Conformal Field Theory as a Nuclear Functor

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with

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Overview

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- Some things are not quite categories,
- but if they were, they would be compact closed: nuclear ideals.
- Conformal field theory is an example of a nuclear functor.

Why compact closure matters

Many mathematical objects have a notion of "dual" object, e.g. vector spaces.
There is a notion of "matrix" representation.
If we can freely move between "input" and "output" we have interesting "transpose" operations.

Typical examples: **Rel**, the category of sets and relations, **FDVect**($\mathbb{C}$), the category of finite-dimensional vector spaces over the complex numbers.

Relations can be turned around at will; we can decide what is "input" and "output." Abramsky exploited this in his theory of SProc, relations extended in time.

Relations

$$\exists x\ R(x, y; x, z)$$

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- Given a relation R(x,y;z,w) we can transpose at will: R(x;y,z,w) or R(x,y,z,w;) or R(x,y,z;w).

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- We can take "traces": R(x,y;w,z) becomes

$$\exists x\ R(x, y; x, z)$$

Vectors and Matrices

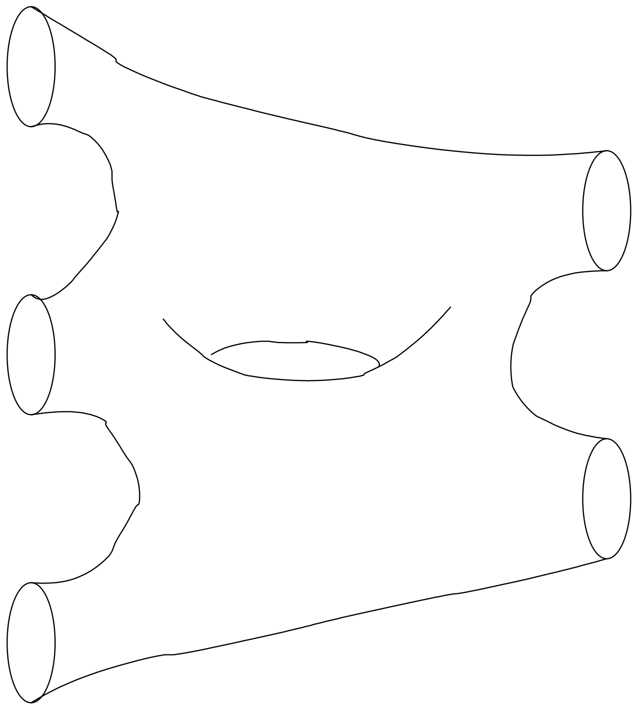
- We can certainly view linear maps as (higher-order) matrices.
- We can transpose at will: from
$$\lambda : V \otimes W^* \to X$$
- to
$$\lambda^t : V \to W \otimes X$$

We can take traces

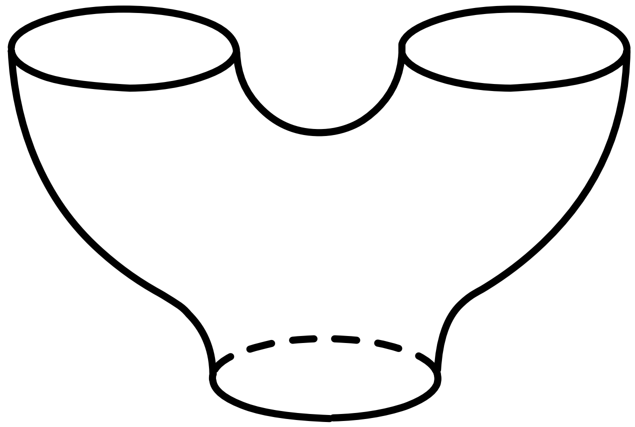
$$\lambda : U \otimes V \to U \otimes W$$

- becomes
- $$tr_U(\lambda) : V \to W$$

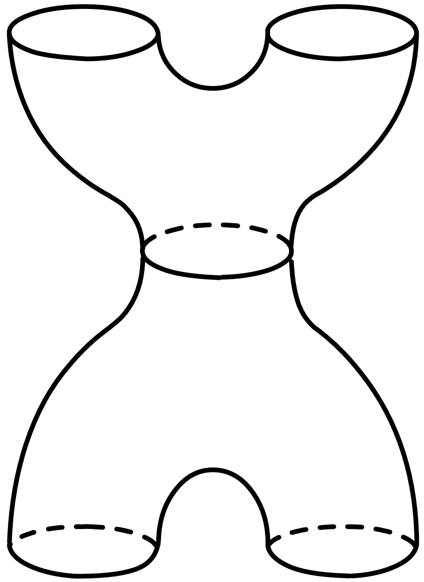
But there are other examples as well.



The category of Cobordisms. Objects are circles (1D compact manifolds), morphisms are 2 manifolds with boundary.



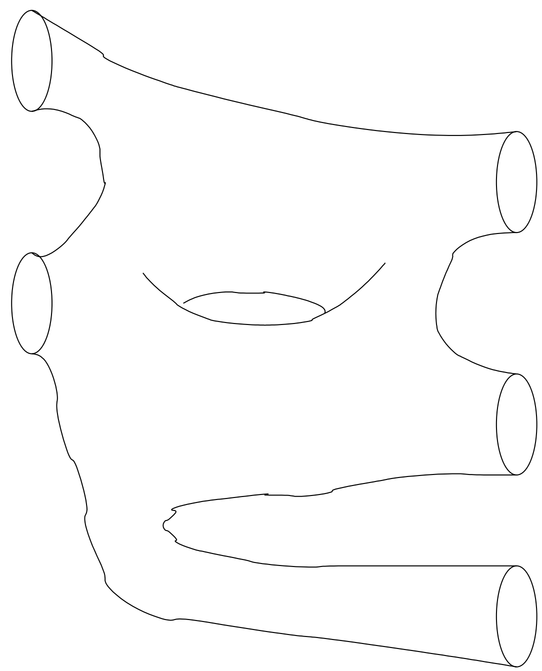
Composition



ig. 2. A composite of two cobordisms

We can deform at will. Thus, we are really looking at manifolds up to homotopy equivalence. A cylinder is the identity.

We can transpose!



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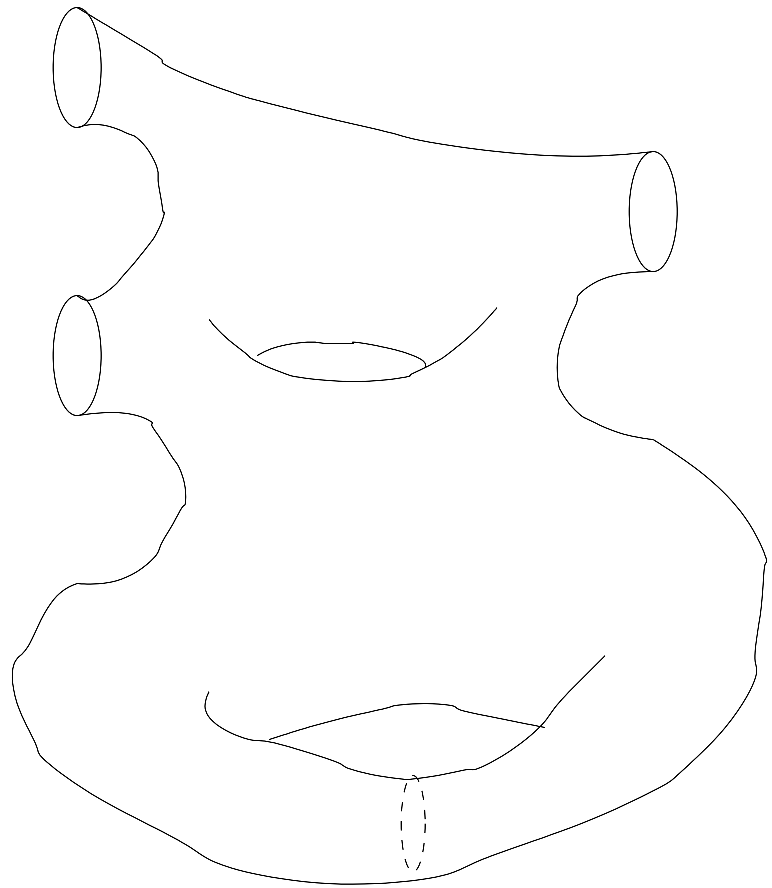

Closed Structure

A symmetric monoidal category is **closed** or **autonomous** if, for all objects A and B, there is an object $A \multimap B$ and an adjointness relation:

$$Hom(A \otimes B, C) \cong Hom(B, A \multimap C)$$

Compact Closure

A *compact closed category* is a symmetric monoidal category such that for each object A there exists a dual object A^*,
and canonical morphisms:

$$\begin{aligned} \nu &\colon I \to A \otimes A^* \\ \psi &\colon A^* \otimes A \to I \end{aligned}$$

such that the usual adjunction equations hold.

Examples: Rel, FDVect, FDHilb, Cob, SProc,...

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- Conjugation and transpose can be combined to give adjoints.
- Sometimes, the conjugation is trivial (Rel) but in QM it is absolutely vital.

Dagger Compact Categories

- Abramsky and Coecke [LICS 2004] introduced strongly compact closed categories to give a categorical axiomatization of QM.
- Selinger [2004] showed how to extend everything to mixed states and axiomatized adjointness as a “dagger” functor.

Dagger Categories

Definition 3.3 *A category $\mathcal{C}$ is a* dagger category *if it is equipped with a functor $(-)^\dagger\colon \mathcal{C}^{op} \to \mathcal{C}$, which is strictly involutive and the identity on objects. In such a category, a morphism f is* unitary *if it is an isomorphism and $f^{-1} = f^\dagger$. An endomorphism is* hermitian *if $f = f^\dagger$. A* symmetric monoidal dagger category *is one in which all of the structural morphisms in the definition of symmetric monoidal category [25] are unitary and dagger commutes with the tensor product.*

Definition 3.4 *A symmetric monoidal dagger category $\mathcal{C}$ is said to* have conjugation *if equipped with a covariant functor $(\ )^*\colon \mathcal{C} \to \mathcal{C}$ (called* conjugation*) which is strictly involutive and commutes with both the symmetric monoidal structure and the dagger operation. Since we have a covariant functor, we denote its action on arrows as follows:*

$$f\colon A \to B \longmapsto f_*\colon A^* \to B^*$$

This is in line with the notation of [33].

So in particular, our $*$-functor satisfies

$$(f_*)^\dagger = (f^\dagger)_*\colon B^* \to A^*$$

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- But they really want to be!

Hilbert-Schmidt Maps

If $f\colon \mathcal{H} \to \mathcal{K}$ is a bounded linear map,
we call f a *Hilbert-Schmidt map* if the sum
$\sum_{i \in I} ||f(e_i)||^2$
is finite for an orthonormal basis $\{e_i\}_{i \in I}$.
The sum is independent of the basis chosen.

Towards Nuclearity

One can easily verify that the Hilbert-Schmidt operators on a space form a 2-sided ideal in the set of all bounded linear operators. Furthermore, if $\mathsf{HSO}(\mathcal{H}, \mathcal{K})$ denotes the set of Hilbert-Schmidt maps from $\mathcal{H}$ to $\mathcal{K}$, then $\mathsf{HSO}(\mathcal{H}, \mathcal{K})$ is a Hilbert space, when endowed with an appropriate norm.

, there is a bijective correspondence:

$$\mathsf{HSO}(\mathcal{H}, \mathcal{K}) \cong Hom(I, \mathcal{H}^* \otimes \mathcal{K})$$

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- HS maps form a two-sided ideal and interact well with the monoidal structure.
- Why not make a compact-closed category out of the Hilbert-Schmidt maps?

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- The identity maps are not Hilbert-Schmidt unless the space is finite dimensional!
- They are too singular to be members of the putative category of Hilbert-Schmidt maps.

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- We show that the morphisms of interest form an ideal and have many of the properties of a dagger compact category.

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- There is a smaller ideal called the "trace class" maps which do have traces.
- Some nuclear maps are too singular to be traced.
- However, the composite of any two nuclear maps is always traced.

$$\begin{pmatrix} 1 & 0 & 0 & \dots \\ 0 & \frac{1}{2} & 0 & \dots \\ 0 & 0 & \frac{1}{3} & \dots \\ \dots & \dots & \dots & \dots \end{pmatrix}$$

This is Hilbert-Schmidt because $\sum_{i=1}^{\infty} i^2 < \infty$ but $\sum_{i=1}^{\infty} i = \infty$.

3.4 Examples

- The category **Rel** of sets and relations is a tensored $*$-category for which the entire category forms a nuclear ideal.
- The category of Hilbert spaces and bounded linear maps maps is a well-known tensored $*$-category, which, in fact, led to the axiomatization [10]. Then the Hilbert-Schmidt maps form a nuclear ideal [2]
- The category **DRel** of tame distributions on Euclidean space [2] is a tensored $*$-category. The ideal of test functions (viewed as distributions) is a nuclear ideal.
- We will define a subcategory of **Rel** called the category of *locally finite relations.* Let $R\colon A \to B$ be a binary relation and $a \in A$. Then $R_a = \{b \in B | aRb\}$. Define R_b similarly for $b \in B$. Then we say that a relation is *locally finite* if, for all $a \in A, b \in B$, R_a, R_b are finite sets.Then it is straightforward to verify that we have a tensored $*$-category which is no longer compact closed. It is also easy to verify that the finite relations form a nuclear ideal.

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- The identities are cylinders; nothing singular about them.

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- Note that everything is trace class in FDHilb.
- Think of this as zero-energy physics.

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- Interested in phenomena that are scale invariant. These arise in statistical mechanics especially in the study of phase transitions.

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- These are closely connected to complex analysis because these transformations are precisely the ones that leave the complex structure invariant.

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- The infinitesimal conformal transformations in 2D form an infinite dimensional Lie algebra (which physicists call the conformal group)
- which can be identified with the functions that leave the complex analytic structure invariant.

Complex Structures

- We need an abstract analogue of i.
- Given a vector space V (not necessarily finite dimensional) we define J: V --> V so that

$$J^2 = -I.$$

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Taking determinants:

$$(det(J))^2 = (-1)^n.$$

So n better be even.
Thus complex structures can only be defined on even-dimensional manifolds.

Riemann Surfaces

A Riemann surface is a topological space X with an open cover $\mathcal{U}$, together with homeomorphisms $\phi_i : U_i \longrightarrow \mathcal{O}$, where $\mathcal{O}$ is an open subset of $\mathbb{C}$ and on the overlap regions $U_i \cap U_j$ the composites (restricted appropriately) $\phi_i \circ \phi_j^{-1}$ are holomorphic.

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- Instead of using cobordisms, the morphisms are required to be manifolds admitting complex structures, Riemann surfaces.
- they can only be squashed by conformal transformations, i.e. transformations that preserve the complex structure.

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- In CFT the set of discs with different conformal structures itself has the structure of a complex manifold.

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- We cannot attach a cylinder and conformally squash it down to a circle. A circle has no complex structure!
- The thing that wants to be the identity is too "singular"!

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- Want to make Segal's "category" live inside a *-tensor category. This involves adding the circles in some principled way.
- There is a way of adding "singular" objects to the collection of curves (Mumford compactification) but this is more fancy than needed.

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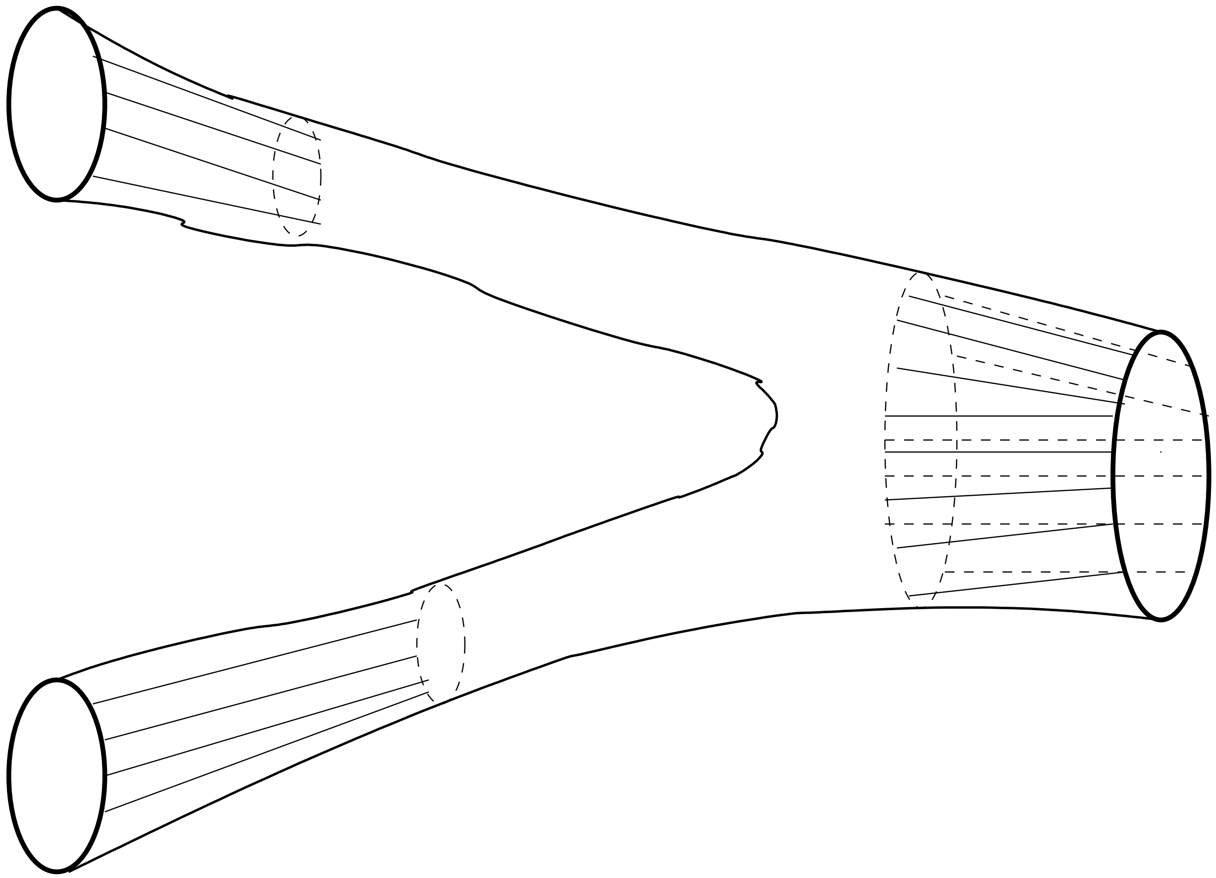
- Neretin defined a category by "collaring" the Riemann surfaces and then allowing the circles to show up as thin collars. This is a symmetric-monoidal dagger category called **Pants.**
- He defined a "volume" in such a way that Riemann surfaces had positive volume and the circles had zero volume.



Positive Volume Surface; collars do not intersect.

The collared regions are conformal images of the regions

$$D^+ = \{z : |z| \le 1\}$$

and

$$D^- = \{z : |z| \ge 1\}.$$

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- A morphism where the collars do not intersect is said to have "positive volume."

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- A morphism where the collars do not intersect is said to have "positive volume."
- The collection of positive volume morphisms forms a nuclear ideal in **Pants**.

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CFT Revisited

- A Conformal Field Theory is just a nuclear functor from **Pants** to **Hilb**.
- In this case it follows that the nuclear maps in **Pants** go to trace-class maps in **Hilb**.
- This gives Segal's definition.
- A generalized CFT is a nuclear functor from **Pants** to any category with a nuclear ideal.

Correct Linear Relations

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- The bulk of the paper is taken up by checking that this example really gives a generalized CFT.

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A linear relation P is called correct *if it is the graph of an operator*

$$\Omega_P\colon V_+ \oplus W_- \to V_- \oplus W_+$$

where the matrix

$$\Omega_P = \begin{pmatrix} K & L \\ L^t & M \end{pmatrix}$$

has the following properties:

(a) $K = -K^t$ *and* $M = -M^t$*;*

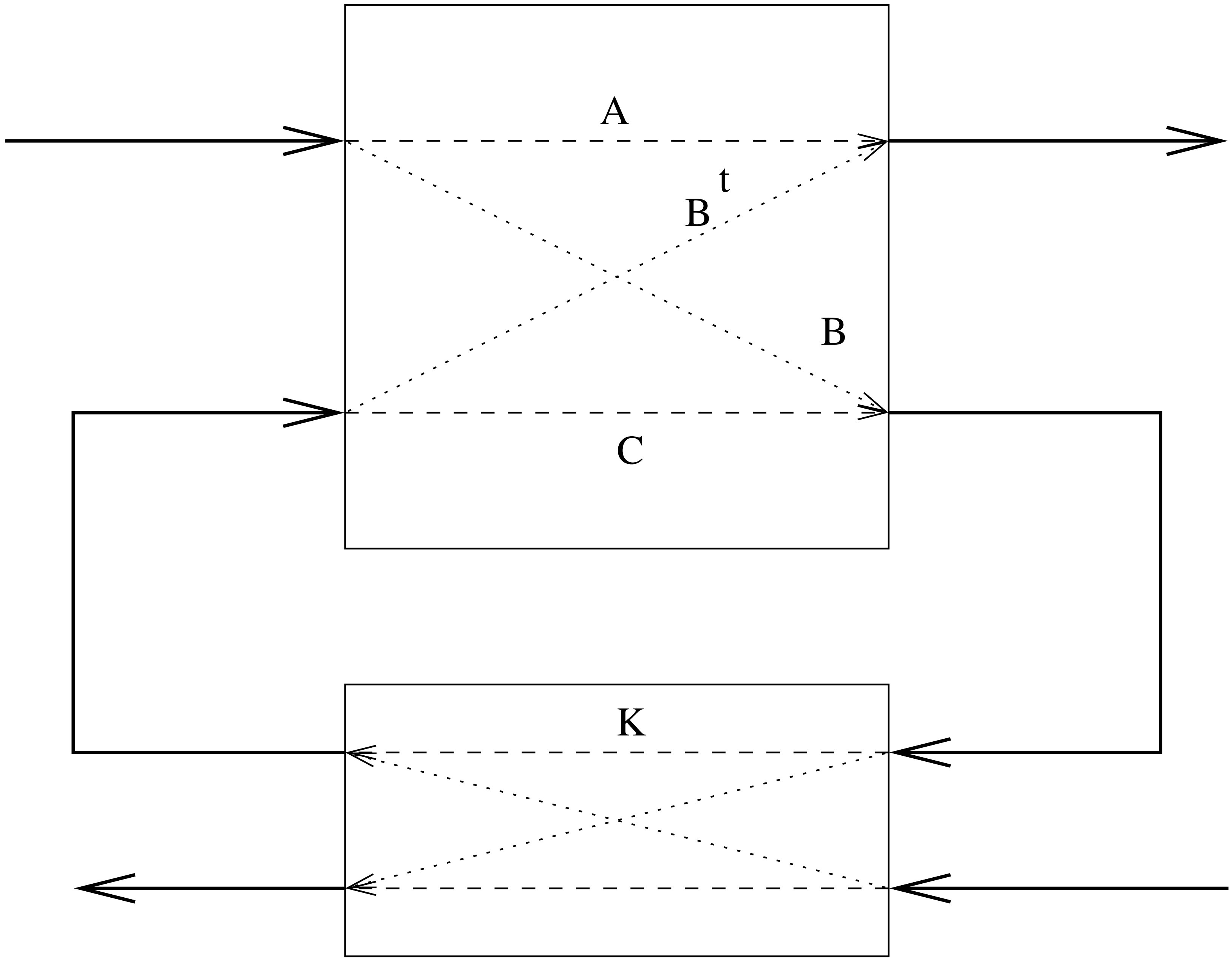
(b) $||\Omega_P|| \leq 1$*;*

(c) $||K|| < 1$ *and* $||M|| < 1$*;*

(d) K *and* M *are Hilbert-Schmidt operators.*

$$\begin{pmatrix} A & B \\ B^t & C \end{pmatrix} * \begin{pmatrix} K & L \\ L^t & M \end{pmatrix} =$$

$$\begin{pmatrix} A + BK(1-CK)^{-1}B^t & B(1-KC)^{-1}L \\ L^t(1-CK)^{-1}B^t & M + L^t(1-CK)^{-1}CL \end{pmatrix}$$


A
B^t
B
C
K

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- The generalized version of CFT could allow one to explore entirely new kinds of CFT, for example, by looking at nuclear functors into the category of Stochastic Relations.