Excursions in Computing Science: Week 4 Two-dimensional Numbers and Turtles

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February 8, 2007

1. We've described a rotation as a product of matrix and vector,

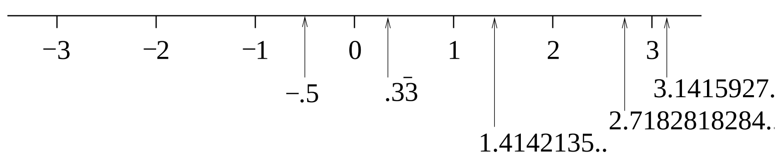
$$R.\vec{v} = \begin{pmatrix} c & -s \\ s & c \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

and a double rotation as a product of two matrices.

$$R_\phi R_\theta = \begin{pmatrix} c & -s \\ s & c \end{pmatrix}_\phi \begin{pmatrix} c & -s \\ s & c \end{pmatrix}_\theta$$

Can these be replaced by just numbers?

2. Numbers so far are 1-dimensional.



A quarter-turn (rotation by $\pi/2$) rotates it by a right angle into the—unknown. E.g.

$$R_{\pi/2} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Two quarter-turns turn 1 into -1

$$R_{\pi/2} R_{\pi/2} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$$

3. On the line, we can do this by

$$-1 \times x = -x$$

 The author gratefully acknowledges support from the taxpayers of Québec and of Canada who have paid his salary and research grants while this work was developed at McGill University, and from his students (who built the implementations and investigated the data structures and algorithms) and their funding agencies.

for any x including $x = 1$.

In the (unknown) two dimensions

$$(R_{\pi/2})^2 \begin{pmatrix} 1 \\ 0 \end{pmatrix} = - \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

So maybe

$$R_{\pi/2} = \sqrt{-1}$$

How can we write $R_{\pi/2}$ (or R_θ) as a number?

4. Try $R_\theta = \cos\theta + \sqrt{-1}\sin\theta$, e.g.,

$$\begin{aligned} R_{\pi/2} &= \sqrt{-1} \\ R_\pi &= R_{\pi/2}R_{\pi/2} = (R_{\pi/2})^2 = -1 \\ R_\phi R_\theta &= (c + \sqrt{-1}s)_\phi (c + \sqrt{-1}s)_\theta \\ &= c_\phi c_\theta - s_\phi s_\theta + \sqrt{-1}(c_\phi s_\theta + s_\phi c_\theta) \end{aligned}$$

Compare this last with

$$\begin{pmatrix} c & -s \\ s & c \end{pmatrix}_\phi \begin{pmatrix} c & -s \\ s & c \end{pmatrix}_\theta = \begin{pmatrix} c_\phi c_\theta - s_\phi s_\theta & -(c_\phi s_\theta + s_\phi c_\theta) \\ c_\phi s_\theta + s_\phi c_\theta & c_\phi c_\theta - s_\phi s_\theta \end{pmatrix}$$

5. So rotations are (weird) numbers.

What about vectors?

$$\begin{pmatrix} c & -s \\ s & c \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} cx - sy \\ sx + cy \end{pmatrix}$$

$$\text{Try} \quad \begin{pmatrix} x \\ y \end{pmatrix} = x + \sqrt{-1}y$$

$$(c + \sqrt{-1}s)(x + \sqrt{-1}y) = cx - sy + \sqrt{-1}(sx + cy)$$

Bingo!

6. $\sqrt{-1}$ is important, if unimaginable.

We'll call it i for "imagine that!".

The best we can do is think of it as a right-angle departure from the known numbers to the unknown.

Think of i = rot90: $(\text{rot}90)^2 = \text{rot}180 = -1$.

(We could call it $\perp$ and write $\perp$ in place of $\sqrt{-1}$ from now on, but that would make it harder for you to read conventional notation.)

(There are in fact *two* conventions. The second convention is to write j for $\sqrt{-1}$. This is used by engineers, perhaps because i also means electric current. i is used by mathematicians and scientists. MATLAB supports both conventions.)

We'll call these two-dimensional numbers "2-numbers" from now on, to keep it short. If we need to, we'll call the old, one-dimensional numbers "1-numbers".

7. These “2-numbers” have the *same formal properties* as the “1-numbers”:

$+, \times$ – closed
– commutative $a+b=b+a$, $a \times b = b \times a$
– associative $(a+b)+c = a+(b+c)$, $(a \times b) \times c = a \times (b \times c)$
– have identities 0 $(+)$, 1 $(\times)$

$+$ each element has an inverse $a + a' = 0 = a' + a$

$\times$ each element except 0 has an inverse $a \times a' = 1 = a' \times a$

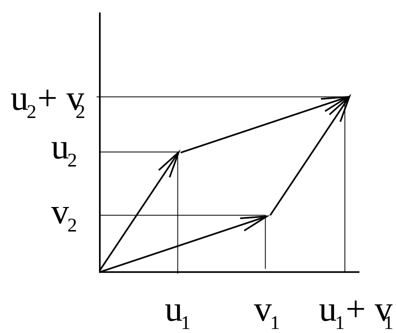
$+, \times$ $\times$ is distributive over $+$: $a \times (b+c) = a \times b + a \times c$

The above are the “field axioms”. 2-numbers and 1-numbers each exemplify what mathematicians call a “field”. (This is quite different from what physicists call a “field”, and neither has anything to do with what a farmer calls a “field”.)

The fact that 2-numbers have the same formal properties as 1-numbers makes them familiar and easy to use. It also justifies our calling them both “numbers”.

8. Adding 2-numbers

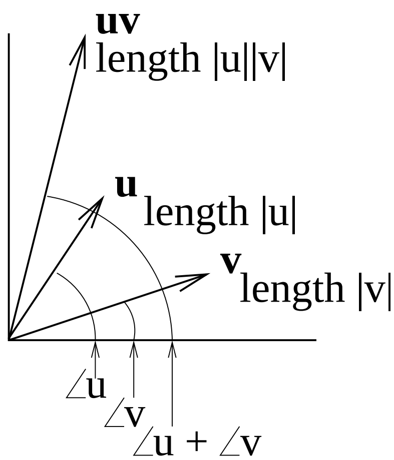
$$\begin{aligned} u+v &= (u_1 + iu_2) + (v_1 + iv_2) \\ &= u_1 + v_1 + i(u_2 + v_2) \end{aligned}$$



9. Multiplying 2-numbers

$$\begin{aligned} uv &= (u_1 + iu_2)(v_1 + iv_2) \\ &= u_1 v_1 - u_2 v_2 + i(u_1 v_2 + u_2 v_1) \\ \text{Try } v &= \mid u \mid (\cos \angle u + i \sin \angle u) = \mid u \mid (c + is) \\ u &= \mid v \mid (\cos \angle v + i \sin \angle v) = \mid v \mid (c' + is') \\ \mid u \mid &= +\sqrt{u_1^2 + u_2^2} \\ \mid v \mid &= +\sqrt{v_1^2 + v_2^2} \\ \text{Now } u_1 v_1 - u_2 v_2 &= \mid u \mid\mid v \mid (cc' - ss') = \mid u \mid\mid v \mid \cos(\angle u + \angle v) \\ u_1 v_2 + u_2 v_1 &= \mid u \mid\mid v \mid (cs' + sc') = \mid u \mid\mid v \mid \sin(\angle u + \angle v) \\ \text{Compare } \begin{pmatrix} c & -s \\ s & c \end{pmatrix} \begin{pmatrix} c' & -s' \\ s' & c' \end{pmatrix} &= \begin{pmatrix} cc' - ss' & -(cs' + sc') \\ (cs' + sc') & cc' - ss' \end{pmatrix} \end{aligned}$$

where the first matrix is $\text{rot}_{\angle u}$, the second is $\text{rot}_{\angle v}$, and the resulting matrix is $\text{rot}_{\angle u + \angle v}$:



Since multiplying these 2-numbers by each other just sums their angles, i.e., the resultant 2-number has an angle which is the sum of the angles of the two elements in the product, the 2-numbers behave just like exponentials.

We can try writing

$$\begin{aligned} u &= \mid u \mid e^{i\angle u} \\ v &= \mid v \mid e^{i\angle v} \\ uv &= \mid u \mid\mid v \mid e^{i(\angle u + \angle v)} \end{aligned}$$

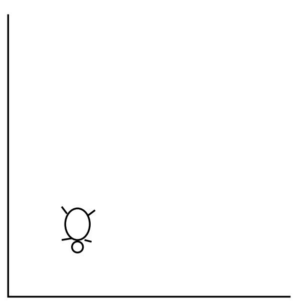
(whatever e is: it doesn't really matter to us).

10. Turtle graphics.

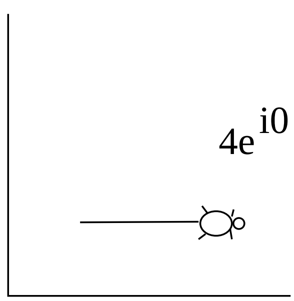
If computers are the wave of the future,
graphics is the surfboard Apologies to [Nel78]

Let's draw a house.

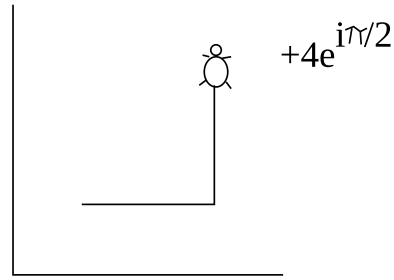
Start.



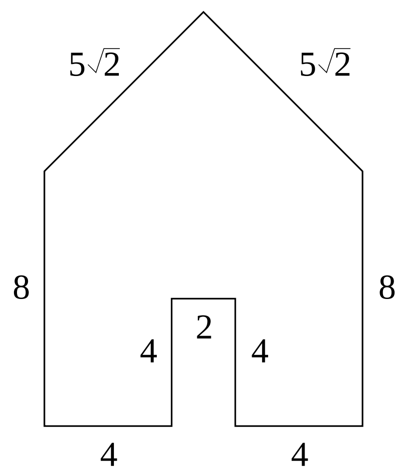
Turn $\pi/2$; Go 4. $4e^{i0}$



Turn $\pi/2$; Go 4. $+4e^{i\pi/2}$



Turn $-\pi/2$; Go 2. $+2e^{i0}$
Turn $-\pi/2$; Go 4. $+4e^{-i\pi/2}$
Turn $\pi/2$; Go 4. $+4e^{i0}$
Turn $\pi/2$; Go 8. $+8e^{i\pi/2}$
Turn $\pi/4$; Go $5\sqrt{2}$. $+5\sqrt{2}e^{i3\pi/4}$
Turn $\pi/2$; Go $5\sqrt{2}$. $+5\sqrt{2}e^{i5\pi/4}$
Turn $\pi/4$; Go 8. $+8e^{i3\pi/2}$



Note: the turtle ends up in the same position and orientation, and the “total turtle turning”, TTT, is 2π.

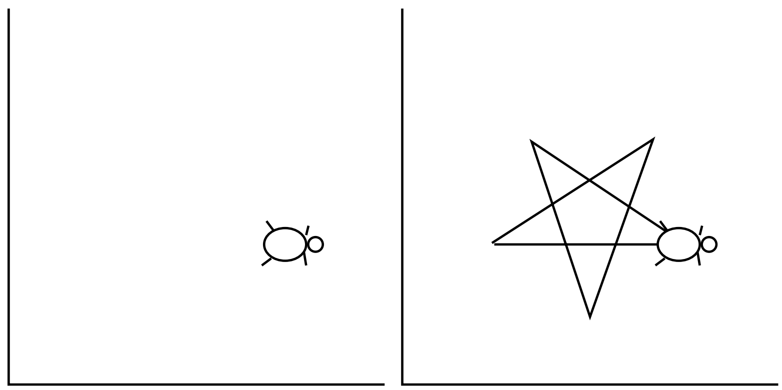
$$4e^{i0} + 4e^{i\pi/2} + 2e^{i0} + 4e^{-i\pi/2} + 4e^{i0} + 8e^{i\pi/2} + 5\sqrt{2}e^{i3\pi/4} + 5\sqrt{2}e^{i5\pi/4} + 8e^{i3\pi/2} = 0$$

11. Closed figures.

Is TTT $= 2\pi$ always?

Let’s try $4\pi/5$ 5 times:

$$e^{i4\pi/5} + e^{i8\pi/5} + e^{i12\pi/5} + e^{i16\pi/5} + e^{i20\pi/5}$$



TTT $= 4\pi$ and note that the turtle winds up where it started and in the same orientation:

$$e^{i4\pi/5} + e^{i8\pi/5} + e^{i12\pi/5} + e^{i16\pi/5} + e^{i20\pi/5} = 0.$$

12. Summary

(These notes show the trees. Try to see the forest!)

- i (Imagine it!) (or $\perp$ or $\sqrt{-1}$ or j) is some weird 2nd dimension.
- $i\times$ rotates a 2-number through $\pi/2$
- $i \times i\times$ rotates a 2-number through π, i.e. *number*$\rightarrow$ $-$*number*
- 2-numbers, $a + ib$ or $a + \perp b$, behave formally exactly like 1-numbers: both satisfy the "field axioms"
- $(\cos\theta + i\sin\theta)\times$ (or $(\cos\theta + \perp \sin\theta)\times$) rotates a 2-number through θ
- $(c + is)(c' + is')\times$ (or $(c + \perp s)(c' + \perp s')\times$) rotates a 2-number through $\angle(c, s) + \angle(c's')$ ...
- ... so this suggests we write $\cos\theta + i\sin\theta$ as $e^{i\theta}$, for some e, just for convenience
- $ae^{i\theta} + a'e^{i\theta'} + ..$ models turtle graphics (except that the turtle makes relative turns, but the angles $\theta, \theta', ..$ are absolute orientations).
- Total Turtle Turning TTT, returning to same position and orientation, is an integer multipls of 2π, and
$$\sum_{k=1}^{m} e^{i2k\pi/m} = 0$$

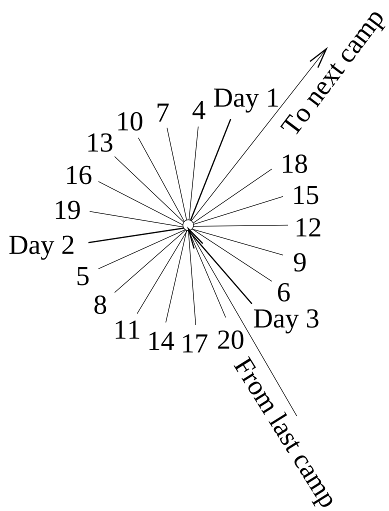
13. **Excursions for Friday and beyond.**

You've seen lots of ideas. Now *do* something with them!

1. For some 1-number, e, (and we do not need to know just now what its value is), $\cos\theta + i\sin\theta = e^{i\theta}$. We need two arguments to show this: a) the expression $\cos\theta + i\sin\theta$ behaves as though it were something raised to the power θ; b) i must also be in this power. What are the arguments? From this, what is the relationship among the five important "numbers", $0, 1, e, i, \pi$?
   (Note. An *expression* is a combination of mathematical symbols, including operators and variables, for which a value can be calculated once values are chosen for the variables.)
2. Use MATLAB to plot e^{ix} (`exp(i*x)`). Compare this with a plot of $\cos(x) + i \times \sin(x)$.

3. Find a running copy of the programming language Logo (politely ask your kid sibling for a go) and explore "turtle graphics". Describe the graphics operators in terms of 2-numbers. Look up "turtle geometry" [AdS81]. What is the turtle's take on general relativity? (Logo not only has neat graphics for Grade 3 use, it also is a superb programming language. Explore all of it!)

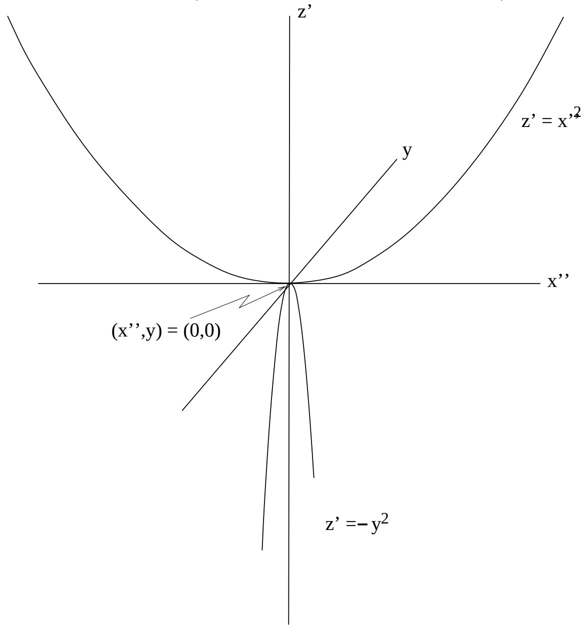
4. A colony of the army ant species Eciton burchellii bivouacs for a period of three weeks, while eggs become pupae, and radiates outward daily in search of food [Mof06]. Their 20 days of hunting (followed by the 21st day, in which they set off for a new bivouac a couple of kilometers distant) cover a region of about 80 m. radius as shown:



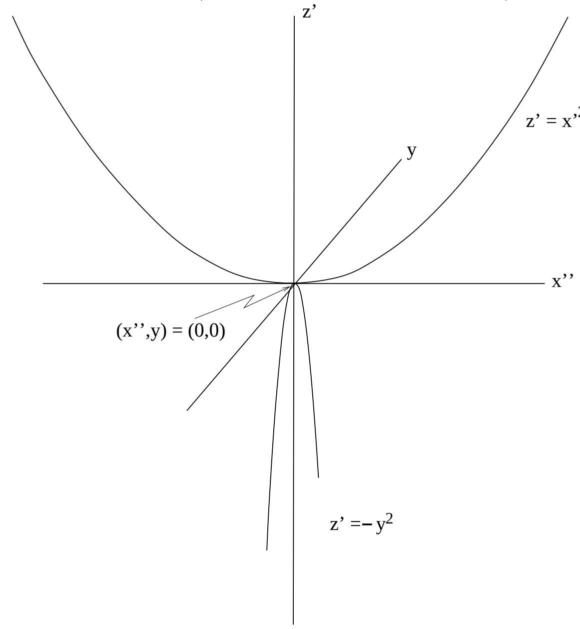
   Describe this pattern using 2-numbers.
   How would you include the path along which the ants arrive from their previous bivouac on day 0?

5. Think about the following two chains of transformations of a parabola, $z = x^2$, and its "shadow" parabola in the perpendicular plane, $z = -y^2$. Relate the final equations for z in each case to $z = ax^2 + bx + c$, and discover the formulas for the "roots", i.e., the values of x and y when $z = 0$.

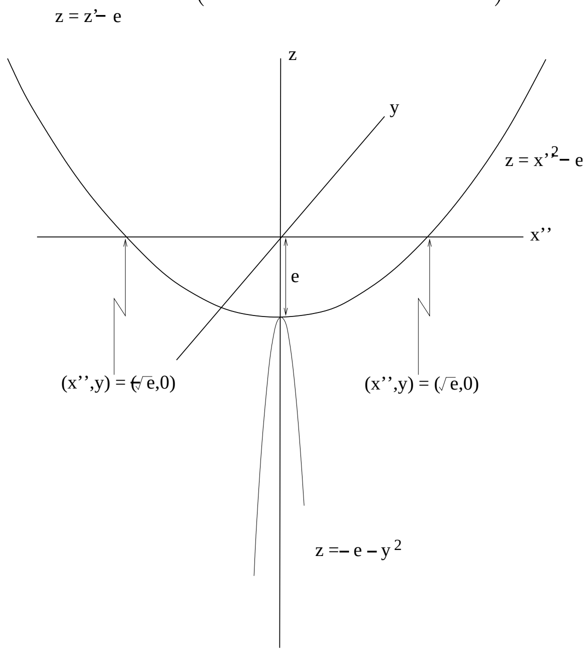
Chain 1 (read down to next page)

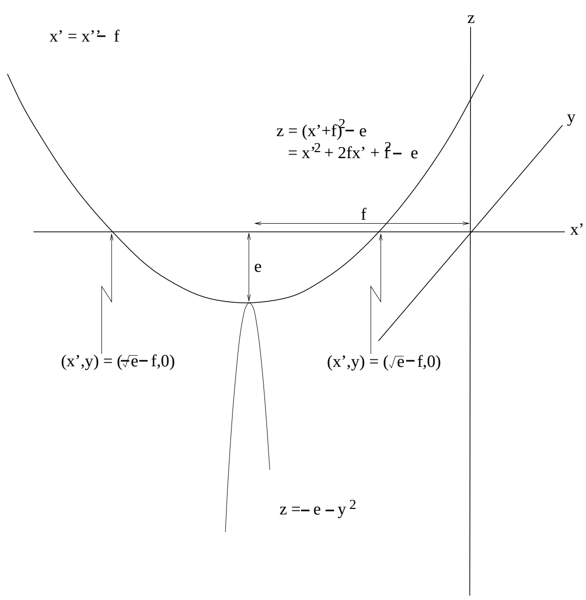


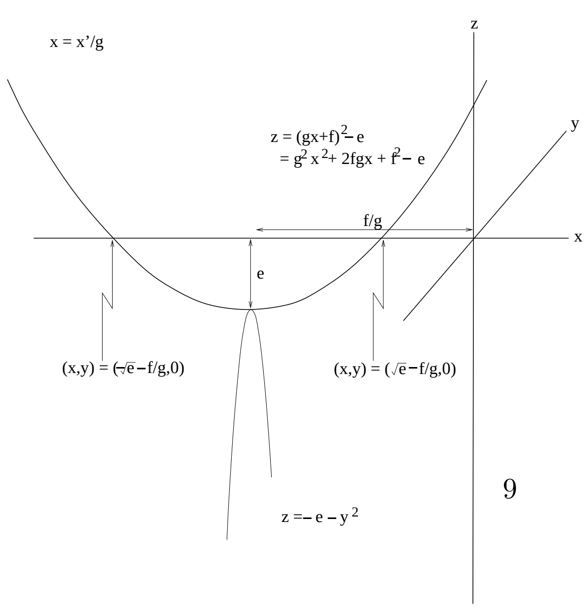
Chain 2 (read down to next page)



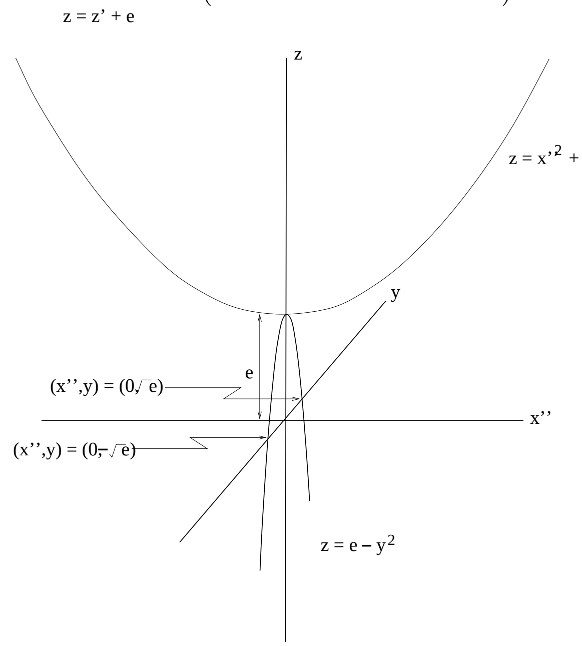
Chain 1 (cont.: read downwards)

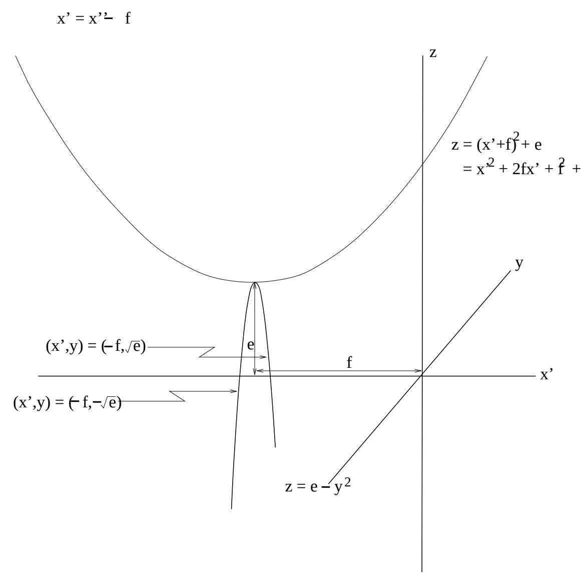


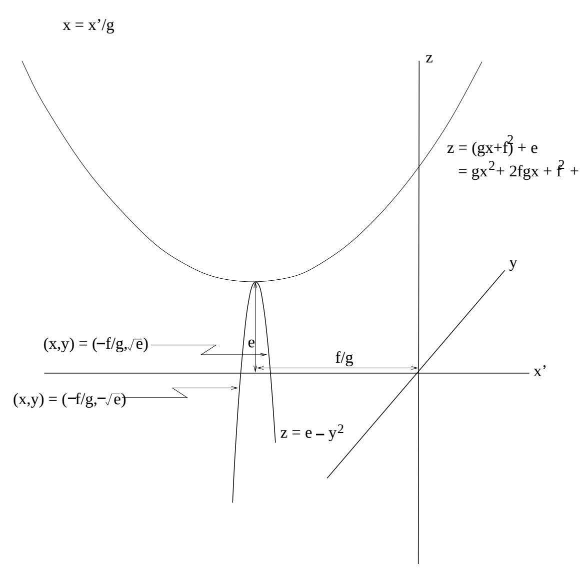




Chain 2 (cont.: read downwards)







6. The parabola $z = x^2$ in the previous excursion is undefined for negative z, but can be extended in direction y at right angles to x with $z = -y^2$. What is the corresponding extension for a circle, $\frac{x^2}{r^2} + \frac{z^2}{r^2} = 1$, or for an ellipse, $\frac{x^2}{a^2} + \frac{z^2}{b^2} = 1$?

7. If $v = x + iy =\mid v \mid e^{i\angle v}$ is any 2-number, its *reflection* in the line of 1-numbers is given by a new operation, the *conjugate*, $v^* = (x + iy)^* = x - iy = (\mid v \mid e^{i\angle v})^* =\mid v \mid e^{-i\angle v}$. Suppose $u = c + is = e^{i\angle u}$ is any other 2-number of magnitude $\mid v \mid= 1$, giving the direction of a line, which we will also call u, and convince yourself that uv^*u is the reflection of v in u.
Using this, go on to show that the *projection* of v in u (that is the component of v that lies in the same direction as u) is $(v + uv^*u)/2$, and that the component of v that is perpendicular to this is $(v - uv^*u)/2$. What are the projections of v on the line of 1-numbers and on $\perp$, the line at right angles to it?

8. Two-dimensional numbers support the rotation

$$\begin{pmatrix} c & -s \\ s & c \end{pmatrix}$$

as $c + is$, with $c^2 + s^2 = 1$ The Lorentz shear

$$\begin{pmatrix} a & b \\ b & a \end{pmatrix}$$

has $a^2 - b^2 = 1$ How might two-dimensional numbers, i in particular, be used to make this shear look like a rotation?
Does it seem from this that there is a 2-number operator for shear, in the way there is a 2-number operator for rotation (and reflection and projection)?

9. On the field axioms (note 7):
a. Show that, if 1-numbers and i satisfy the field axioms, then so do 2-numbers.
b. Find two matrices whose product does not commute.
c. Are matrix products associative?
d. Compare uv^T and v^Tu for a vector u and a transposed vector v^T. Is this operation closed?
e. Find an operation on 1-numbers which is commutative but not associative.
f. Find an operation on strings, such as “hello ” and “world ”, which is associative but not commutative. (Note. A *string* is a sequence of characters, often letters and spaces.)
g. Are the field axioms consistent, i.e., do any of them contradict any others? (Other questions one can ask about axioms: Are they independent—can any be derived from others? Are they complete—is anything missing?)

10. These axioms are also given by W. W. Sawyer *A Concrete Approach to Abstract Algebra* [Saw59]. Discuss his demonstration, at the end of the book, that an angle cannot be trisected using only straight lines and lengths (“ruler and compass”).

11. If a set is not closed under an operation, can the operation be associative? Revisit the question of the independence of the field axioms.

12. The integers do not form a field but a “ring”: which field axiom must be omitted to describe integers? Are the real numbers a ring? Are there more fields than rings or vice-versa?

13. Look up Leonhard Euler, 1707–83. What are Euler’s identity and Euler’s number? How did he come across them?

14. Look up Johann Carl Friedrich Gauss, 1777–1855, and his doctoral thesis on the fundamental theorem of algebra [Gau99]. How does he use two-dimensional numbers? Does he refer to $\sqrt{-1}$? Given the completeness of 2-numbers established by Gauss' proof, how likely is it that there are "3-numbers", with the same properties (i.e., satisfying the field axioms) and describing three dimensions? What property(ies) should be changed to describe 3D?
What mathematical progression did Gauss discover at the age of 10?

15. Expand the terms of the expressions describing the two closed figures of Notes 10 and 11,

$$4e^{i0} + 4e^{i\pi/2} + 2e^{i0} + 4e^{-i\pi/2} + 4e^{i0} + 8e^{i\pi/2} + 5\sqrt{2}e^{i3\pi/4} + 5\sqrt{2}e^{i5\pi/4} + 8e^{i3\pi/2}$$

and

$$e^{i4\pi/5} + e^{i8\pi/5} + e^{i12\pi/5} + e^{i16\pi/5} + e^{i20\pi/5}$$

and show that they each sum to zero in both dimensions.

16. Any part of the lecture that needs working through.

References

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[Saw59] W. W. Sawyer. *A Concrete Approach to Abstract Algebra.* W. H. Freeman & Co., San Francisco, 1959.