$

So we try

$$
\begin{aligned}
\text{error}_{\text{length}} &\simeq 1 * (x - 0.5) + (-1 * (y - 0.5)) + (-1/2) \\
&= x - y - 1/2
\end{aligned}
$$

We now have two linear equations to solve.

$$
\begin{aligned}
\text{error}_{\text{length}} &= x - y - 1/2 \\
\text{error}_{\text{slope}} &= x + y - 1
\end{aligned}
$$

That is

$$0 = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} -1 \\ -1/2 \end{pmatrix}$$

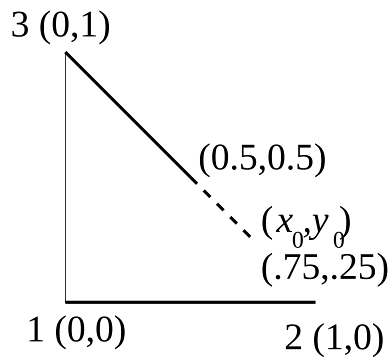
$$\begin{pmatrix} 1 \\ 1/2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

Gaussian elimination gives

$$\begin{pmatrix} 1 \\ -1/2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3/4 \\ 1/4 \end{pmatrix}$$

So (x_0, y_0) changes position from (1/2,1/2) to (3/4,1/4).



This keeps the slope at -1, but the length is not equal to 1.
In fact we can figure out that (x_0, y_0) should be at $(1/\sqrt{2}, 1 - 1/\sqrt{2}) = (0.707, 0.293)$.

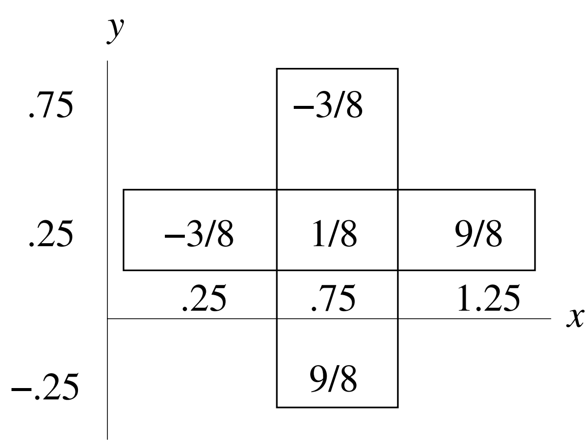
9. **Iterate!**

Since the first step did not get all the way, we take another step.
It will get us closer. (No amount of steps will ever get us all the way because each step will give numbers which are fractions, i.e., rational, while $\sqrt{2}$ is irrational.)

But we can go on taking steps until we are close enough.

Iteration is the essence of nonlinearity.

$$\begin{aligned} \text{error}^x_{\text{length}} &= x^2 + y_0(y_0 - 2) \\ &= x^2 + 1/4(1/4 - 2) \\ &= x^2 - 7/16 \end{aligned} \qquad \begin{aligned} \text{error}^y_{\text{length}} &= x_0^2 + y(y - 2) \\ &= (3/4)^2 + y(y - 2) \\ &= y(y - 2) + 9/16 \end{aligned}$$



$$\begin{aligned} \text{error}_{\text{length}} &\simeq \frac{9 - (-3)}{8}(x - 3/4) + \frac{-3 - 9}{8}(y - 1/4) + 1/8 \\ &= 3/2(x - 3/4) - 3/2(y - 1/4) + 1/8 \\ &= 3/2x - 3/2y - 5/8 \end{aligned}$$

Again, two linear equations.

$$\begin{aligned} \text{error}_{\text{length}} &= 3/2x - 3/2y - 5/8 \\ \text{error}_{\text{slope}} &= x + y - 1 \end{aligned}$$

That is

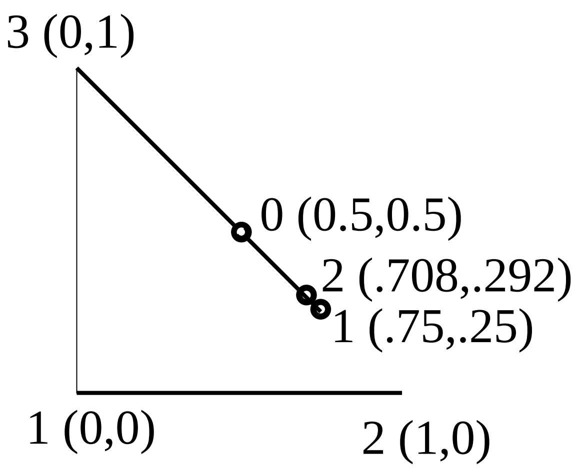
$$0 = \begin{pmatrix} 1 & 1 \\ 3/2 & -3/2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} -1 \\ -5/8 \end{pmatrix}$$

$$\begin{pmatrix} 1 \\ 5/8 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 3/2 & -3/2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

Gaussian elimination gives

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 17/24 \\ 7/24 \end{pmatrix} = \begin{pmatrix} 0.708 \\ 0.292 \end{pmatrix}$$

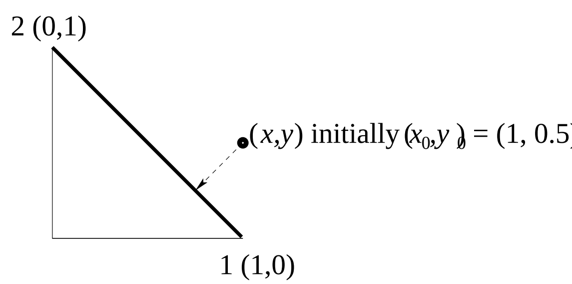
This answer is wrong by only 1 in the third decimal place.
Not enough to show on the drawing.
So two iterations were enough this time, although we could keep going.



10. **Underdetermined equations**

Sketchpad example 3 (point collinear with line):

$$\begin{aligned} 0 &= \begin{vmatrix} x_1 & x_2 & x \\ y_1 & y_2 & y \\ 1 & 1 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 0 & x \\ 0 & 1 & y \\ 1 & 1 & 1 \end{vmatrix} \\ &= 1 - x - y = (-1, -1) \begin{pmatrix} x \\ y \end{pmatrix} + 1 \end{aligned}$$



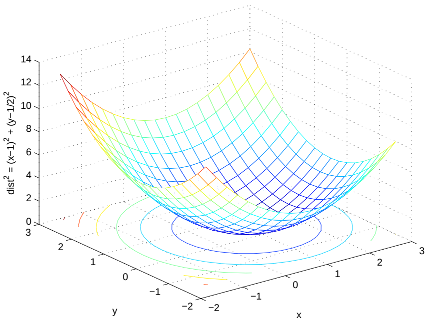
One equation, two unknowns

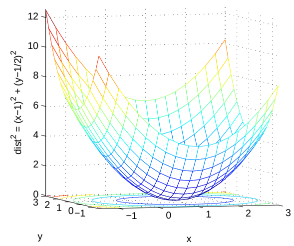
$$(1, 1) \begin{pmatrix} x \\ y \end{pmatrix} = 1$$

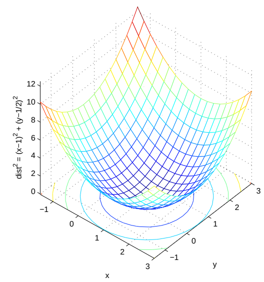
Let's minimize how far (x, y) moves from (x_0, y_0).

$$\begin{aligned} \text{dist}^2 &= (x - x_0)^2 + (y - y_0)^2 \\ &= (x - 1)^2 + (y - 1/2)^2 \\ &= x^2 - 2x + y^2 - y + 5/4 \end{aligned}$$

On minimizing: slope = 0 means slope^x = 0 and slope^y = 0







On slopes:

$$\begin{aligned}\text{slope } x^2 &= \frac{(x-\Delta x)^2 - x^2}{\Delta x} = \frac{(x^2 - 2x\Delta x + \cdots) - x^2}{\Delta x} \simeq 2x \\ \text{slope } x &= \frac{(x-\Delta x) - x}{\Delta x} = 1 \\ \text{slope } c &= \frac{c-c}{\Delta x} = 0\end{aligned}$$

So
$$0 = \begin{pmatrix} \text{slope}^x \\ \text{slope}^y \end{pmatrix} \text{dist}^2 = \begin{pmatrix} \text{slope}^x \\ \text{slope}^y \end{pmatrix} (x^2 - 2x + y^2 - y + 5/4) = \begin{pmatrix} 2x-2 \\ 2y-1 \end{pmatrix}$$
and so $x = 1, y = 1/2$.

Of course: (x, y) is closest to (x_0, y_0) when $(x, y) = (x_0, y_0) = (1.1/2)$

11. Now let's minimize *subject to the constraint* $x + y = 1$

$$\begin{aligned}0 &= \begin{pmatrix} \text{slope}^x \\ \text{slope}^y \\ \text{slope}^\lambda \end{pmatrix} (\text{dist}^2 + \lambda * (x + y - 1)) \\ &= \begin{pmatrix} \text{slope}^x \\ \text{slope}^y \\ \text{slope}^\lambda \end{pmatrix} (x^2 - 2x + y^2 - y + 5/4 + \lambda * (x + y - 1)) \\ &= \begin{pmatrix} 2x - 2 + \lambda \\ 2y - 1 + \lambda \\ x + y - 1 \end{pmatrix} \\ &= \begin{pmatrix} 2 & 0 & 1 \\ 0 & 2 & 1 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ \lambda \end{pmatrix} + \begin{pmatrix} -2 \\ -1 \\ -1 \end{pmatrix} \\ \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} &= \begin{pmatrix} 2 & 0 & 1 \\ 0 & 2 & 1 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ \lambda \end{pmatrix} \\ \begin{pmatrix} x \\ y \\ \lambda \end{pmatrix} &= \begin{pmatrix} 3/4 \\ 1/4 \\ 1/2 \end{pmatrix}\end{aligned}$$

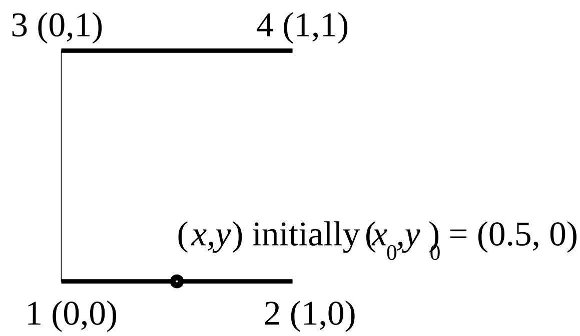
The λ is called a "Lagrange multiplier", and the trick makes sense if you think about it.

(3/4,1/4) is where the arrow points in the figure: (x, y) moved perpendicularly towards the line.

12. **Overdetermined equations**

Sketchpad example 4 (point collinear with two lines):

Constrain (x, y) to be collinear with both 12, 34:
$0 = \text{error}_{12}$ and
$0 = \text{error}_{34}$



We'll start simply, by constraining (x, y) to be collinear with 12

$$0 = \text{error}'_{12} = \begin{vmatrix} x_1 & x_2 & x \\ y_1 & y_2 & y \\ 1 & 1 & 1 \end{vmatrix} = \begin{vmatrix} 0 & 1 & x \\ 0 & 0 & y \\ 1 & 1 & 1 \end{vmatrix} = y$$

This is underdetermined: x is not specified at all.

So instead we mimimize how far (x, y) moves from (x_0, y_0), as before, subject to the constraint $y = 0$.

$$(x - x_0)^2 + (y - y_0)^2 + \lambda_1 y = x^2 - x + y^2 + 1/4 + \lambda_1 y$$

$$\begin{aligned} 0 &= \text{error}_{12} \\ &= \begin{pmatrix} \text{slope}^x \\ \text{slope}^y \\ \text{slope}^\lambda_1 \end{pmatrix} (x^2 - x + y^2 + \lambda_1 y + 1/4) \\ &= \begin{pmatrix} 2x - 1 \\ 2y + \lambda_1 \\ y \end{pmatrix} \\ &= \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ \lambda_1 \end{pmatrix} + \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} \end{aligned}$$

Similarly we mimimize how far (x, y) moves from (x_0, y_0), to get to line 34: i.e., subject to the constraint given by the determinant below. Minimize

$$(x-x_0)^2+(y-y_0)^2+\lambda_2 \begin{vmatrix} x_3 & x_4 & x \\ y_3 & y_4 & y \\ 1 & 1 & 1 \end{vmatrix} = (x-0.5)^2+(y-0)^2+\lambda_2 \begin{vmatrix} 0 & 1 & x \\ 1 & 1 & y \\ 1 & 1 & 1 \end{vmatrix} = x^2-x+y^2+1/4+\lambda_2 * (y-1)$$

$$\begin{aligned} 0 &= \text{error}_{34} \\ &= \begin{pmatrix} \text{slope}^x \\ \text{slope}^y \\ \text{slope}^\lambda_2 \end{pmatrix} (x^2 - x + y^2 + 1/4 + \lambda_2 * (y - 1)) \end{aligned}$$

$$= \begin{pmatrix} 2x-1 \\ 2y+\lambda_2 \\ y-1 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ \lambda_2 \end{pmatrix} + \begin{pmatrix} -1 \\ 0 \\ -1 \end{pmatrix}$$

13. This gives six equations in four unknowns, which thus are overdetermined.

$$\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ \lambda_1 \\ \lambda_2 \end{pmatrix}$$

To avoid getting confused with all these numbers, let's call this

$$B = AX$$

If we minimixe

$$(AX-B)^T(AX-B)$$

we will get a "least squares" solution: error^2 is minimized
(that is, the error that we get by forcing X to be the solution given
by choosing any four elements of B and the corresponding rows of A).

It is plausible that slope$^X((AX-B)^T(AX-B))$ is proportional to

$$A^T(AX-B)$$

so I am going to suppose it (but it really needs to be shown).

This slope must be set to zero for the mimimization: so we must solve

$$A^TAX = A^TB$$

$$A^TA = \begin{pmatrix} 2 & 0 & 0 & 2 & 0 & 0 \\ 0 & 2 & 1 & 0 & 2 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 8 & 0 & 0 & 0 \\ 0 & 10 & 2 & 2 \\ 0 & 2 & 1 & 0 \\ 0 & 2 & 0 & 1 \end{pmatrix}$$

$$A^TB = \begin{pmatrix} 2 & 0 & 0 & 2 & 0 & 0 \\ 0 & 2 & 1 & 0 & 2 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

MATLAB:

```
A = [8,0,0,0;0,10,2,2;0,2,1,0;0,2,0,1];
B = [4;1;0;0];
X = A\B

X =

    0.5000
    0.5000
   -1.0000
   -1.0000
```

This gives $(x, y) = (1/2, 1/2)$, halfway between the two lines.
This seems a good compromise. It minimizes the error^2 from both lines if we put the point closer to one than the other line (or even right on it).

14. **Summary**

(These notes show the trees. Try to see the forest!)

A.
- A Web page is considered to have some importance, which it shares equally among all the web pages it links to.
- A Web page derives its importance from the importance of the Web pages that link to it.
- Some preliminary importance may be assigned to each Web page in a way that sustains some sort of importance even though no other page may link to it.
- The importances of all pages can be thought of as a vector over the Web the sum of whose terms is 1, taken to be the overall importance of the Web.
- The links of the Web can be thought of as a matrix, which includes both the sharing of importance and the sustaining importance.
- The final importances of the Web pages are in the vector that is unchanged by operating on it with the matrix: the eigenvector of the matrix.
- Iteration is one way to find the eigenvector associated with the largest eigenvalue.

B.
- Elementary row operations on the matrix and the target vector in a matrix equation do not change the meaning of the equation.
- They can be used systematically to solve matrix equations by "gaussian elimination".
- Gaussian elimination has a "triangularization" step costing $n^3/3$ operations, (+,*), and a "back-substitution" step costing $n^2/2$.
- Sketchpad uses matrix equations to enforce constraints on a drawing.
- Linear constraints, such as requiring that a point lie on the intersection of two lines, give simple matrix ("linear") equations.
- Nonlinear constraints, such as requiring a line of a given slope to have the same length as another line, require linear approximation to the nonlinear equation, which is iterated until the solution is close enough.
- Underdetermined constraints, such as requiring a point to be collinear with one line, are solved by also minimizing how far the point must move from its initial position.

- Distance , which is quadratic, can be minimized by setting its slope, which is linear, to zero, giving a linear constraint.
- Minimization subject to a constraint can be done by multiplying the error from this constraint by a "Lagrange multiplier", adding it all to the expression to be minimized, and minimizing as before and also with respect to the Lagrange multiplier.
- Overdetermined constraints, such as requiring a point to be collinear with two parallel lines, can be solved by minimizing the sum of squares of all the measures of how wrong any particular position of the point might be, all subject to the constraint.

15. **Excursions for Friday and beyond.**
You've seen lots of ideas. Now *do* something with them!

1. Do the relative importances of documents p, q and r make sense in the first, cyclic Web example? What strikes you most? If the link from q to r is removed, what changes? Why?
2. Why does repeating `b = A*b` eventually find the eigenvector of A with the largest eigenvalue for almost any starting value for b? Which starting value for b would fail to give this eigenvector?
3. Using a source vector, S, with 1-norm 0.15, what are the final ranks of p, q and r given the Web

$$\begin{array}{c|ccc} \surd & \mathrm{p} & \mathrm{q} & \mathrm{r} \\ \mathrm{p} & 0 & 0 & 0 \\ \mathrm{q} & 1 & 0 & 1 \\ \mathrm{r} & 0 & 1 & 0 \end{array}$$

   (This matrix describes the Web, but is not necessarily W: check this!) Don't forget to normalize R before each iteration. (Write a MATLAB script to do an iteration.) Is there a cycle? What would happen if $S = 0$?
4. Are R and S *really* vectors? (Week 7c)
5. Look up Page, Brin, Motwani & Winograd, 1999 *The PageRank Citation Ranking: Bringing Order to the Web* [PBMW99]. What other forms do they suggest for the source vector?
6. Use the same iteration technique to find an eigenvector of

$$\frac{5}{3}\begin{pmatrix} 1 & -4/5 \\ -4/5 & 1 \end{pmatrix}$$

   How fast does it converge? (If you got convergence on the first iteration, try it again with some other starting vector.) Once you have figured out the eigenvector, find a vector orthogonal to it and use that as the starting vector. What are the eigenvalues? What is their product? (You do not need to normalise before each iteration for this question. In fact, you should not.) In MATLAB, run `characteristic = poly(A)`, where A is the above matrix, then run `roots(characteristic)`. Make connections! NB `poly(A)` gives a polynomial in e which is `det(A - e*I)`. Try the code `syms e; det(A - e*eye(2))`. Then try `solve(ans)`.
7. Try the iteration technique on

   $\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$ How about $\frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$?

   Comment! Explore as in the previous excursion.

8. (Factor analysis/Principle components analysis in the measurement of personality)

(This is a long discussion which you can follow by plotting the points, and doing the calculations. Factor analysis is widely used, not just in psychology.)

A "social attitude inventory" asks psych. test subjects if, on a scale of 1 to 5, they strongly agree (1), agree (2), don't care (3), disagree (4) or strongly disagree (5) with statements such as the following.
x *Birth control, except when recommended by a doctor, should be made illegal*
y *It is immoral for men and women to find out whether they are sexually suited before marriage (e.g., by trial marriage).*
z *Conscientious objectors (pacifists) should be excused military service without stigma and certainly not treated as traitors to their country.*
(These questions are adapted from chapter 7 of [Eys58], published in 1957. How should this questionnaire have changed in the intervening fifty years to capture the same aspects of personality today?)

Suppose that five subjects responded as follows to the above three questions.

Question\Subject	a	b	c	d	e
x	1	5	3	2	4
y	1	5	3	2	4
z	2	2	3	4	4

Factor analysis tries to find out how many different types of personality are represented by the answers to these questions. It does this by calculating the extent to which different questions are independent of each other. Each question may be treated as a "vector" of responses, in this case a 5-dimensional vector. If the vectors are normalized, their scalar product is the cosine of the angle between them. (We saw this for two dimensions in Week 2; it is true for any number of dimensions. Show this! Use the invariance of $\vec{u}^T\vec{v} = (R\vec{u})^T R\vec{v}$ for any transformation matrix R, such as a rotation, whose inverse is its own transpose, to reduce the general case to the two-dimensional case.) This cosine is close to 1 if the vectors point in nearly the same direction, and close to 0 if the vectors are nearly at right angles, which we will call "orthgonal" and can consider to be "independent".

Before we normalize the three 5-vectors for **x**, **y** and **z**, we subtract their average values from each term. (This is not essential, but it is conventionally done in statistics, which then calls the results of the scalar products either the "variance" (product of a vector with itself) or the "co-variance" (product of a vector with another). It has the advantage of making the numbers smaller.) Here are the three vectors so reduced, with their lengths written in the first column. (The average values are 3 for each vector.)

Lengths	Question\Subject	a	b	c	d	e
$\sqrt{10}$	**x**	-2	2	0	-1	1
$\sqrt{10}$	**y**	-2	2	0	-1	1
2	**z**	-1	-1	0	1	1

The nine scalar products of these three vectors with each other may be written as a matrix.

$$\begin{pmatrix} -2 & 2 & 0 & -1 & 1 \\ -2 & 2 & 0 & -1 & 1 \\ -1 & -1 & 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} -2 & -2 & -1 \\ 2 & 2 & -1 \\ 0 & 0 & 0 \\ -1 & -1 & 1 \\ 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 10 & 10 & 0 \\ 10 & 10 & 0 \\ 0 & 0 & 4 \end{pmatrix}$$

and if all the vectors had been normalized, we get the "variance-covariance" matrix

$$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

We see that **x** and **y** are completely correlated, which is pretty obvious from the original data. It is also obvious that there are only two dimensions in this data, which we could have seen just by plotting the original data as five points in 3-D. There are just two independent "factors" or "principle components". The next step is to extract these, using techniques which will work even if the data is much more complex and much less obvious.

Before going on to the next step, let's alter the original data so that the correlation between **x** and **y** is a little less obvious. Suppose the answers to question **x** are `2, 4, 3, 3, 3` instead, so the (co-)variance matrix is now found from

$$\begin{pmatrix} -1 & 1 & 0 & 0 & 0 \\ -2 & 2 & 0 & -1 & 1 \\ -1 & -1 & 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} -1 & -2 & -1 \\ 1 & 2 & -1 \\ 0 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 4 & 0 \\ 4 & 10 & 0 \\ 0 & 0 & 4 \end{pmatrix}$$

and, when normalized, this is the (co-)variance matrix

$$\begin{pmatrix} 1 & 2/\sqrt{5} & 0 \\ 2/\sqrt{5} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Now we proceed to finding the two important dimensions. For the first case, with (co-)variance

$$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

z is one of the important dimensions because it is clearly independent of the others: its covariance with them is zero. But **x** and **y** are mixed up together, and so we must be able to find some *combination* of **x** and **y** which gives an independent state. (The "state" is some mixture of questions **x** and **y**. "Mixture of questions" is really no stranger than quantum mechanics, where states can be mixtures of other states. In factor analysis, it can be taken as a handy fiction; in quantum mechanics, we must accept it being closer to physical reality.) In fact, we want to find mixtures of **x** and **y** so that the (co-)variance matrix, for these mixtures and **z**, has zeros everywhere except on the diagonal.

Now we are on familiar ground. We want the *eigenvectors* of the (co-)variance matrix. A little trial and error gives (1,1,0), (1,-1,0) and (0,0,1) as these eigenvectors. (Normalizing them, that would be $(1/\sqrt{2}, 1/\sqrt{2}.0), (1/\sqrt{2}, -1/\sqrt{2}, 0)$ and (0,0,1).)

What is the (co-)variance matrix for these new states, $\mathbf{x}+\mathbf{y}$, $\mathbf{x}-\mathbf{y}$ and **z**?

$$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} a/\sqrt{2} & b/\sqrt{2} & 0 \\ a/\sqrt{2} & -b/\sqrt{2} & 0 \\ 0 & 0 & c \end{pmatrix},$$

where actually doing the matrix multiplication shows that $a = 2, b = 0$ and $c = 1$. These eigenvectors are orthogonal to each other, and since they are normalized,

$$\begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

(The first matrix is actually the transpose of the second, but the entries in this particular matrix conceal the fact.)

Thus

$$
\begin{aligned}
&\begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \\
= &\begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a/\sqrt{2} & b/\sqrt{2} & 0 \\ a/\sqrt{2} & -b/\sqrt{2} & 0 \\ 0 & 0 & c \end{pmatrix} \\
= &\begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix} \\
= &\begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.
\end{aligned}
$$

The way to diagonalize a (symmetric) matrix, V, is to multiply $E^T V E$, where E is the array of (orthogonal) normalized eigenvectors of V.

This diagonalization effectively transforms the question-space into a space of mixed questions, so of course it transforms the questions themselves. Just as $V \to E^T V E$, so the questions $Q \to E^T Q$. Here are the new, mixed questions.

$$
\begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -2 & 2 & 0 & -1 & 1 \\ -2 & 2 & 0 & -1 & 1 \\ -1 & -1 & 0 & -1 & 1 \end{pmatrix} = \begin{pmatrix} -2\sqrt{2} & 2\sqrt{2} & 0 & -\sqrt{2} & \sqrt{2} \\ 0 & 0 & 0 & 0 & 0 \\ -1 & -1 & 0 & 1 & 1 \end{pmatrix}
$$

One whole dimension disappears completely. This is because $b = 0$. Since the original $\mathbf{x}$ and $\mathbf{y}$ values lay entirely in a plane, all the variance is in that plane, giving $a = 2$, and none is orthogonal to the plane, giving $b = 0$.

Plot these results as five points in a two-dimensional space.

If the orthogonal variance is not zero, it might still be quite small. Show that the eigenvectors for the second case, with (co-)variance matrix

$$
\begin{pmatrix} 1 & 2/\sqrt{5} & 0 \\ 2/\sqrt{5} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}
$$

are also (1,1,0), (1,-1,0) and (0,0,1), but this time with "eigenvalues" $a = 1 + 2/\sqrt{5} = 1.894$, $b = 1 - 2/\sqrt{5} = 0.106$ and $c = 1$. The variance given by b, while not zero anymore, is much smaller than the other variances, and we can decide to discard it. We won't even use this eigenvector in transforming the questions.

$$
\begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -1 & 1 & 0 & 0 & 0 \\ -2 & 2 & 0 & -1 & 1 \\ -1 & -1 & 0 & 1 & 1 \end{pmatrix} = \begin{pmatrix} -3/\sqrt{2} & 3/\sqrt{2} & 0 & -1/\sqrt{2} & 1/\sqrt{2} \\ -1 & -1 & 0 & 1 & 1 \end{pmatrix}
$$

We again have two factors, having decided that the variance of the third is too small to count.

Plot these results as five points in a two-dimensional space. This is the factor space. Can you think of names to give the axes to suggest the kinds of personalities that would give high scores on each of the two axes?

Typically such personality measurements would consist of thousands of subjects answering tens of questions. Five dimensions of personality traits and fifty questions to assess them are in [Har05]. The long discussion in this excursion should encourage you to take it all with a pinch of salt.

9. Show that
$$\begin{pmatrix} a & c \\ b & d \end{pmatrix}$$
is the sum of a symmetric and an antisymmetric matrix. What are they? What about any 3×3 matrix? Find the components also for the special cases of the Web matrices.
In MATLAB, run `characteristic = poly(A)`, where A is any of the Web matrices, then run `roots(characteristic)`. Discuss the results in the light of this and the last two excursions.

10. Given
$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} a & c \\ b & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix},$$
show the effect of doing the three elementary operations on the *columns*.

11. The *binomial coefficient* is defined as $n!k \stackrel{\text{def}}{=} \frac{n!}{(n-k)!k!} = \frac{n(n-1)..(n-k+1)}{k(k-1)..1}$, and prononced "$n$ choose k", where $k! \stackrel{\text{def}}{=} k(k-1)..1$.
a) Draw $k!2$ as a triangle of rows of dots, one row for each value of $k = 2, 3, 4, ..$
b) Draw $k!3$ as a tetrahedron of triangular planes of dots, one plane for each value of $k = 3, 4, ..$
c) Show that $n!k = (n-1)!(k-1) + (n-1)!k$ and hence that binomial coefficients form *Pascal's triangle*, in which any binomial coefficient is the sum of the two directly above it.

0!0
1!0 1!1
2!0 2!1 2!2
3!0 3!1 3!2 3!3
4!0 4!1 4!2 4!3 4!4
5!0 5!1 5!2 5!3 5!4 5!5
⋰ ⋱⋰ ⋱⋰ ⋱⋰ ⋱⋰ ⋱⋰ ⋱

d) Given $0! \stackrel{\text{def}}{=} 1$ (and hence $0!0 = 1$), convert this triangle into a triangle of integers which shows these sums of pairs.
e) Use the two versions of Pascal's triangle to discover that $0!k + 1!k + 2!k + .. + n!k = (n+1)!(k+1)$, given $n!k \stackrel{\text{def}}{=} 0$ for any $n < k$.
f) Use this sum to show that $1 + 2 + 3 + 4 + .. + n = \frac{(n+1)n}{2}$. What is an even more direct way to show this result?
g) Use the sum similarly to show that $1^2 + 2^2 + 3^2 + .. + n^2 = \frac{(n+1)(n+1/2)n}{3}$. Hint: $k!2 = (k^2 - k)/2 = (k^2 - k!1)/2$

12. Write a MATLAB function, `function x = GaussElim(A,b)`, using for-loops, which solves the linear equation $Ax = b$ for square matrix A and vectors x and b.

Suppose the matrix and vectors are of matching sizes (or write code to check this and give a special answer if not). Do not try to swap rows to avoid dividing by small numbers or zero:

your function may fail or give strange results for some matrices. You can check your result on MATLAB using `A*x`; if this is b you are OK; if not, and you are stuck and would like to see what x is, use `A\b`.

Modify your function to `function [x,det] = GaussElim(A,b)`, which also returns the determinant of A (the product of all your denominators).

Is there a MATLAB construct which allows you to avoid specifying the order in which the inner loops are executed?

13. Using Gauss elimination by hand, find the determinants of

$$\begin{matrix} 0 & 1 & x \\ 0 & 1 & y \\ 1 & 1 & 1 \end{matrix} \qquad \text{and} \qquad \begin{matrix} 1 & 0 & x \\ 0 & 1 & y \\ 1 & 1 & 1 \end{matrix}$$

and compare with the diagonals method of finding these 3×3 determinants.

14. We defined the determinant of a $d \times d$ matrix to be the d-dimensional hypervolume of the hyperparallelogram whose edges are the target vectors of $(1,0,..,0)^T$, $(0,1,0,..,0)^T$, etc., i.e., the columns of the transforming matrix; and we said that swapping any two of these vectors changes the sign of the volume and of the determinant. How should we interpret the *rows* of the matrix as a volume, and should there be a similar rule for sign change on swapping rows?

15. What is the determinant of the projection transformation,

$$\begin{matrix} c^2 & cs \\ cs & s^2 \end{matrix}$$

(from Week 1 on polarized light)? Why?

16. Use MATLAB's `det()` function to explore the determinants of some 2×2 and 3×3 matrices. Try deliberately to create some sign changes and some zeros, and look carefully at the matrices that give those results.

17. Representing $x^2 - 3/4$ as `ex = [1,0,-3/4]` and $y^2 - 2y + 1/4$ as `ey = [1,-2,1/4]`, use MATLAB's `polyval(ex,x)` and `polyval(ey,y)` to evaluate these polynomials at different values of x (such as 0, 0.5, 1) and y (such as 0, 0.5, 1), respectively. Use one of MATLAB's plotting routines to plot each of these curves.

18. How many additions and how many multiplications are needed to calculate $ax^2 + bx + c = axx + bx + c$? How many are needed for $c + x(b + x(a))$? How about a polynomial of degree d, with $d + 1$ coefficients?

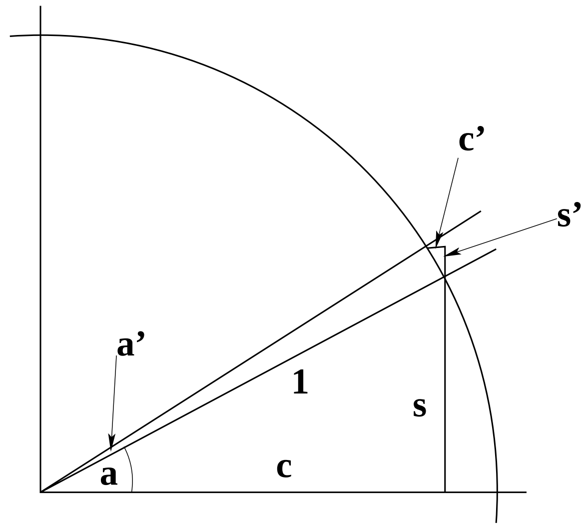
19. Plot $(x-1)^2 + (y-1/2)^2$ using MATLAB's `mesh()`, and use the plot window controls to move the plot around until you are convinced that slope^x and slope^y are both 0 at the minimum. For some other function (for instance, the negative of the function you just plotted), 0 values for slope^x and slope^y have another meaning: what is it?

20. Given a polynomial (in MATLAB representation) `[a,b,c]`, what is its slope in the same representation?

21. What does the binomial theorem tell you about the slope of x^n for any positive integer n?.

22. What function is its own slope everywhere, $f(x) = \text{slope}^x f(x)$?
Two things to try:
1. Draw curves until you get a feel for the shape of such a function.
2. Find coefficients a_i in the series $f(x) = a_0 + a_1 x^1 + a_2 x^2 + ..$
Note that slope$^x f(x)$ is conventionally written $\mathrm{d}\, f(x)/\mathrm{d}x$, and that this excursion is exploring the differential equation that gives the fixed point of the derivative operator, $\mathrm{d}/\mathrm{d}x$.

23. Use the figure



to find the slopes of sin() and cos() everywhere: **s** is sin(**a**), **s'** is the difference between sin(**a**+**a'**) and sin(**a**), and similarly for **c**, **c'** and cos(). Hint: in the **c'**–**s'** triangle, which angle is the same as **a**?
What is the slope everywhere of cos() + $i\times$sin()? How does this relate to the function that is its own slope everywhere of the previous excursion?
What series, $a_0 + a_1x^1 + a_2x^2 + ..$, might apply to sin()? To cos()? How might you then write a MATLAB program to calculate sin() and cos(), pretending that MATLAB does not have these as builtin functions?

24. For the underdetermined equation, $x + y = 1$, what does MATLAB's `pinv(A)*b` give? (`A = [1,1], b = [1]`) What distinguishes this solution from any other of the infinite number of solutions? Why is it not satisfactory as a Sketchpad solution?

25. Why do we minimize the square of the error, $AX - B$, giving least square error, rather than minimizing the error, giving, "least error"?

26. For ordinary variables, a, b and x, show that $\text{slope}^x((ax - b)^2) = 2a(ax - b)$.
For matrices, A, B, X, why does $A^TAX = A^TB$ not imply $AX = B$?
For A, B and X from the overdetermined equation of the lecture, show that $A^TAX = A^TB$ does result from minimizing on all variables.

27. Solve the overdetermined equation of the lecture, $AX = B$, using `A\B` in MATLAB.

28. What are the constraints to make Sketchpad turn six points, approximately located by the user, into the vertices of a hexagon? Work through an example, taking the first two points drawn as immovable.

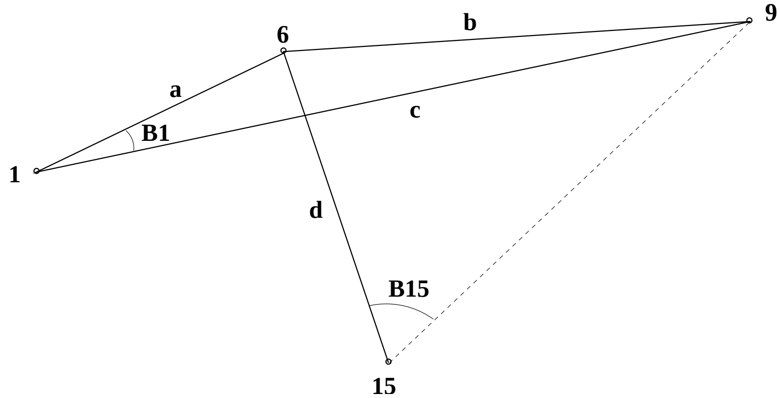
29. Look up Ivan Sutherland, 1938–, and his 1963 MIT Ph.D. thesis, *Sketchpad: A Man-machine Graphical Communications System* [Sut80]. (See also sketchpad.ps in the notes for this course.) How does Sketchpad simulate linkages such as the Peaucellier linkage for drawing straight lines?

30. Least squares solutions to overdetermined equations are used in surveying to find the coordinates of a location after taking redundant measurements. This is used by GPS (global positioning systems) to locate a receiver given its distances from the several NAVSTAR satellites that may be visible at any one time.

Here is a two-dimensional, ground-based surveying problem which is simpler to treat than 3-D satellite positioning with all the complexities of doing orbit calculations (taking account orbit changes due to solar radiation), spherical trigonometry (adjusting for the imperfect ellipsoidal shape of the "geoid"), and very precise clocking needed to convert radio transmission times through space and the atmosphere into distances (dealing with the irregularities in the Earth's rotation, the movement of the north pole, and continental drift). This example is taken from chapter 5 of [Lei95], an informative (especially chapters 2 and 3) book but sadly inaccessible to beginners.

There are four locations, two with known coordinates

$$x_1 = 603.11, y_1 = 3102.44, x_{15} = 833.52, y_{15} = 2975.50.$$



The coordinates of the other two are only guesses

$$x_6 = 900, y_6 = 3600, x_9 = 900, y_9 = 3600.$$

But four distances and two angles have been measured

$$a = 179.7, b = 309.7, c = 480.4, d = 218.3, \cos B_1 = 0.9683, \cos B_{15} = 0.4203.$$

If we specify five distances (all nonlinear)

$$\begin{aligned}
\alpha(x_6, y_6) &= (x_6 - x_1)^2 + (y_6 - y_1)^2 \\
\beta(x_6, y_6, x_9, y_9) &= (x_6 - x_9)^2 + (y_6 - y_9)^2 \\
\gamma(x_9, y_9) &= (x_9 - x_1)^2 + (y_9 - y_1)^2 \\
\delta(x_6, y_6) &= (x_6 - x_{15})^2 + (y_6 - y_{15})^2 \\
\epsilon(x_9, y_9) &= (x_9 - x_{15})^2 + (y_9 - y_{15})^2
\end{aligned}$$

we can write six equations in the four unknowns, x_6, y_6, x_9, y_9

$$\begin{aligned}
\alpha(x_6, y_6) &= a^2 \\
\beta(x_6, y_6, x_9, y_9) &= b^2 \\
\gamma(x_9, y_9) &= c^2 \\
\delta(x_6, y_6) &= d^2 \\
\alpha(x_6, y_6) + \gamma(x_9, y_9) - \beta(x_6, y_6, x_9, y_9) &= 2 \times \sqrt{\alpha(x_6, y_6)\gamma(x_9, y_9)} \times \cos B_1 \\
\delta(x_6, y_6) + \epsilon(x_9, y_9) - \beta(x_6, y_6, x_9, y_9) &= 2 \times \sqrt{\delta(x_6, y_6)\epsilon(x_9, y_9)} \times \cos B_{15}
\end{aligned}$$

The rest of this excursion is for you: linearize these equations about the guessed values $x_6 = 900, y_6 = 3600, x_9 = 900, y_9 = 3600$; set up the linear equations $\vec{B} = A\vec{X}$ with A a 6×4 matrix and $\vec{X}$ the four unknown coordinates; solve the least-squares version, $A^T A\vec{X} = A^T\vec{B}$ for the next approximation to the coordinates in X; repeat this until the values for the unknown coordinates converge to the same kind of precision that the known coordinates started out with; compare your answers with the locations given in Table F.3 of [Lei95, page 532] (those data are in degrees, minutes and seconds of longitude and latitude; yours will come out in seconds to be added to $(291°, 44°)$, a location in Maine).

31. Any part of the lecture that needs working through.

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