The pumping lemma for context-free languages

Tuesday, March 9, 2021 11:21 AM

Context - free languages

$$S \rightarrow aSb \mid \varepsilon$$

$$L(G) = \{a^n b^n \mid n \geq 0\}$$

$(ab)^*$ is a regular language with equal # of a's & b's.

$\{a^n b^n c^n \mid n \geq 0\}$ is <u>NOT</u> context-free.

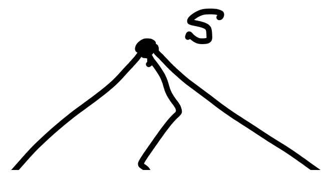
<u>Recall</u> every CFL has a CFG in Chomsky Normal Form

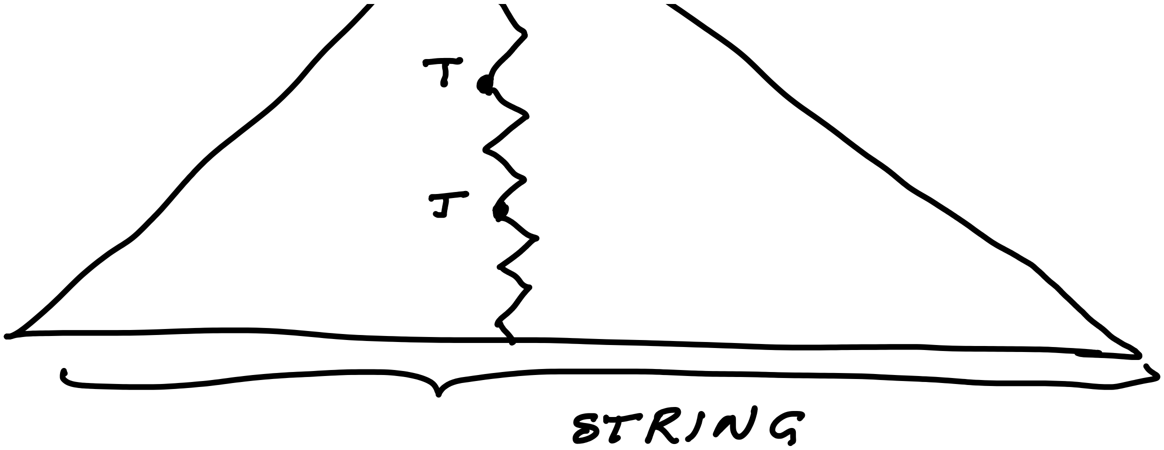
$$A \rightarrow BC$$

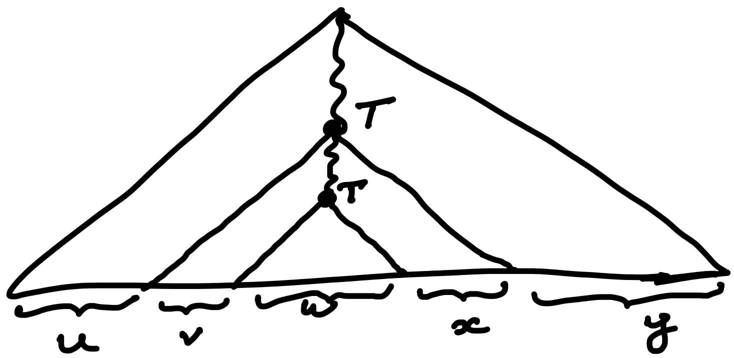
$$A \rightarrow a$$

When we are using a CNF grammar the parse <u>tree</u> is binary.

A grammar has only <u>finitely</u> many non-terminals. So if we generate a sufficient <u>deep</u> tree we must start repeating the non-terminals.

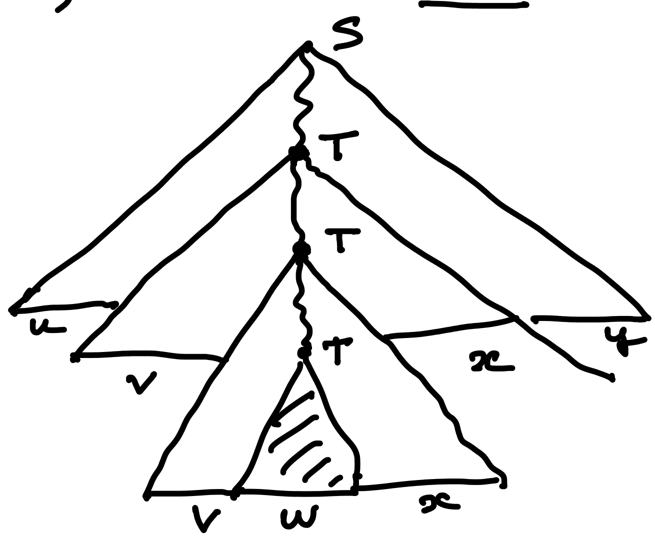




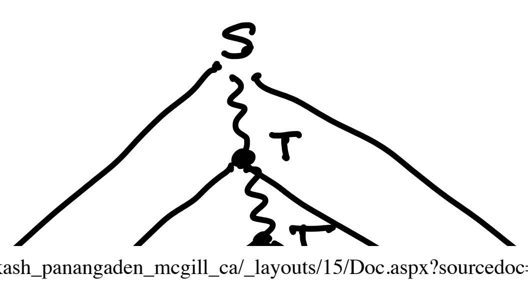


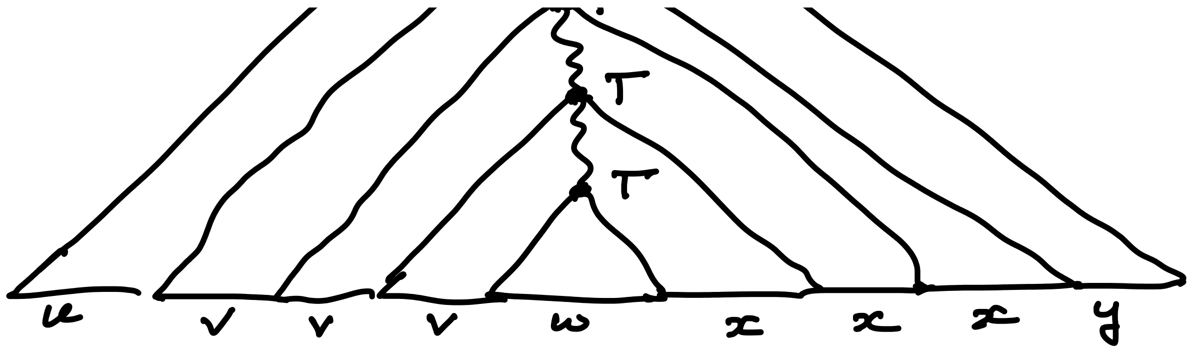
$S \xRightarrow{*} uvwxy$ $\quad T \xRightarrow{*} vwx$ $\quad T \xRightarrow{*} w$

v, x cannot <u>both</u> be empty (CNF)



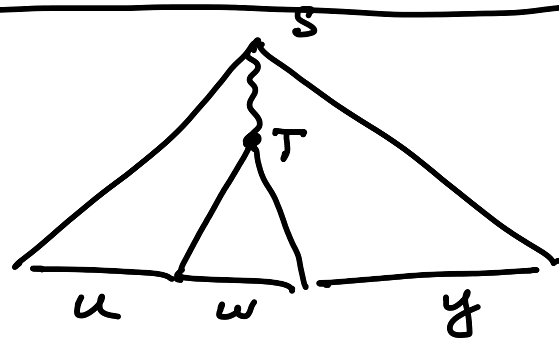
so $u\, vv\, w\, xx\, y \in L$





$u\,v\,v\,v\,w\,x\,x\,x\,y \in L$

$$\boxed{\forall i \geq 0 \quad u v^i w x^i y \in L}$$



$u\,w\,y \in L$

Full statement of the pumping lemma:

$\forall$ CFLs L $\quad \exists\, p > 0$

$\forall s \in L \quad |s| \geq p$

$\exists\, u, v, w, x, y \in \Sigma^*$ such that

- $s = u v w x y$
- $|vx| \neq 0$
- $|vwx| \leq p$

$\forall i \geq 0 \quad uv^i w x^i y \in L$

CFL $\Rightarrow$ every string can be pumped

CONTRAPOSITIVE:

if there is even one string that cannot be pumped

$\Rightarrow$ not a CFL

Fix L some language

If $\forall p > 0$

$\exists s \in L$ s.t. $|s| \geq p$

$\forall u, v, w, x, y \in \Sigma^*$ s.t.

- $s = uvwxy$
- $|vx| > 0$
- $\boxed{|vwx| \leq p}$

$\exists i \geq 0 \quad uv^i w x^i y \notin L$

then L is NOT a CFL.

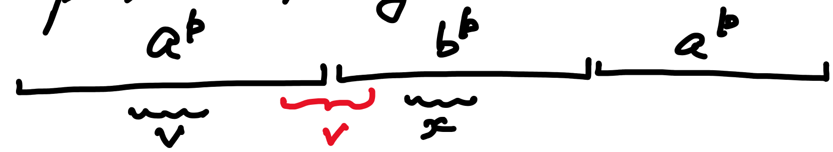
EXAMPLES

(1) $L = \{a^n b^n a^n \mid n \geq 0\}$

- Demon chooses $p > 0$

- I choose $a^p b^p a^p$

Have to analyze how the demon may break up the string into u, v, w, x, y:



- v (or x) crosses a boundary between two blocks: $i = 2$, now a's and b's are out of order

- v, x both contain only a's they have to be in the <u>same</u> block: now choose $i = 2$ (or 3 (2?)) the 3 blocks do not have the same length.

- same argument if both v, x contain only b's

- v contains only a's and x contains only b's: now one block is untouched by pumping: $i = 2$ and the block lengths cannot match

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EXAMPLE 2 Σ: arbitrary $|\Sigma| \geq 2$

→ $L = \{ww \mid w \in \Sigma^*\}$ $L' = \{ww^{rev} \mid w \in \Sigma^*\}$

L is not context free

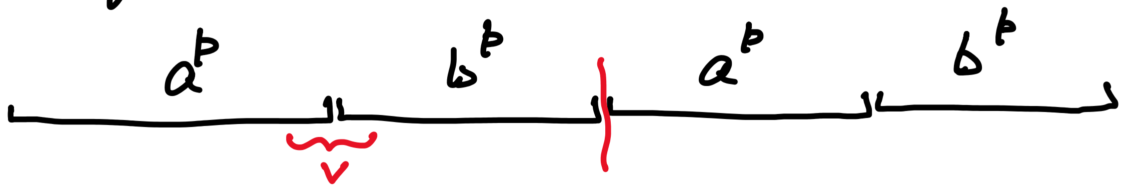
<u>FACT</u>: if L is a CFL & R is <u>regular</u> then $L \cap R$ is a CFL.

$$\hat{L} = L \cap a^* b^* a^* b^*$$

$$\hat{L} = \{ \underline{a^i} b^j \underline{a^i} b^j \mid i, j \geq 0 \}$$

Demon picks $p > 0$

I pick $a^p b^p a^p b^p$



- v or x straddles a block boundary: $i=2$ letters are out of order
- If v, x are in the same block: choose $i=2$. Either 1^{st} & 3^{rd} blocks are mismatched or 2^{nd} & 4^{th} are mismatched
- If v is in the first block, x is in the 2^{nd} block: choose $i=2$, now 1^{st} & 3^{rd} no longer match / also 2^{nd} & 4^{th} blocks don't match

Note : v, x must be in the same block or in adjacent blocks.

$\overline{L}$ is context-free : we have seen a PDA already and there is a grammar in the <u>supplementary notes</u>.

CFL's are not closed under complement.

- $L_1 = \{a^n b^n c^m \mid n, m \geq 0\}$ $L_2 = \{a^m b^n c^n \mid n, m$