Discrete Quantum Causal Evolution

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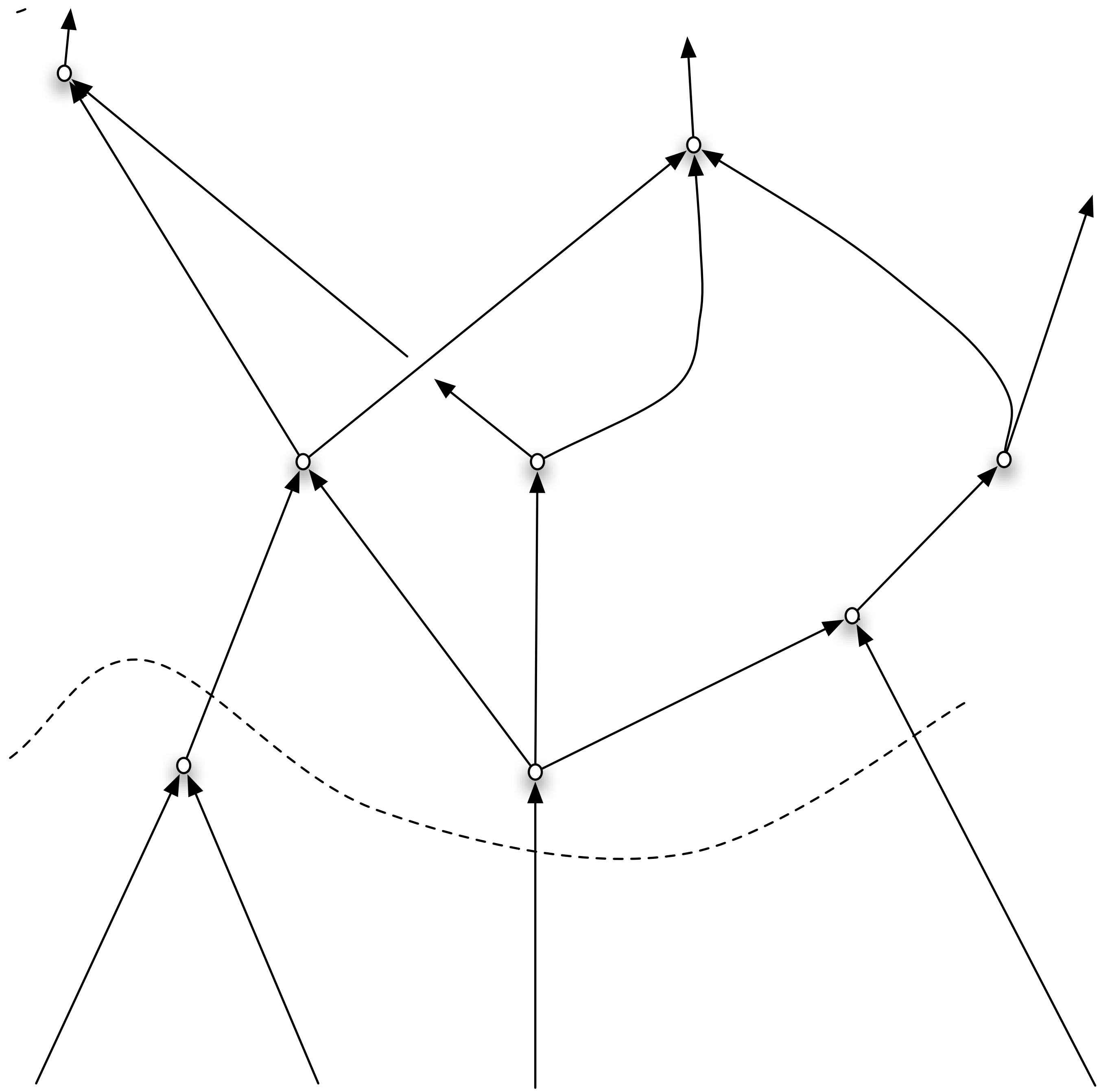
- Joint work with Richard Blute and Ivan T. Ivanov
- Ongoing work with Blute, Ivanov, Alessio Guglielmi and Lutz Strassburger
- Motivated by an early paper by Fotini Markopoulou on causal evolution
- Also motivated by desire to modify consistent histories to work on causal structures rather than sequences

The Challenge

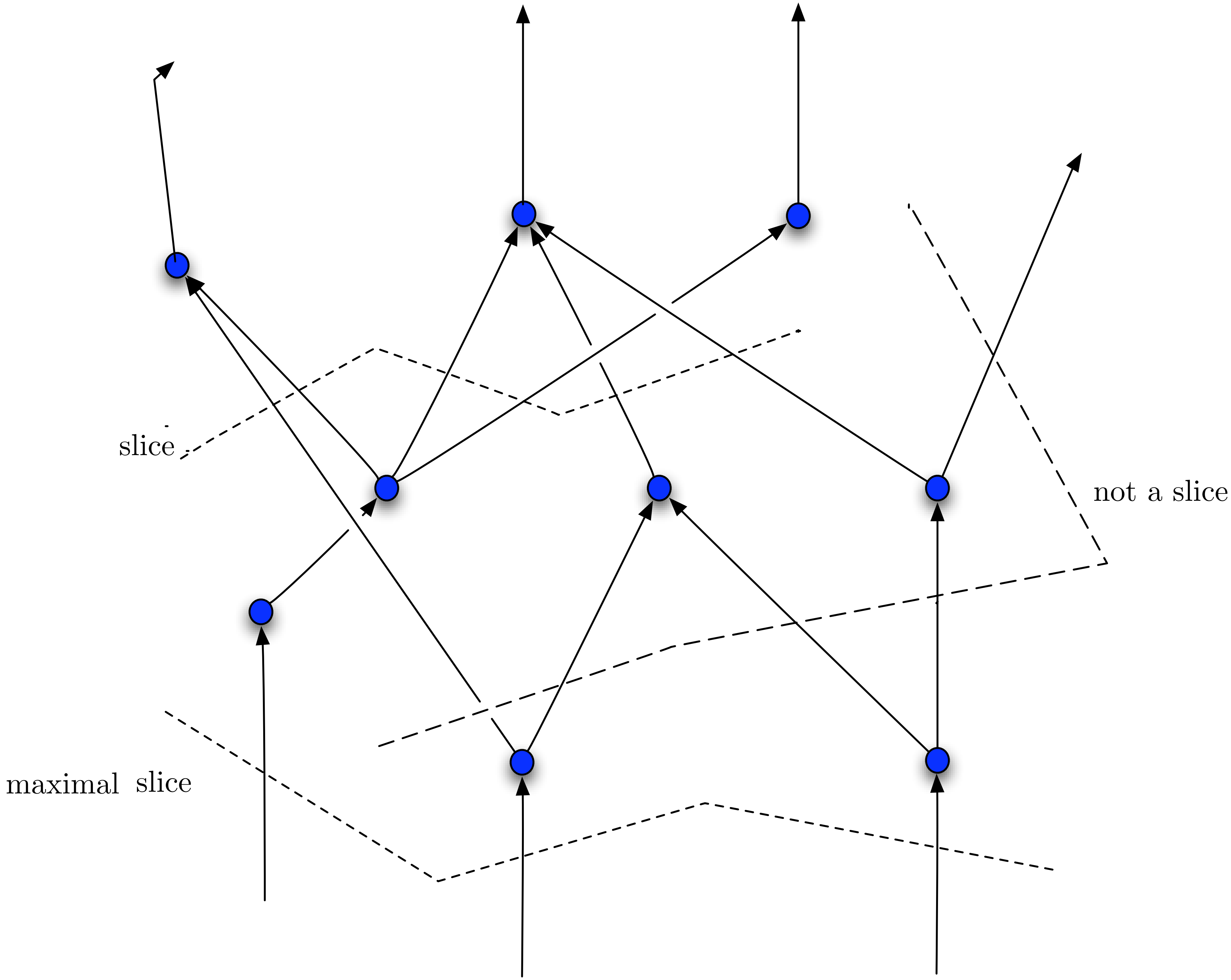
- Causality is limited to the light cone: no superluminal communication possible
- state is not local: entanglement is possible.
- How do we guarantee causality automatically while allowing non-locality?

Causal Structure



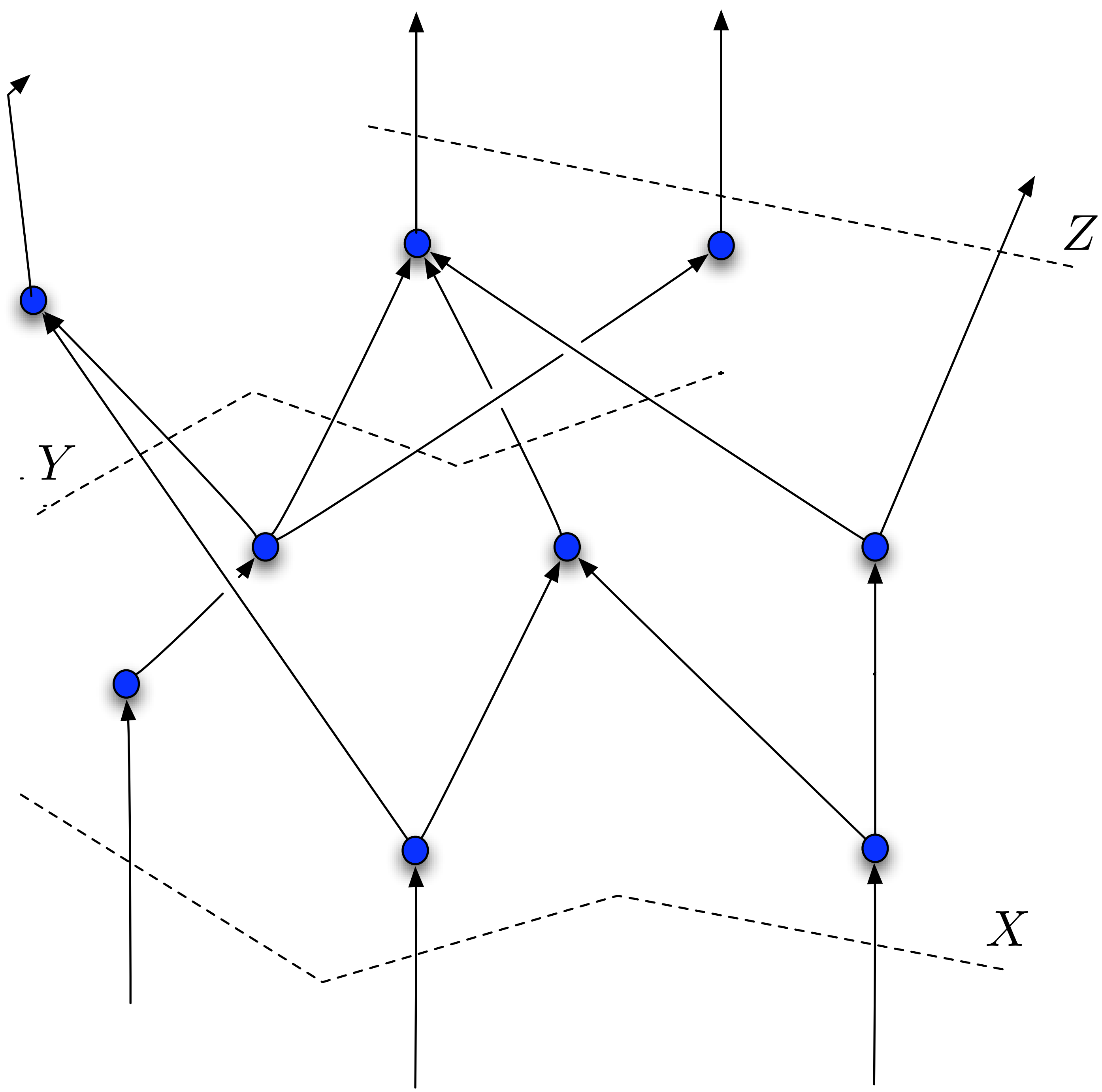
A partial order between events

Slices in Discrete Spacetime



Slices are sets of edges.

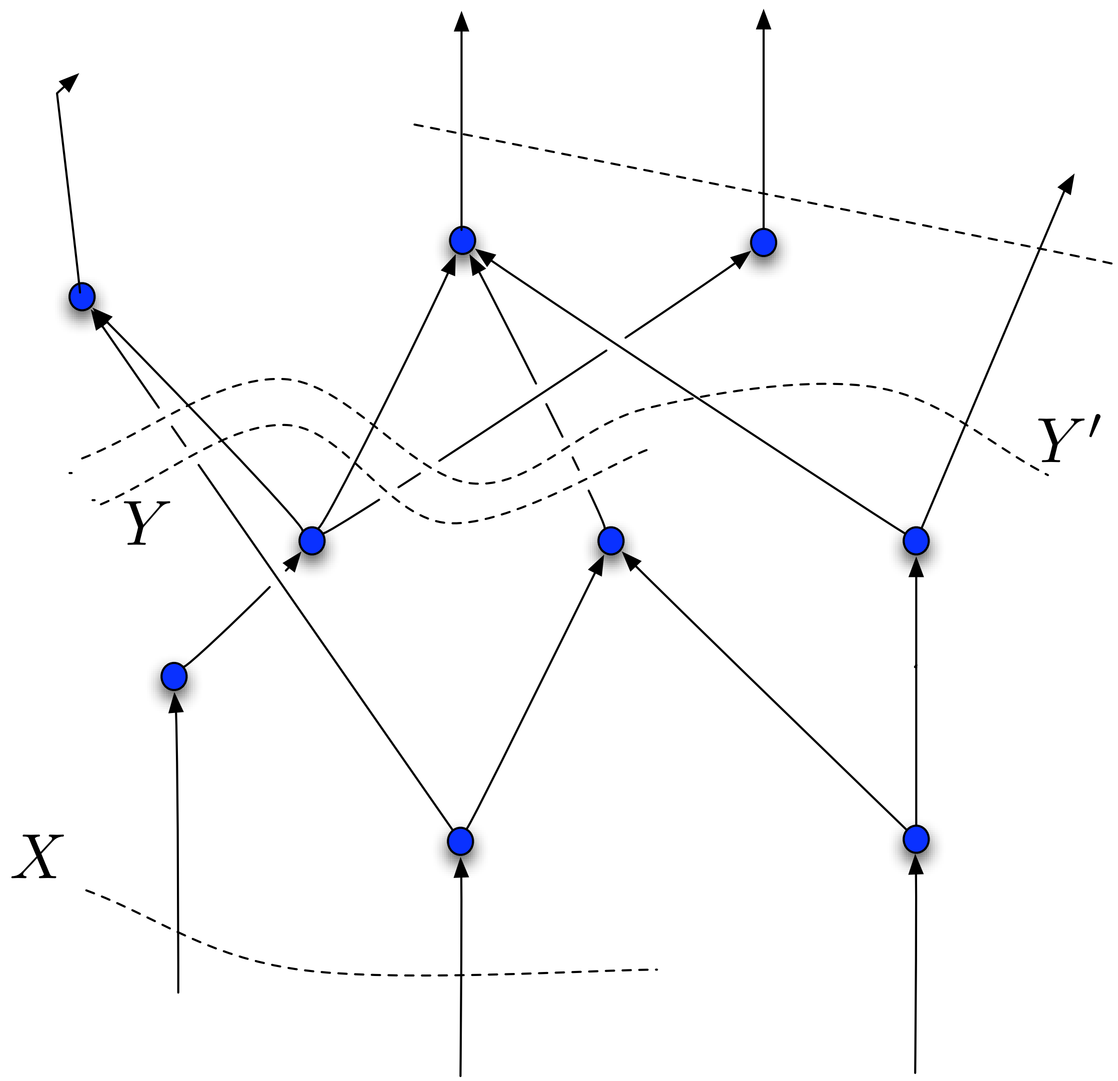
One can order antichains: $X \preceq Y$ means $\forall x \in X \exists y \in Y \;\; x \leq y$.



Here $X \preceq Y, Z$ but not $Y \preceq Z$.

Another order on antichains

$$X \sqsubseteq Y \text{ if } X \preceq Y \text{ and } \forall y \in Y\ \exists x \in X\ \ x \leq y$$

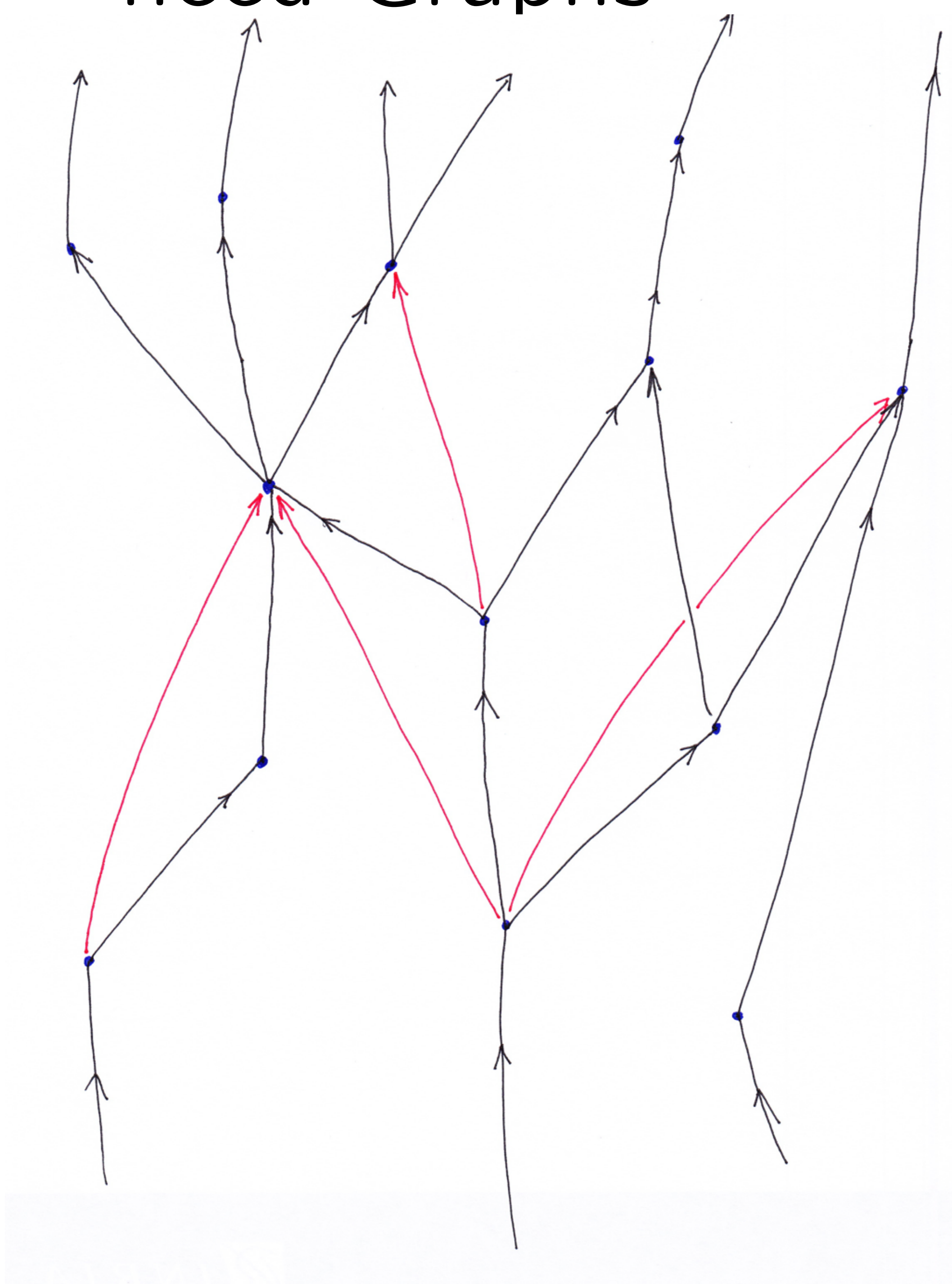


also $X \preceq Y'$
but not $X \sqsubseteq Y'$

Here $X \preceq Y$
and $X \sqsubseteq Y$

This is called the Egli-Milner order.

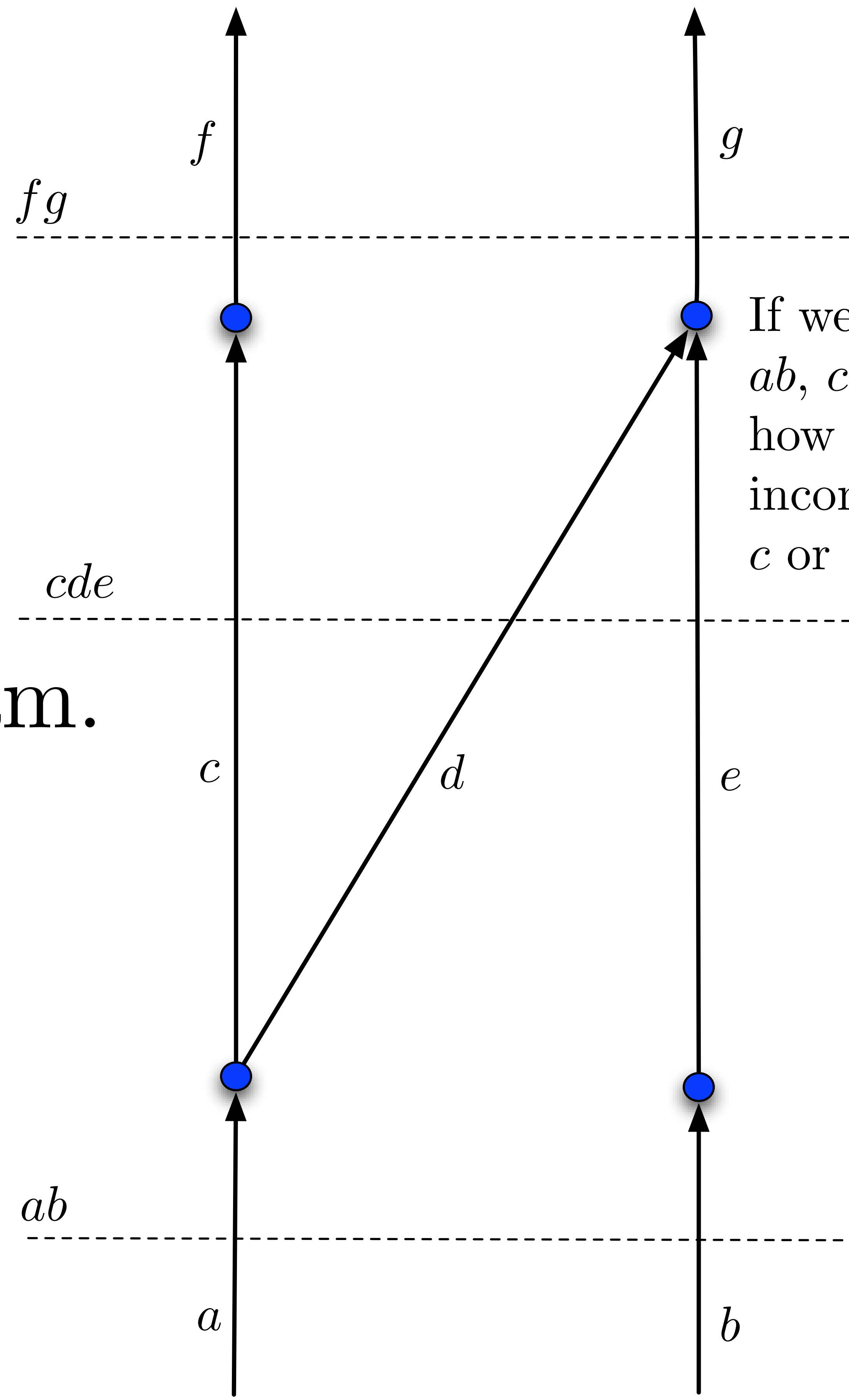
Posets are not enough: we need Graphs



Here the red lines would not be necessary if we were just talking about posets.

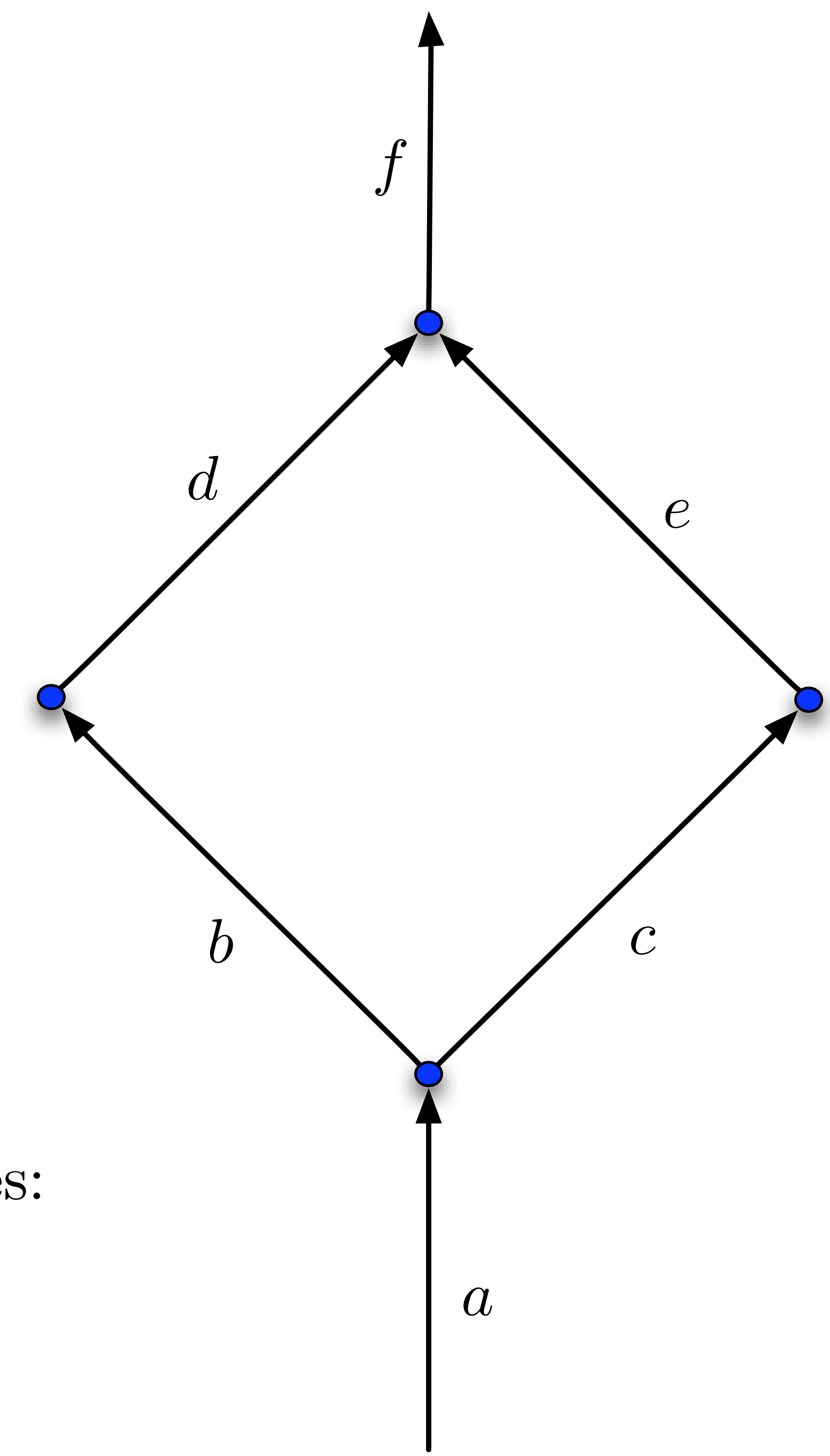
We need to keep track of "how effects propagate".

The N diagram.



If we propagate along global slices: ab, cde and fg, how do we ensure that the incoming b does not influence c or f?

The Diamond picture



If we propagate along local slices:
a, b, c, d, e, f,
how do we ensure that the
correlations between b and c
is not lost?

Evolution of a Quantum System

Evolution occurs at vertices, "observers" sit at edges and "see" the local subsystem as described by a density matrix. The observers are only a figure of speech, they do not interact with the system.

At each vertex

- **either** one has ordinary quantum evolution,
- **or** there is an *interaction* with
  - **either** another quantum system,
  - **or** a classical system (measurement),
- **or** the system breaks up into subsystems that fly apart.

What happens at vertices?

- A purely quantum evolution is described by a unitary operator U acting on ρ by $U^\dagger \rho U$.

- A measurement is described by a projection operator (actually by a POVM).

- A system breaking up is described by tracing.

- A number of *independent* systems coming together is described by tensoring.

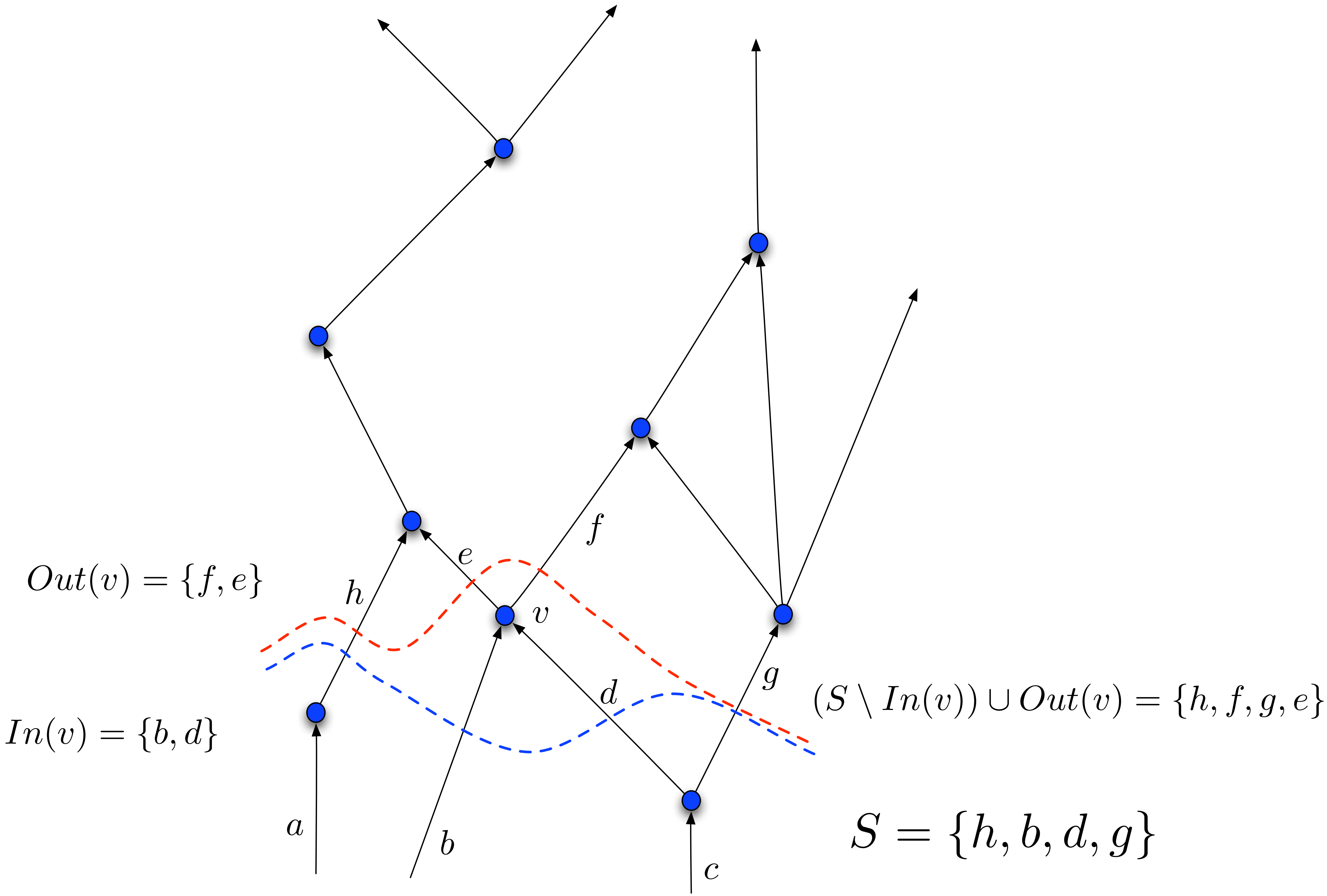
Locative Slices 1

Fix any subset of incoming edges. These always form a slice.

Suppose S is a slice and v is a vertex such that all the incoming edges of v are in S. Then

$$(S \setminus \mathsf{In}(v)) \cup \mathsf{Out}(v)$$

is always a slice. It is the slice obtained by *propagating* S through v.


$Out(v) = \{f, e\}$
$In(v) = \{b, d\}$
$(S \setminus In(v)) \cup Out(v) = \{h, f, g, e\}$
$S = \{h, b, d, g\}$
e
f
h
v
g
d
a
b
c

Locative Slices 2

Def: A **locative** slice is defined by induction.

- Any subset of the incoming edges of the graph forms a locative slice.

- If S is a locative slice and v is a vertex with $\mathsf{In}(v) \subset S$ then the slice obtained by propagating S through v is locative.

Intuition: If S is locative then the density matrix on S can be computed without ever computing partial traces: no information is lost.

POVMs

Measurements are described by positive operator-valued measures - the usual projective measurements are a special case. Outcomes labelled by $\mu \in \{i, \ldots, N\}$, to every outcome we have an operator F_μ. The transformation of the density matrix is

$$\rho' = \frac{1}{\kappa_\mu} F_\mu \rho F_\mu^\dagger.$$

Let $E_\mu := F_\mu^\dagger F_\mu$; these are positive operators. For a measurement they satisfy $\sum_\mu E_\mu = I$ and the probability of observing outcome μ is $Tr(E_\mu \rho) = \kappa_{mu}$. Henceforth we write p_μ rather than κ_μ.

Intervention Operators

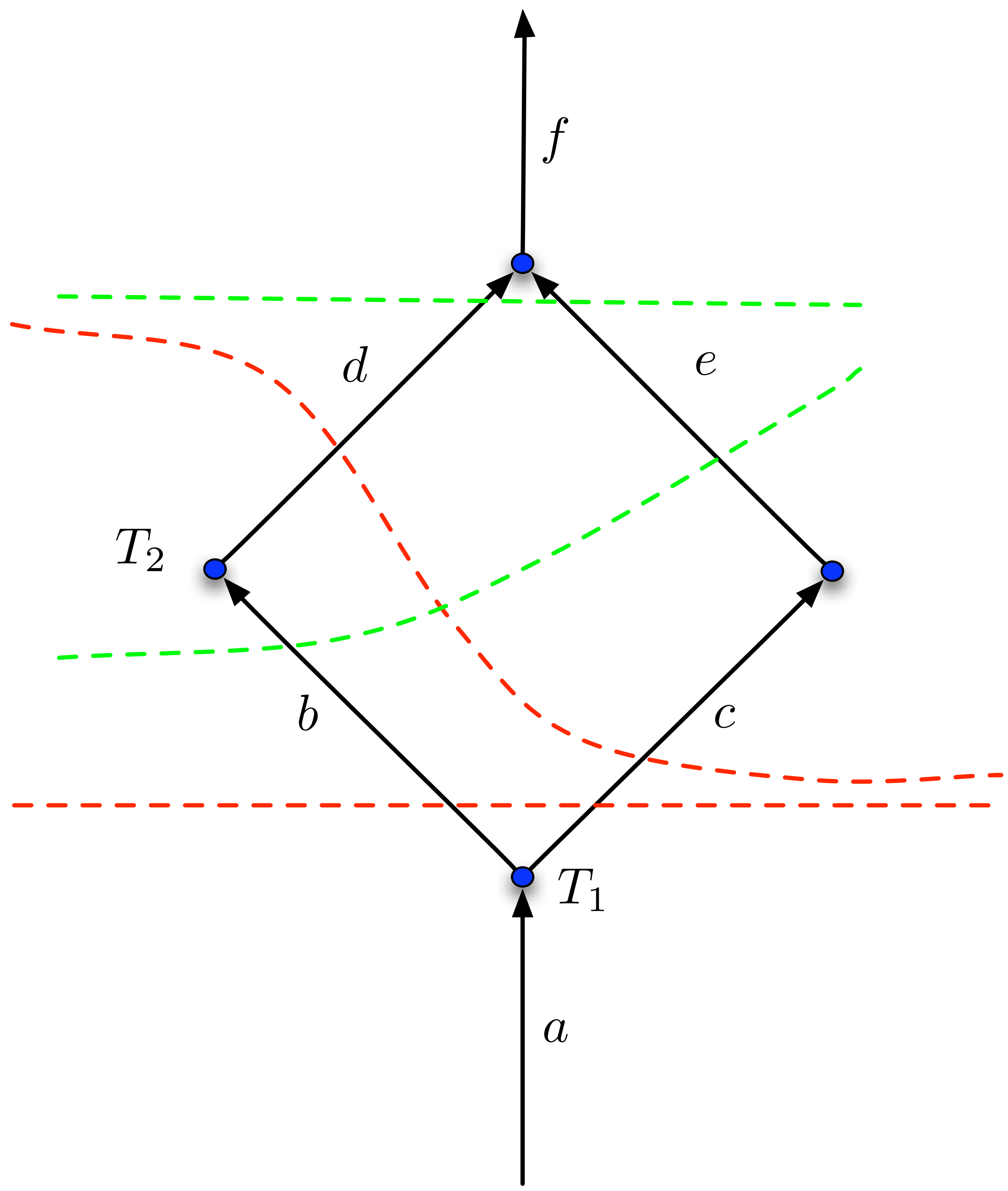
More general interaction: part of the quantum system gets discarded during the measurement. The transformation of the density matrix is given by:

$$\rho'_\mu = \frac{1}{p_\mu} \sum_m A_{\mu m} \rho A^\dagger_{\mu m}$$

where μ labels the degrees of freedom observed, m labels the degrees of freedom discarded and each $A_{\mu m}$ now maps between two Hilbert spaces of (perhaps) different dimensionality.

Propagating Density Matrices on Locative Slices

Each edge - more generally, each slice - has a density matrix. **In a given family** of slices each vertex has an intervention operator.


f
d
e
T_2
b
c
T_1
a

Propagating through T_1 gives:

$$T_1 : \mathcal{DM}(\mathcal{H}_a) \rightarrow \mathcal{DM}(\mathcal{H}_b \otimes \mathcal{H}_c)$$

Propagating through T_2 using the red slice gives:

$$T_2 : \mathcal{DM}(\mathcal{H}_b \otimes \mathcal{H}_c) \rightarrow \mathcal{DM}(\mathcal{H}_d \otimes \mathcal{H}_c)$$

Propagating through T_2 using the green slice gives:

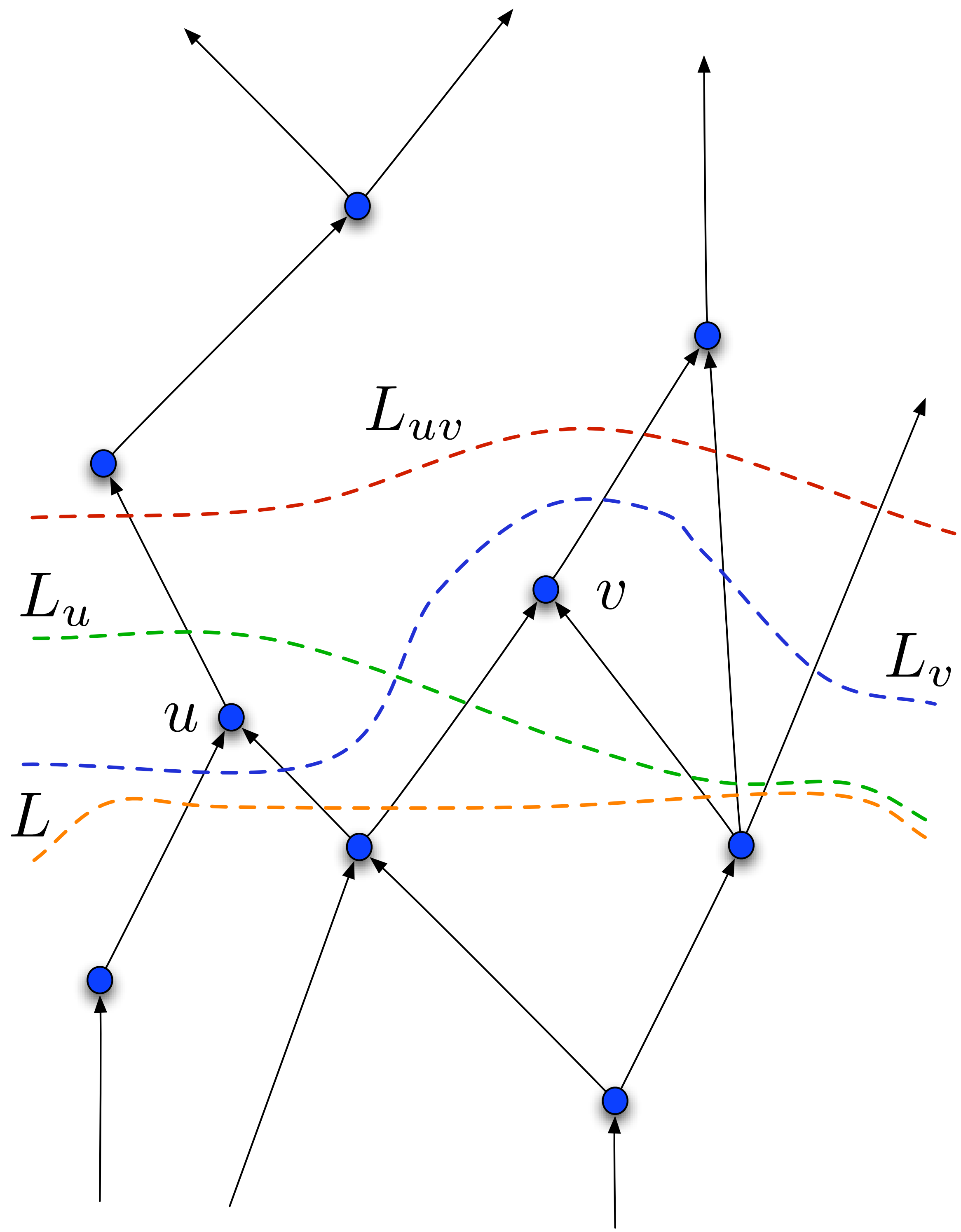
$$T_2 : \mathcal{DM}(\mathcal{H}_b \otimes \mathcal{H}_e) \rightarrow \mathcal{DM}(\mathcal{H}_d \otimes \mathcal{H}_e)$$

Each version of T_2 is padded out with the appropriate identity operators, the "real" action of the of T_2 is to transform the b-piece of the density matrix into the d-piece.

Our Proposal - Summary

- Work with dags not just posets
- Density matrices on edges
- Propagation (interventions) at vertices
- Keep track of density matrices on "special" (locative) slices
- Evolve along locative slices
- Compute the density matrix for an edge by first computing the density matrix on the **minimal** locative slice containing that edge, then take the appropriate partial traces.

Slicing Independence



Intervention operators at spacelike related vertices commute

Slicing independence

Suppose L is a locative slice and u and v are two minimal vertices above L. Clearly u and v are acausal with respect to each other so the intervention operators commute. Thus we can go $L \longrightarrow L_u \longrightarrow L_{uv}$ or $L \longrightarrow L_v \longrightarrow L_{uv}$. Clearly L_u, L_v and L_{uv} are all locative and the density matrix on L_{uv} will be the same calculated either way. We can piece together such "diamonds" inductively.

Evolution: Proposal 1

To obtain the density matrix on an edge e (or any slice S): evolve along locative slices up to the (unique) minimal locative slice containing e (S) then project down to e (S) using partial traces.

We know that this is independent of the slicing.

Minimality captures causality.

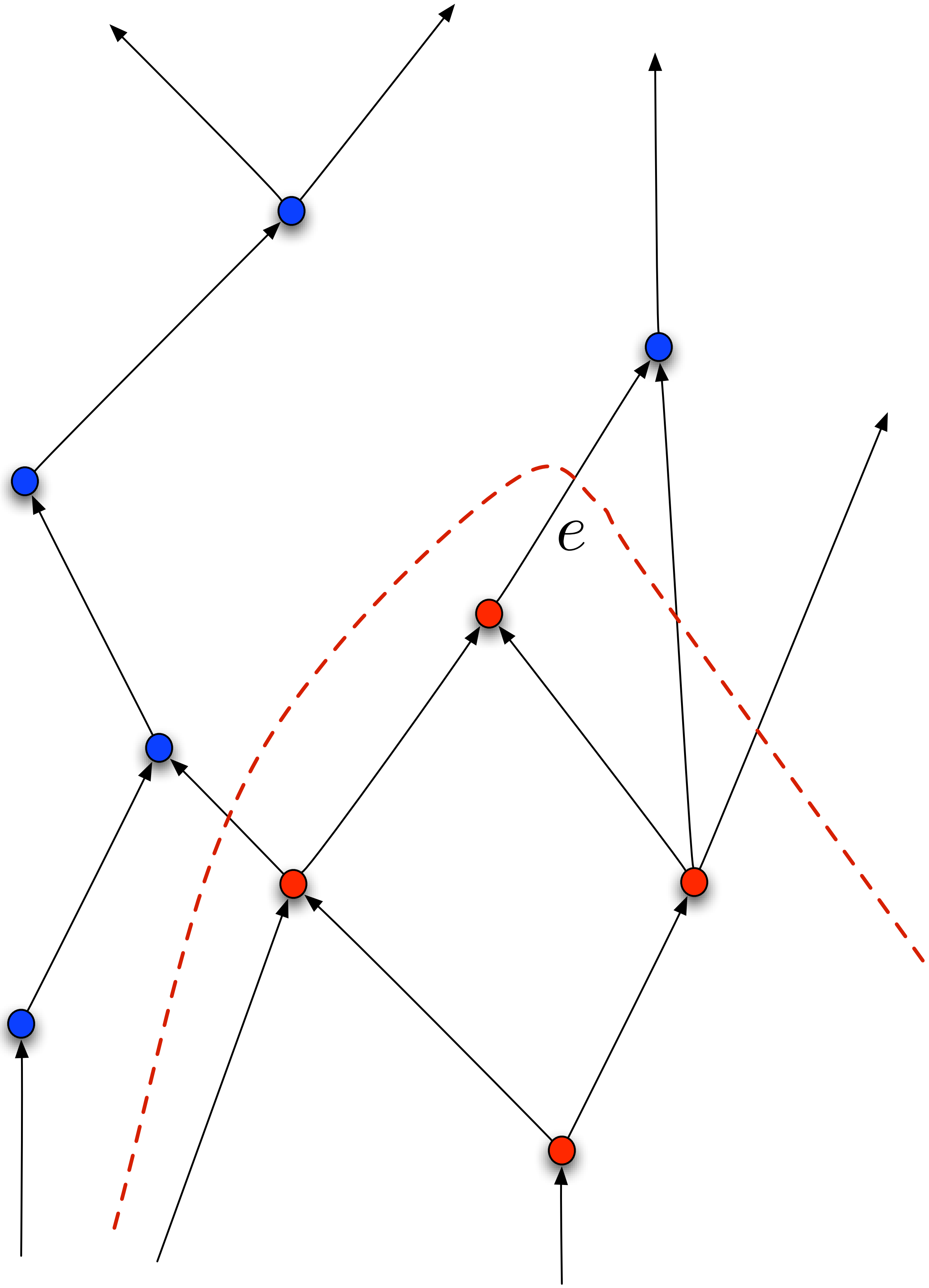
Evolution: Proposal 2

Evolve – using a different rule – along locative slices up to *any* locative slice containing e then project using partial traces.

We have to do different things according as whether an event is to the past of an edge or not.

Causality is built into the evolution prescription.

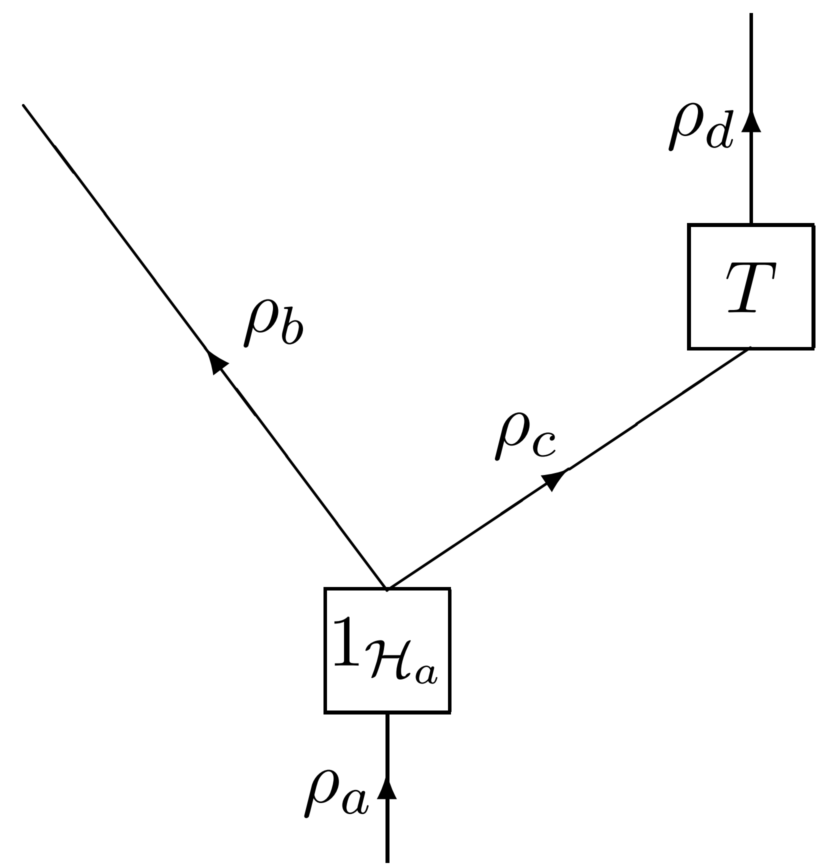
Why causality holds with proposal 1.



Only vertices to the past of e will be covered by the minimal locative slice.

And we get all of them.

A simple scenario



At T we measure the spin of the second sub-system and find it to be up.

Let $\rho_a = |\psi_a\rangle\langle\psi_a|$ where

$$\psi_a = 1/\sqrt{2}\ (\psi_1^{\uparrow} \otimes \psi_2^{\uparrow} + \psi_1^{\downarrow} \otimes \psi_2^{\downarrow}).$$

Now $\rho_{bc} = \rho_a$. Since bc is the least locative slice for b we have

$$\rho_b = Tr^c \rho_{bc} = 1/2\ (|\psi_1^{\uparrow}\rangle\langle\psi_1^{\uparrow}| + |\psi_1^{\downarrow}\rangle\langle\psi_1^{\downarrow}|).$$

At T we measure the spin of the second subsystem and find it to be up.

$$T(\rho) = 2P_2^{\uparrow} \rho P_2^{\uparrow}.$$

Thus

$$\rho_{bd} = T(\rho_{bc}) = (|\psi_1^{\uparrow}\rangle \otimes\ |\psi_2^{\uparrow}\rangle)(\langle\psi_1^{\uparrow}|\ \otimes \langle\psi_2^{\uparrow}|).$$

Now bd is **not** the minimal locative slice for b if we attempt

$$\rho_b = Tr^d(\rho_{bd}) = |\psi_1^\uparrow\rangle\langle\psi_1^\uparrow|$$

which is incorrect. We need to sum over all possible outcomes since b is causally independent of the intervention T and cannot be influenced by the outcome.

$$\tilde{\rho}_{bd} = \tilde{T}(\rho_{bc}) = \sum_{s=\uparrow,\downarrow} P_2^s \rho_{bc} P_2^s$$

now if we trace over the degrees of freedom at d we get the right answer.

Why causality holds 2

Proposal 2: The density matrix is computed using variants of the intervention operators depending on causal relations. Want to compute a density matrix for an edge e from an arbitrary locative slice L - not necessarily the minimal one - containing e. Compute $\tilde{\rho}_L$ using:

- $\rho \mapsto \frac{1}{p_\mu} A_\mu \rho A_\mu^\dagger$
  if the vertex is to the causal past of e

- $\rho \mapsto \frac{1}{p_\mu} \sum_\mu A_\mu \rho A_\mu^\dagger$
  if the vertex is not to the causal past of e.

Now the causality is explicit in the evolution prescription.

The two proposals give the same result

Proof Idea: The vertices that are not to the past of e can be systematically "peeled off" by first using commutativity to move them outermost and then using the cyclic property of trace to rewrite

$$Tr(\sum_{\mu} A_{\mu}\rho A_{\mu}^{\dagger})$$

as

$$Tr(\sum_{\mu} A_{\mu}^{\dagger} A_{\mu}\rho)$$

and then using the identity for POVMs

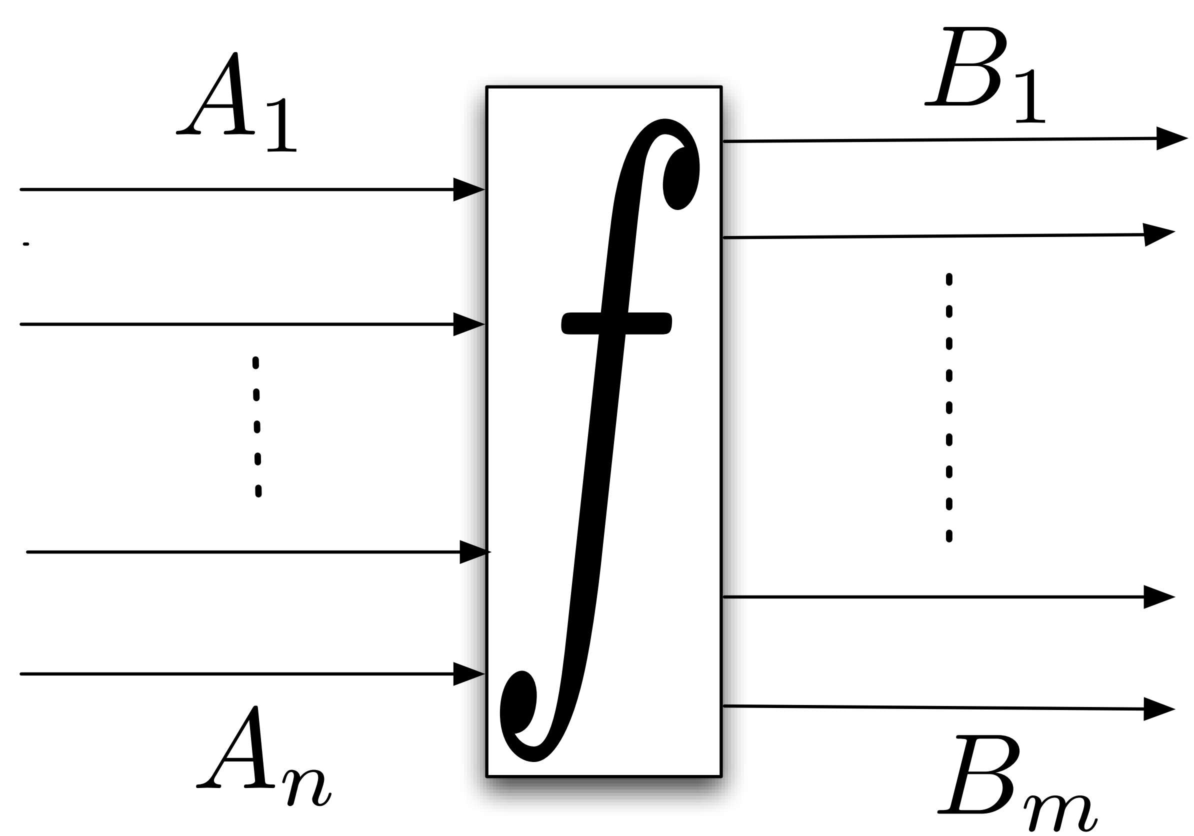
$$\sum_{\mu} A_{\mu}^{\dagger} A_{\mu} = I.$$

Peeling off successively we reduce from L to M.

Polycategories

Morphisms may connect many objects:

$$f \colon A_1, A_2, \ldots, A_n \longrightarrow B_1, B_2, \ldots, B_m$$



Any monoidal category can be viewed as a polycategory:

$$f\colon A_1 \otimes A_2 \otimes \ldots \otimes A_n \longrightarrow B_1 \otimes B_2 \otimes \ldots \otimes B_m$$

but the polycategory view emphasizes the relationship to (classical) deductive systems.

However, this is a degenerate example. In polycats the comma on the right of the arrow may be a "par" $\invamp$ as in linear logic.

Polycategories generated by DAGs

Each edge is an object.
Each vertex is a (poly)morphism.

Given a finite dag G. The *free polycategory generated by* G, denoted $P(G)$, is defined as follows: given vertex v has incoming edges $A_1, A_2, \ldots, A_n$ and outgoing edges $B_1, B_2, \ldots, B_m$ then the polycategory will have a polymorphism of the form $f_v \colon A_1, A_2, \ldots, A_n \longrightarrow B_1, B_2, \ldots, B_m$.

One imposes closure under composition, existence of identities and other algebraic conditions.

Polycats of Interventions

The usual category of Hilbert spaces is monoidal and hence defines a polycategory *but this is not the one that we use.*

In our category called Conj: Objects are finite-dimensional Hilbert spaces. A morphism from $\mathcal{H}_1$ to $\mathcal{H}_2$ is a finite family of maps $\{A_i\}_{i\in I}$ of linear maps $A_i\colon \mathcal{H}_1 \longrightarrow \mathcal{H}_2$.

Composition is then described as follows. If we have the following pair of maps:

$$\mathcal{H}_1 \xrightarrow{\{A_i\}_{i\in I}} \mathcal{H}_2 \xrightarrow{\{B_j\}_{j\in J}} \mathcal{H}_3$$

then the composite is:

$$\mathcal{H}_1 \xrightarrow{\{B_j \circ A_i\}_{\langle i,j\rangle \in I\times J}} \mathcal{H}_3$$

The objects are actually just labeled by the Hilbert spaces; they are really the set of **density matrices** on the Hilbert space.

A morphism in **Conj** acts as follows:

$$\rho \mapsto \sum_m A_m \rho A_m^\dagger.$$

We restrict the class of morphisms so that the conjugation is trace preserving. Thus, we have **superoperators**. We call this category **Supops**. We write $\mathcal{P}(\mathbf{Supops})$ for the associated polycategory.

Dynamics is a Polyfunctor

The evolution of density matrices is given by a rule for calculating them on a locative slice given the density matrix on earlier locative slices. A polyfunctor from the polycategory generated by the DAG to $\mathcal{P}(\mathbf{Supops})$ gives exactly such a correspondence.

Polyfunctors are – in essence – monoidal and thus polyfunctoriality states that one has slicing invariance.

Conclusions

Locativity captures exactly the slices needed to guarantee causal evolution

There is a way of presenting all this as a deductive system in a logic called BV.

Edges are atomic propositions.

Slices are formulas

Vertices describe inferences

Locative slices are deducible formulas

Connectives express whether the edges are entangled or independent.

Subtleties arise with induced correlations.

We have not dealt with beam-splitting experiments. (Ben Sprott is working with me on this now).