Anyons, Braids and Topological Quantum Computing

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Quantum Computing

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- requires implementing precise transformations on the qubits.

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- A tall order!

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- The topology will keep the configuration from coming apart.
- Where do we find quantum braids or knots?

Quantum Statistics

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- Depending on the type of particle the answer could be 1/3 (bosons) or 0 (fermions).

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- Symmetries form a group.

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- $$\rho : G \to GL(H)$$

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- The permutation group is a symmetry of a quantum system: the system looks the same if you interchange particles of the same type.

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- or the alternating representation: a permutation P is mapped to +1 or -1 according to whether P is odd or even.

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- The state vector of a system either changes sign under an interchange of any pair of identical particles (fermions) or does not (bosons).

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- If the state vector changes sign under an interchange of identical particles but must also look the same if they are in the same state we have v = -v; where v is the state vector describing two identical particles in the same state.

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- In short v = 0!
- With fermions two particles cannot be in exactly the same state: Pauli exclusion principle. The reason for chemistry!!

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- The explanation of lasers, superconductivity and many other collective phenomena.

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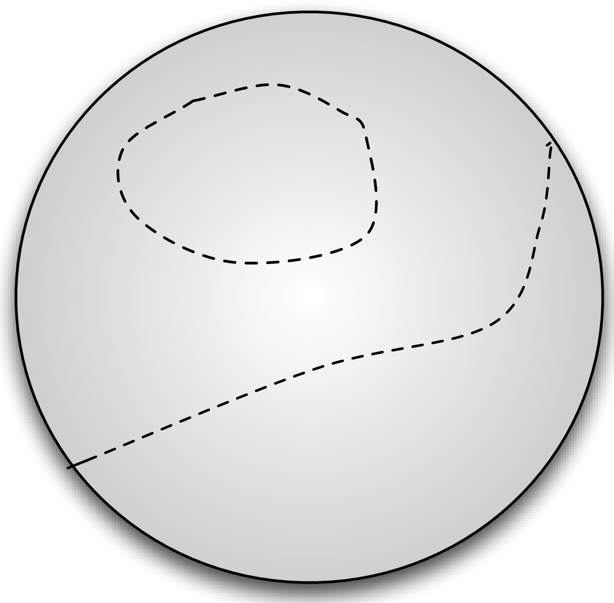
- The resulting group is not simply connected: there are loops that cannot be continuously deformed to a point.

A picture of $SO(3)$ showing a loop that can be shrunk to a point and one that cannot.

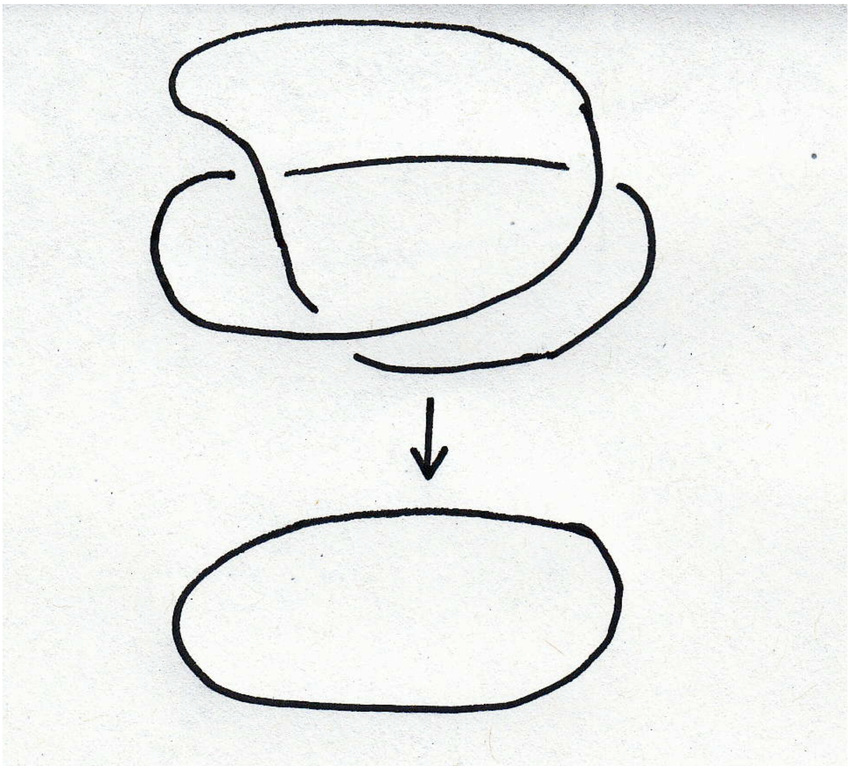


$SO(3)$ is not simply connected.

There is another group $SU(2)$: the group of unitary 2×2 matrices with determinant 1.

There is a homomorphism from $SU(2)$ to $SO(3)$ which is onto and 2 to 1 and which locally looks just like $SO(3)$ but globally is simply connected.

Now which is the relevant symmetry group for quantum mechanics?



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Nature has two types of particles: those for which a 2π rotation is the identity and those for which a 4π rotation is the identity.

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What happens in two dimensions?

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Such entities are called *anyons*.

What happened to the Spin-Statistics theorem?

It still holds in two dimensions! The relevant group is no longer the permutation group but the braid group.

To understand why we need to think about the physics of two dimensional entities.

In the laboratory we get 2D physics with a thin gas of free electrons trapped between two semiconductor layers.

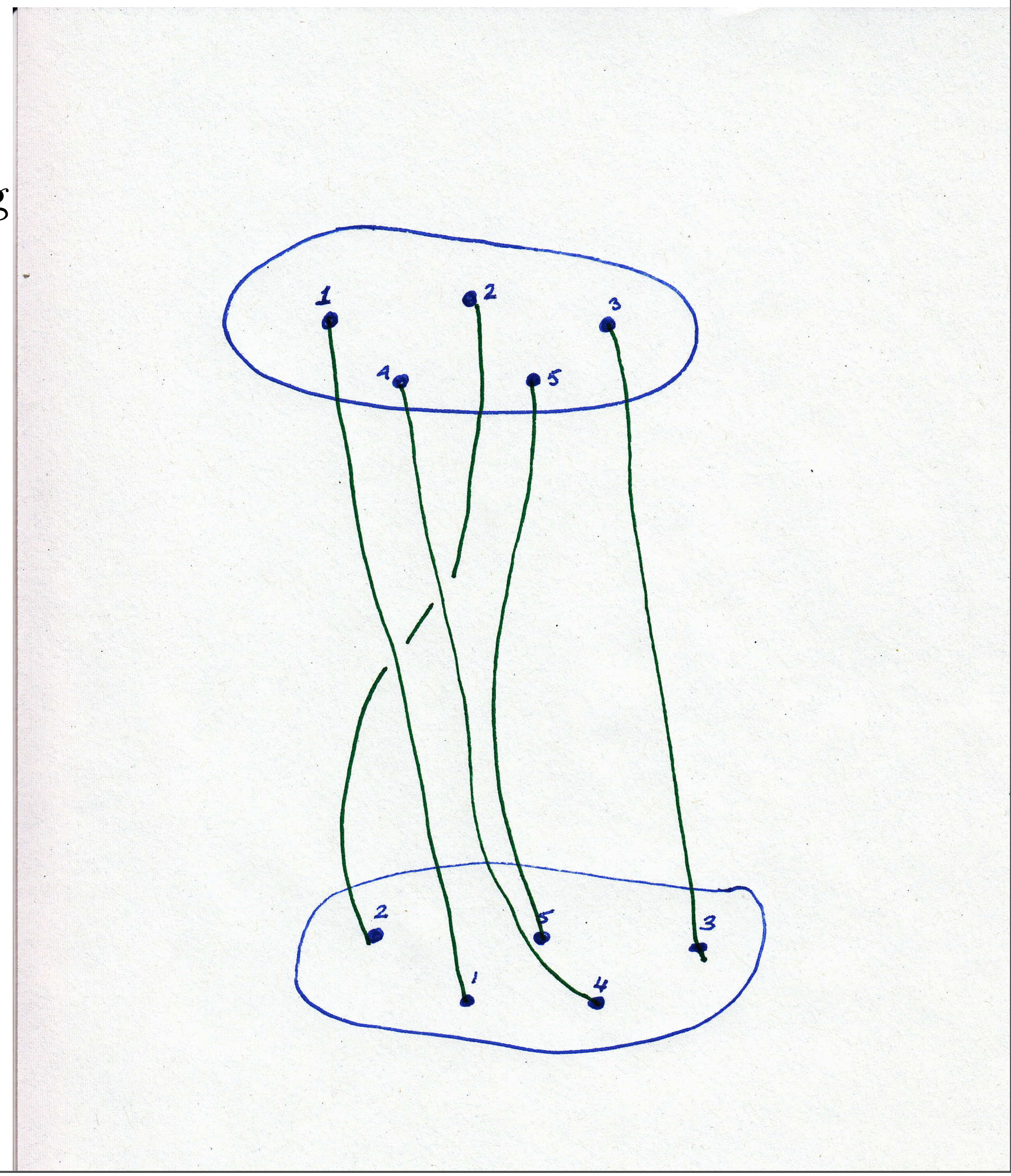
A strong magnetic field is applied in the perpendicular direction confining the "gas" to a 2D layer.

Excited states of this system are not electrons but *virtual* particles with strange properties.

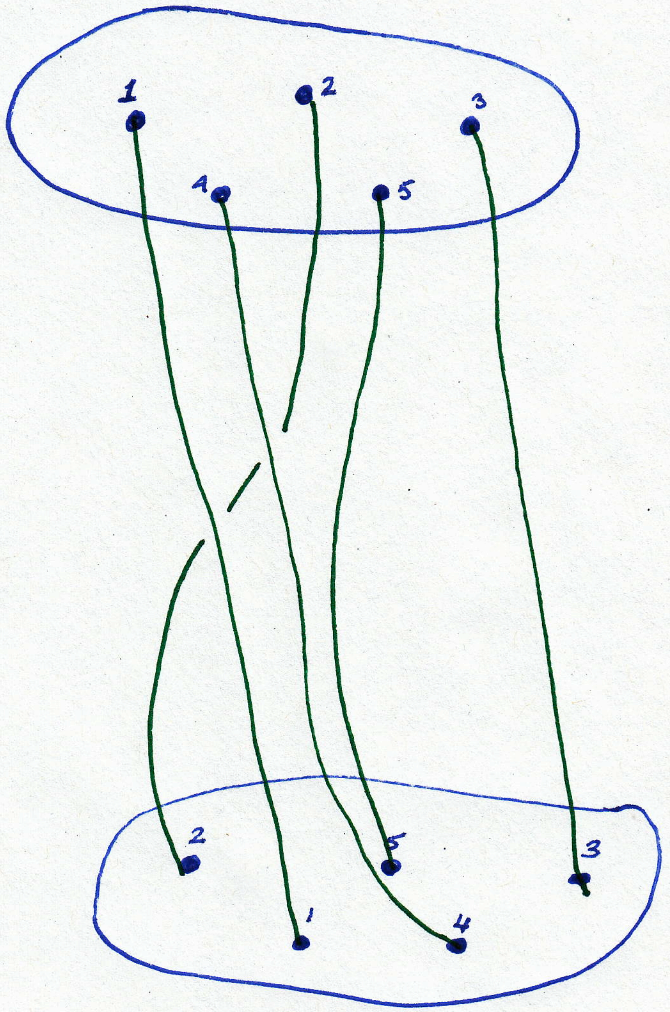
Imagine some (5 in the picture) particles and consider what happens when some of them are exchanged.

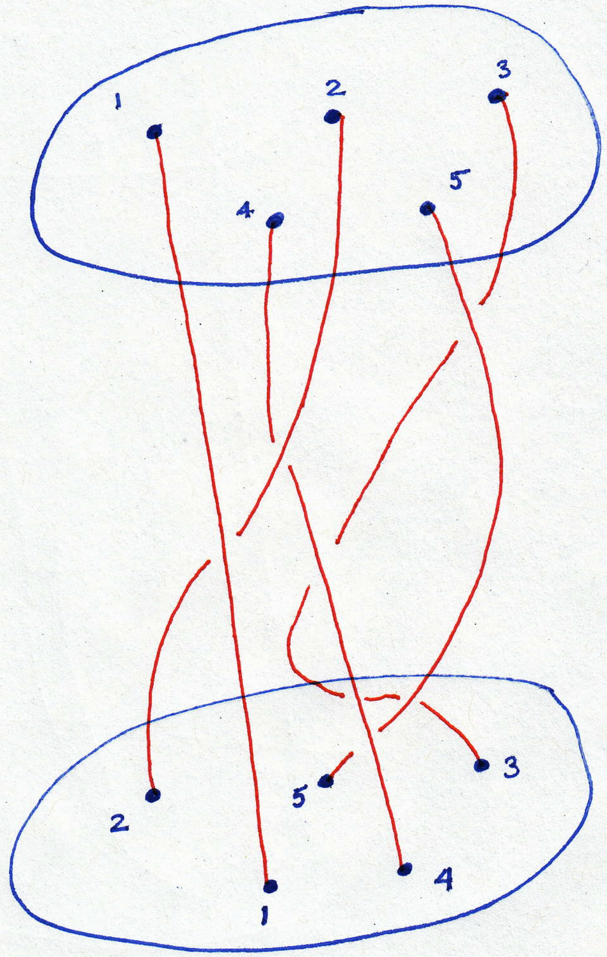
Here $1 \mapsto 4, 2 \mapsto 1, 3 \mapsto 3, 4 \mapsto 5$ and $5 \mapsto 2$

In 3D the strands can always be disentangled; the only thing that matters is the start and end point. So we can describe the effect just by giving a permutation.



In 2D the entangling matters. One has to distinguish between different braidings.

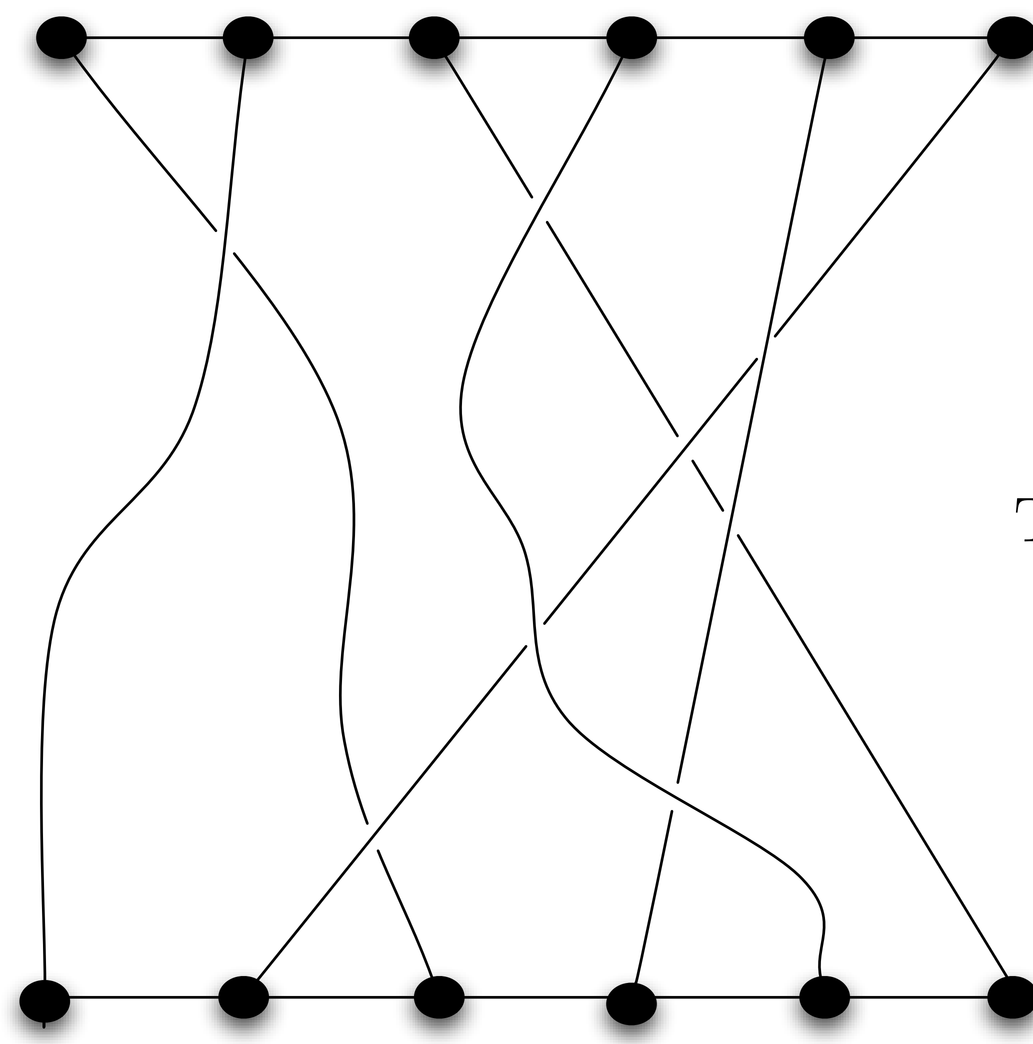




Here the permutations are the same but the braiding is different.

The Braid Group

Fix n and consider n points on a line with another n points on a line below. We connect them with strands. The generators of the group are interchanges of adjacent strands.



This is an element of B_6.

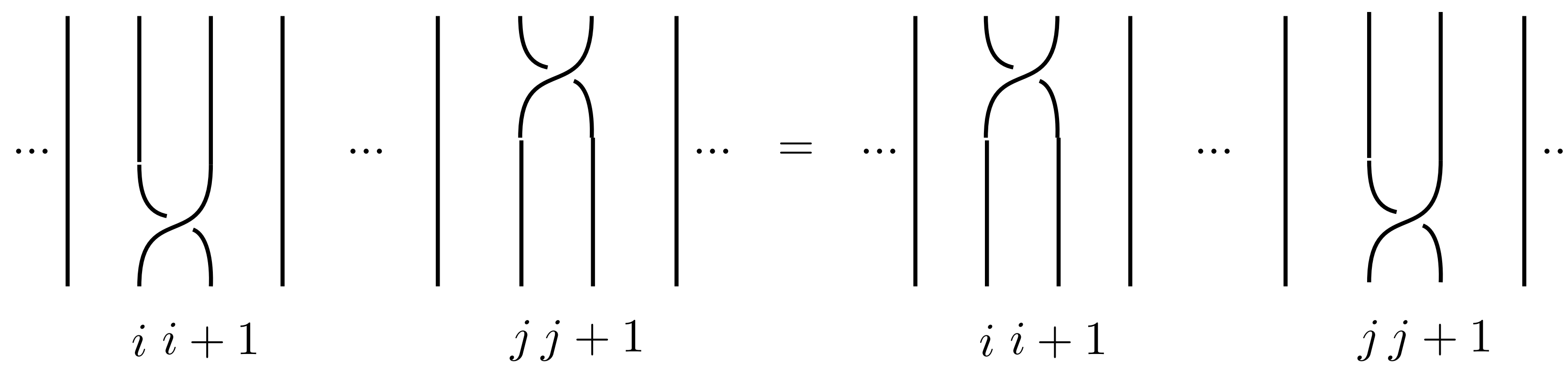
Much richer theory than the permutation group.

For n points the generators are b_1 to b_{n-1} and their inverses. The generators obey the following equations:

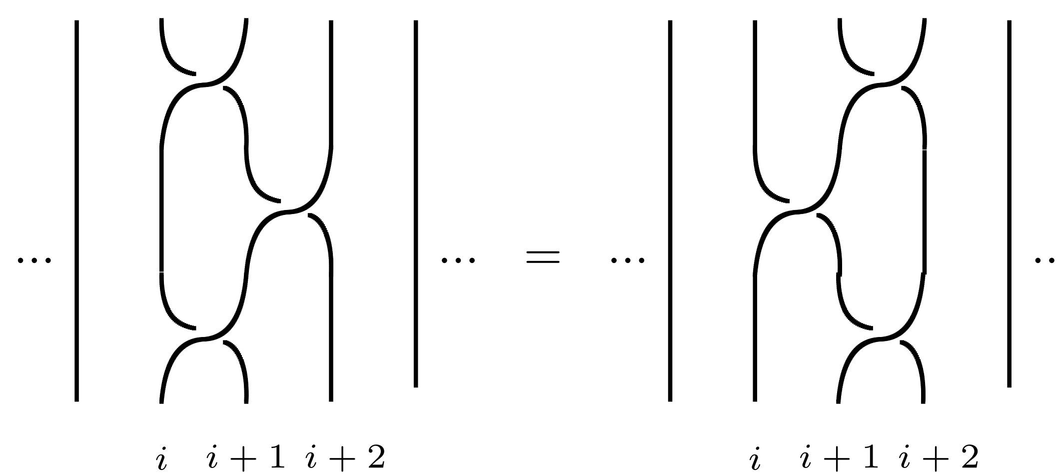
$$b_i b_j = b_j b_i \quad \text{for} \quad |i-j| \geq 2 \tag{1}$$

$$b_i b_{i+1} b_i = b_{i+1} b_i b_{i+1} \quad \text{for} \quad 1 \leq i \leq n-1. \tag{2}$$

which respectively depicts as:



and



Generalized Spin-Statistics theorem holds in dimensions 2 and 3.

See the paper by Froelich and Gabbiani : Local Quantum Theory and Braid Group Statistics.

There is a lot more to be said about knots, braids, physics and related things but we need to get on with the main story.

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Not all anyons are so simple!

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However, there are more interesting representations.

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There are candidates but there are no definite laboratory demonstrations of non-abelian anyons.

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How do we describe all this complicated algebra? There are different types of things that combine in non-trivial ways. We have essentially an exotic type theory.

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We need braided monoidal categories. The tensor product structure gives the fusion possibility. The additive structure gives the different possibilities.

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We need to capture the idea of combining types and getting new types as a result. We also need the idea of "putting together" and "or".

We need to have the ability to describe braids.

In fact, the anyons are extended objects with more than "string-like" structure. We need braided *ribbons* that may have *twists* in them.

We need braided monoidal categories. The tensor product structure gives the fusion possibility. The additive structure gives the different possibilities.

To accomodate everything we use what are called *modular tensor categories.*

An example: Fibonacci anyons

Two basic types: 1 and τ.

$$\left.\begin{array}{l} \mathbf{1} \otimes \mathbf{1} \simeq \mathbf{1} \\ \mathbf{1} \otimes \tau \simeq \tau \otimes \mathbf{1} \simeq \tau \\ \tau \otimes \tau \simeq \mathbf{1} \oplus \tau \end{array}\right\} \text{Here are the fusion rules.}$$

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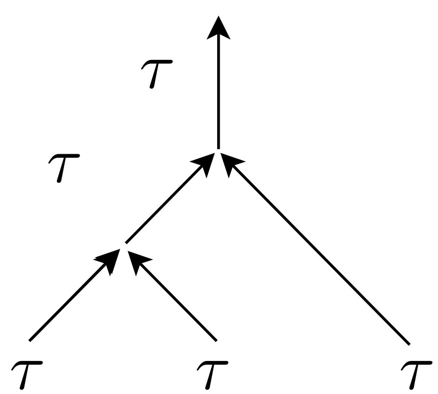
In pictures



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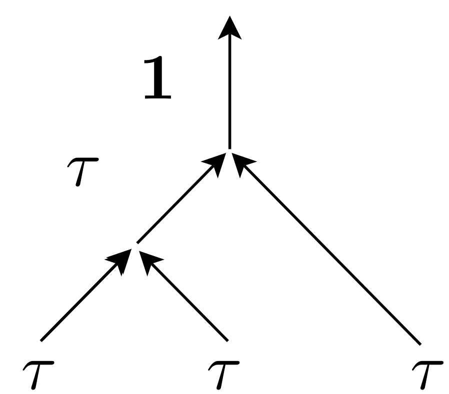

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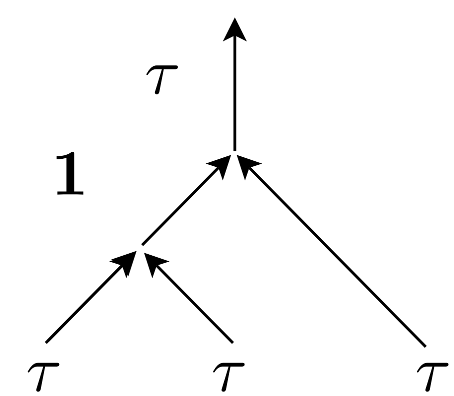

The basic idea to simulate quantum computation with anyons is given by the following steps:

1. Consider a compound system of anyons. We initialise a state in the splitting space by fixing the charges the subsets of anyons according to the way they will fuse. This determines the basis state in which the computation starts.
2. We braid the anyons together, it will induce a unitary action on the chosen splitting space.
3. Finally, we let the anyon fuse together and the way they fuse determines which state is measured and this constitutes the output of our computation.

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In fact it is possible to show that the Fibonacci anyons are *universal* for quantum computation.

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The one-dimensional space represents possible "leakage."

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We are almost there, but we need at least one *two*-qubit gate.

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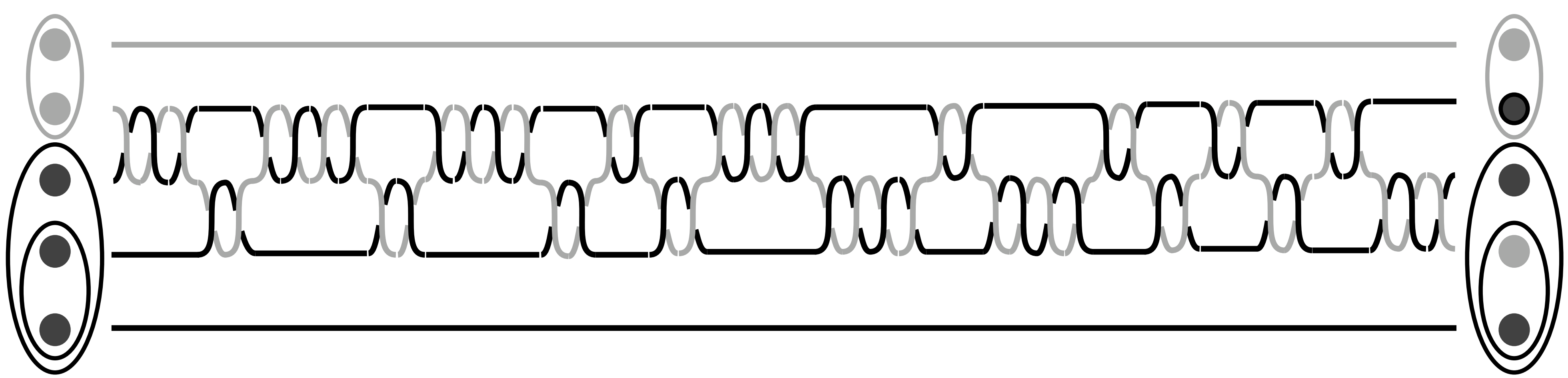

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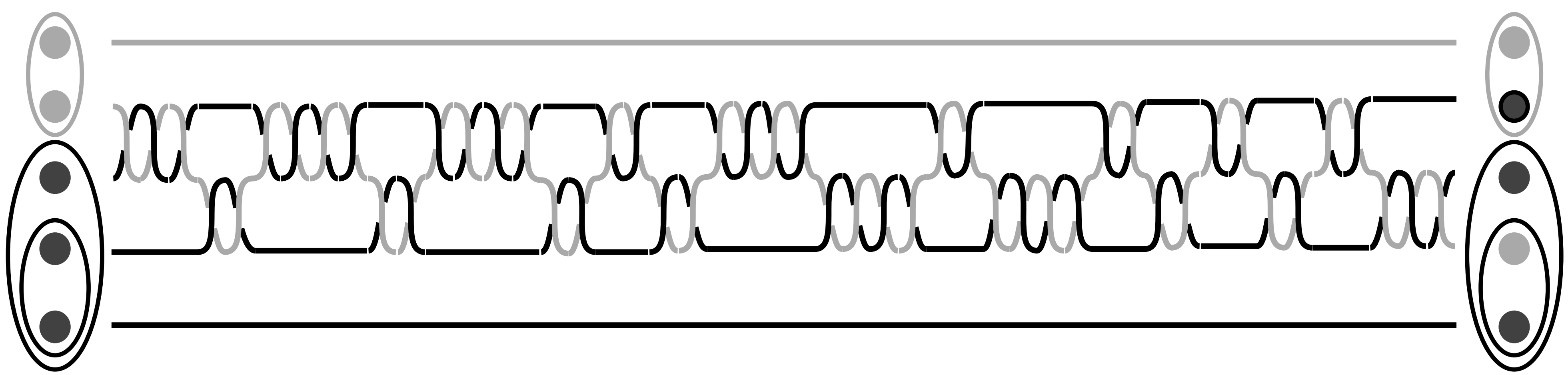

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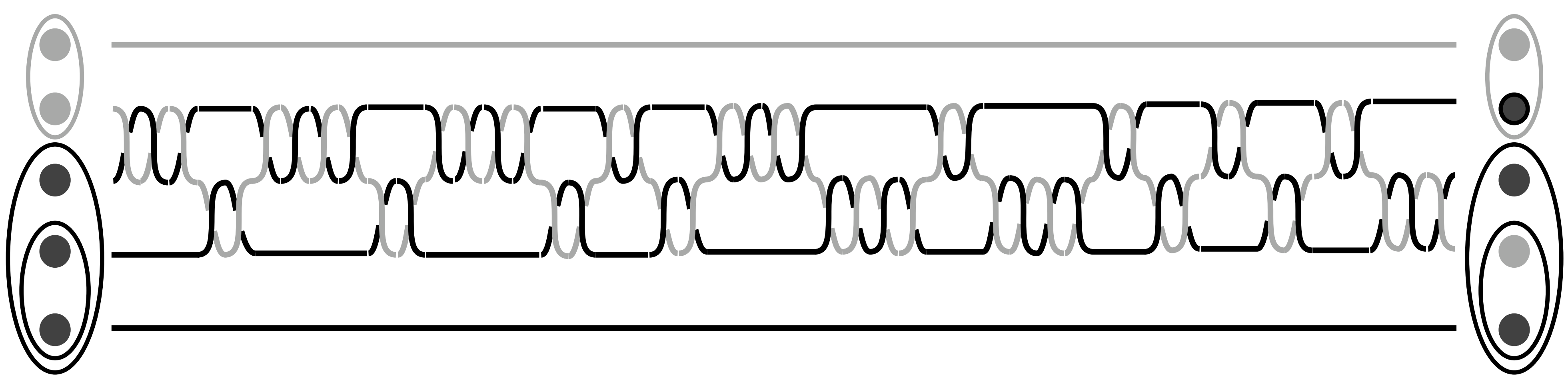

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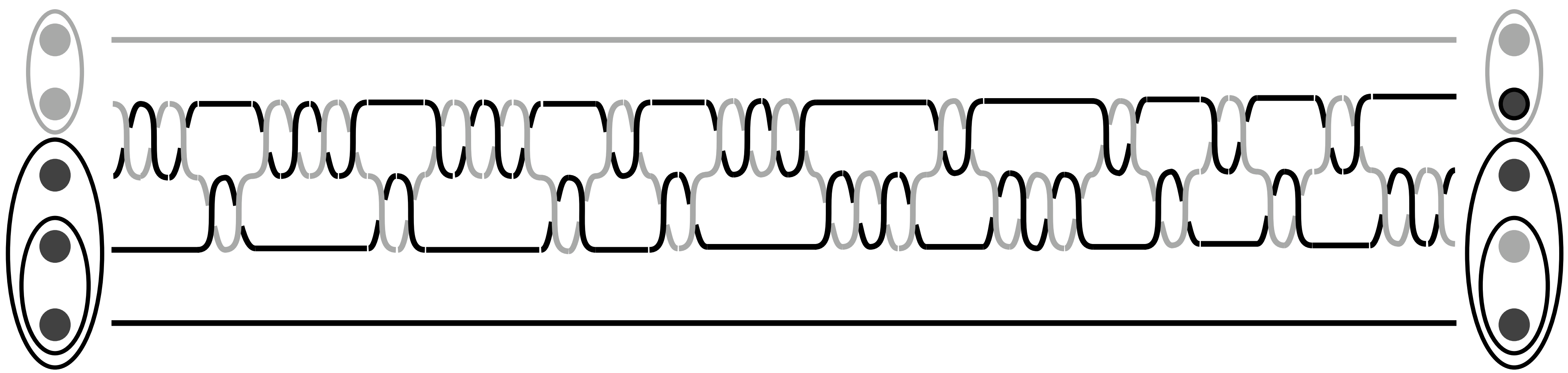

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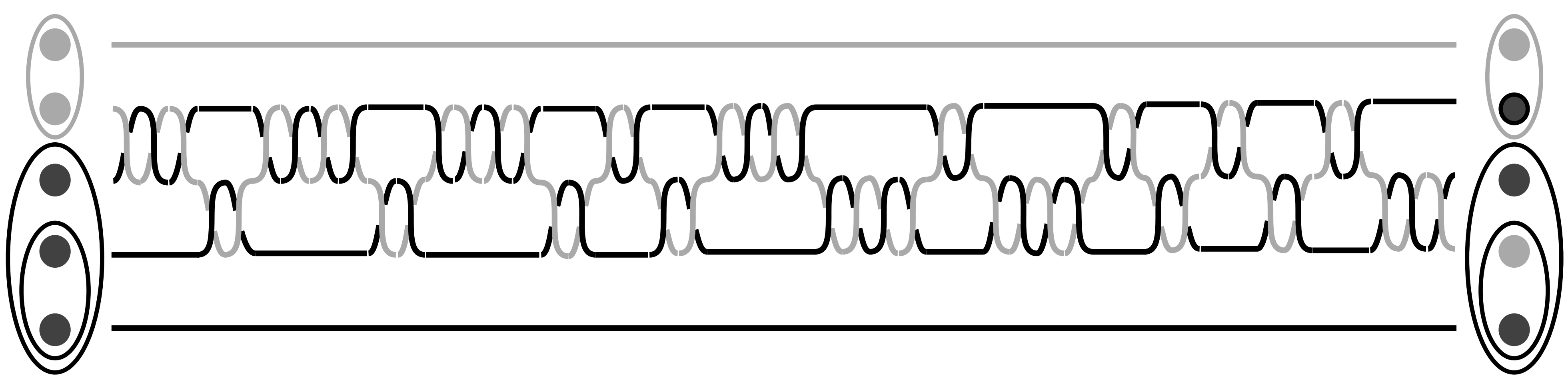

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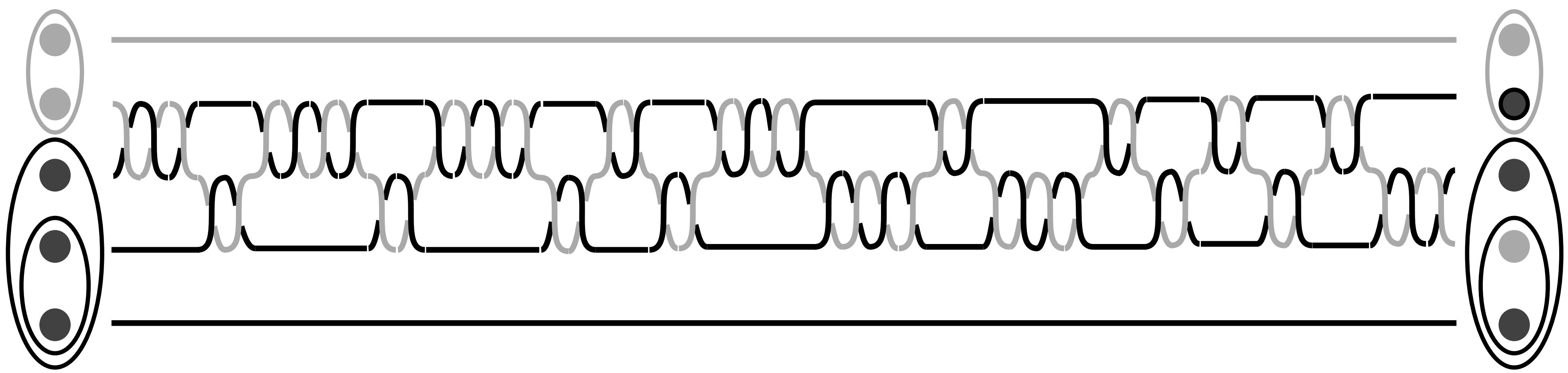

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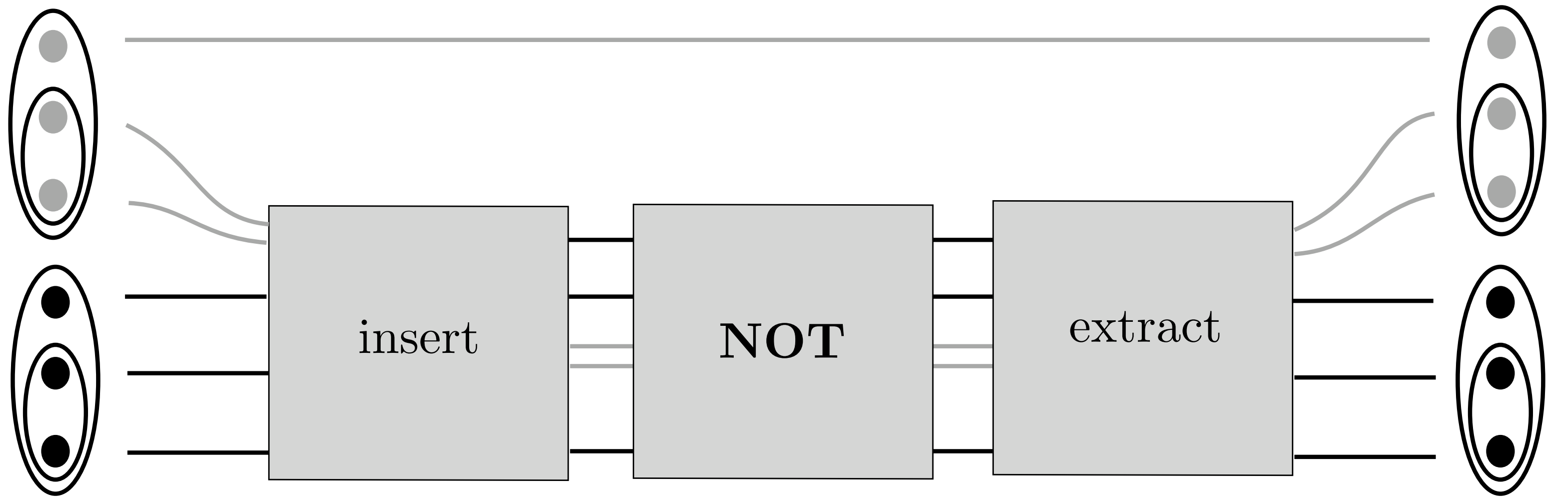
How do they come up with this? By being clever!



The above shows the general scheme.

A **NOT** can be implemented as a one-qubit unitary. We insert a *pair* of test anyons. They fuse to produce a τ or a $\mathbf{1}$.

If the fusion produces a 1 then any tensoring with the other anyons has no effect. If it produces a τ the **NOT** will have an effect. At the end we restore the state of the control triplet.

Details are admittedly hairy and formalizing all this is daunting.

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Tremendously exciting synergy between the three communities.

Some references

J. Preskill, Lectures notes in quantum computation, chapter 9. Avialable at `http://www.theory.caltech.edu/people/preskill/ph229`

S. Das Sarma, M. Freedman, C. Nayak, S. Simon and A. Stern, Non-abelian anyons and topological quantum computing, in Rev. Mod. Phys.

P. Panangaden and E. O. Paquette, A categorical presentation of quantum computation with anyons. To appear. Available on my web page.