Equational reasoning for probabilistic programming
Part II: Quantitative equational logic

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Probabilistic Programming Languages 29th May - 2nd June 2017

The basic idea

- Approximate equations: $s =_\varepsilon t$, s is within ε of t.
- Definitely not an equivalence relation;
- it defines a *uniformity* (but we won't stress this point of view).
- Quantitative analogue of equational reasoning.
- completeness results, universality of free algebras, Birkhoff-like variety theorem, monads ....

History

- Moggi 1988: How to incorporate “effects” into denotational semantics?
- (Strong) Monads!
- Plotkin, Power (and then many others): view effects algebraically. Monads are given by operations and equations.
- Categorically: equational presentations are Lawvere theories (but we won’t talk about them here either).
- A monad of great interest: Lawvere (1964) The category of probabilistic mappings.
- Giry (1981): monad on measure spaces and also on Polish spaces.

Metrics

- Probabilistic reasoning requires measure theory but,
- measure theory works best on Polish spaces (topological space underlying separable complete metric spaces).
- Metric ideas present in semantics from the start: Jaco de Bakker's school.
- Mardare, P., Plotkin (2016): Develop the theory of effects in a metric setting (motivated by probability).
- Algebras will come with metric structure and quantitative equational theories will define monads on $\mathbf{Met}$.

Quantitative equations

- Signature Ω, variables X we get terms $\mathbb{T}X$.
- Quantitative equations: $\mathcal{V}(\mathbb{T}X)$:

$$s =_\varepsilon t, \quad s, t \in \mathbb{T}X, \quad \varepsilon \in \mathbb{Q} \cap [0, 1]$$

- A substitution σ is a map $X \longrightarrow \mathbb{T}X$; we write $\Sigma(X)$ for the set of substitutions.
- Any σ extends to a map $\mathbb{T}X \longrightarrow \mathbb{T}X$.
- Quantitative inferences: $\mathcal{E}(\mathbb{T}X) = \mathcal{P}_{\text{fin}}(\mathcal{V}(\mathbb{T}X)) \times \mathcal{V}(\mathbb{T}X)$

$$\{s_1 =_{\varepsilon_1} t_1, \ldots, s_n =_{\varepsilon_n} t_n\} \vdash s =_\varepsilon t$$

Deducibility relations

- (Refl) $\emptyset \vdash t =_0 t$
- (Symm) $\{t =_\varepsilon s\} \vdash s =_\varepsilon t$.
- (Triang) $\{t =_\varepsilon s, s =_{\varepsilon'} u\} \vdash t =_{\varepsilon+\varepsilon'} u$.
- (Max) For $e' > 0$, $\{t =_\varepsilon s\} \vdash t =_{\varepsilon+\varepsilon'} s$.
- (Arch) For all $\varepsilon \geq 0$, $\{t =_{\varepsilon'} s \mid \varepsilon' > \varepsilon\} \vdash t =_\varepsilon s$. Infinitary!
- (NExp) For $f : n \in \Omega$,
  $\{t_1 =_\varepsilon s_1, \ldots, t_n =_\varepsilon s_n\} \vdash f(t_1, ..t_i, ..t_n) =_\varepsilon f(s_1, ..s_i, ..s_n)$
- (Subst) If $\sigma \in \Sigma(X)$, $\Gamma \vdash t =_\varepsilon s$ implies $\sigma(\Gamma) \vdash \sigma(t) =_\varepsilon \sigma(s)$.
- (Cut) If $\Gamma \vdash \phi$ for all $\phi \in \Gamma'$ and $\Gamma' \vdash \psi$, then $\Gamma \vdash \psi$.
- (Assumpt) If $\phi \in \Gamma$, then $\Gamma \vdash \phi$.

Quantitative equational theories

- Given $S \subset \mathcal{E}(\mathbb{T}X)$, $\vdash_S$: smallest deducibility relation containing S.
- Equational theory: $\mathcal{U} = \ \vdash_S \bigcap \mathcal{E}(\mathbb{T}X)$.

Quantitative algebras

- Ω: signature; $\mathcal{A} = (A, d)$,
  A an Ω-algebra and (A, d) a metric space.
- All functions in Ω are nonexpansive.
- Morphisms are Ω-algebra homomorphisms that are nonexpansive.
- $\mathbb{T}X$ is an Ω-algebra. $\sigma : \mathbb{T}X \longrightarrow A$, Ω-homomorphism.
- (A, d) **satisfies** $\{s_i =_{\varepsilon_i} t_i / i = 1, \ldots, n\} \vdash s =_\varepsilon t$ if

$$\forall \sigma,\ d(\sigma(s_i), \sigma(t_i)) \leq \varepsilon_i,\ i = 1, \ldots, n$$

implies

$$d(\sigma(s), \sigma(t)) \leq \varepsilon.$$

- We write $\{s_i =_{\varepsilon_i} t_i / i = 1, \ldots, n\} \models_{\mathcal{A}} s =_\varepsilon t$.
- We write $\mathbb{K}(\mathcal{U}, \Omega)$ for the algebras satisfying $\mathcal{U}$.

A metric on $\mathbb{T}X$

$$d^{\mathcal{U}}(s,t) = \inf\{\varepsilon \mid \emptyset \vdash s =_\varepsilon t \in \mathcal{U}\}$$

- Why not use the following?

$$d^{\mathcal{U}}(s,t) = \inf\{\varepsilon \mid \forall V \in \mathcal{P}_f(\mathcal{V}(X)), V \vdash s =_\varepsilon t \in \mathcal{U}\}$$

- They are the same!
- The (pseudo)metric can take on infinite values.
- The kernel is a congruence for Ω.
- If we take the quotient we get an (extended) metric space.
- The resulting algebra is in $\mathbb{K}(\Omega, \mathcal{U})$.
- We can do this for any set M of generators and produce a "free" quantitative algebra.

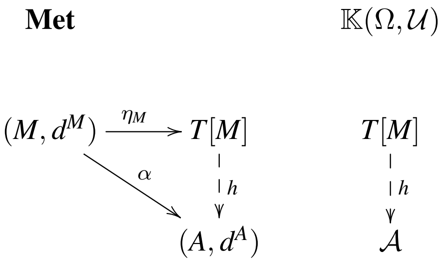
Completeness

$\forall \mathcal{A} \in \mathbb{K}(\mathcal{U}, \Omega)$, $\Gamma \models_{\mathcal{A}} \phi$ if and only if $[\Gamma \vdash \phi] \in \mathcal{U}$.

- Analogue of the usual completeness theorem for equational logic.
- Right to left is by definition.
- Left to right is by a model construction.
- The proof needs to deal with quantitative aspects and uses the archimedean property.

Free construction from a metric space

- Starting from a **metric space** (M, d) we can define $\mathbb{T}M$ by adding constants for each $m \in M$
- and axioms $\emptyset \vdash m =_e n$ for every rational e such that $d(m,n) \leq e$.
- Call this extended signature Ω_M and the extended theory $\mathcal{U}_M$.
- Any algebra in $\mathbb{K}(\mathcal{U}_M, \mathcal{U}_M)$ can be viewed as an algebra in $\mathbb{K}(\Omega, \mathcal{U})$ by forgetting about the interpretation of the constants from M.
- Given any $\alpha : M \longrightarrow A$ non-expansive we can turn $\mathcal{A} = (A, d)$ into an algebra in $\mathbb{K}(\Omega_M, \mathcal{U}_M)$ by interpreting each $m \in M$ as $\alpha(m) \in A$.



$\mathcal{U}_M$ is consistent if and only if the map η_M is an isometry.

We have a monad on **Met**.

Birkhoff Variety Theorems

- Three kinds of equations: (a) unconditional equations
- (b) basic equations : assumptions of the form $x =_\varepsilon y$, x, y variables.
- (c) Horn clauses, assumptions may involve terms.
- Usual variety theorem says: a class of algebras is equationally definable if and only it it is closed under products, homomorphic images and subalgebras.
- We have to consider a new kind of closure property.

Reflexive homomorphisms

- A $\mathfrak{c}$-reflexive homomorphism f between QA's $\mathcal{A}, \mathcal{B}$, where $\mathfrak{c}$ is a cardinal number, is a homomorphism with the property that for any subset $B' \subset B$ with $|B'| < \mathfrak{c}$, there is a subset $A' \subset A$ with $f(A') = B'$ and f restricted to A' is an *isometry*.
- If $\mathcal{U}$ is an unconditional theory then $\mathbb{K}(\Omega, \mathcal{U})$ is closed under homomorphic images.
- If $\mathcal{U}$ is a basic equational theory with every conditional equation having only finitely many assumptions then $\mathbb{K}(\Omega, \mathcal{U})$ is closed inder $\aleph_0$-reflexive homomorphisms.
- If $\mathcal{U}$ is a basic equational theory then $\mathbb{K}(\Omega, \mathcal{U})$ is closed inder $\aleph_1$-reflexive homomorphisms.
- A $\mathfrak{c}$-variety is a class of algebras closed under products, subalgebras and $\mathfrak{c}$-reflexive homomorphisms.
- A $\mathfrak{c}$-equational class is a class of algebras defined by $\mathfrak{c}$-basic conditional equations.

The main theorem

$\mathcal{K}$ is a $\mathfrak{c}$-variety if and only if it is a $\mathfrak{c}$-basic equational class.

- $\mathcal{K}$ is an unconditional equational class iff it is a variety.
- $\mathcal{K}$ is a finitary-basic equational class iff it is an $\aleph_0$-variety.
- $\mathcal{K}$ is a basic equational class iff it is an $\aleph_1$-variety.