McGill University COMP360 Winter 2011 **Instructor: Phuong Nguyen**

Some non-credit exercises on network flow and linear programming

Network Flow

Question 1 We define the *Escape Problem* as follows. We are given a directed graph $G = (V, E)$ (picture a network of roads). A certain collection of nodes $X \subset V$ are designated as *populated nodes*, and a certain other collection $S \subset V$ are designated as *safe nodes*. (Assume that X and S are disjoint.) In case of an emergency, we want evacuation routes from the populated nodes to the safe nodes. A set of evacuation routes is defined as a set of paths in G so that (i) each node in X is the tail of one path, (ii) the last node on each path lies in S, and (iii) the paths do not share any edges. Such a set of paths gives a way for the occupants of the populated nodes to "escape" to S, without overly congesting any edge in G.

(a) Given G, X, S, show how to decide in polynomial time whether such a set of evacuation routes exists.

(b) Suppose we have exactly the same problem as in (a), but we want to enforce an even stronger version of the "no congestion" condition (iii). Thus we change (iii) to say "the paths do not share any *nodes*."

With this new condition, show how to decide in polynomial time whether such a set of evacuation routes exists.

Also, provide an example with the same G, X, S, in which the answer is yes to the question in (a) but no to the question in (b).

Question 2 Let $G = (V, E)$ be a bipartite graph where V is partitioned into $X \cup Y$. Suppose that M is a matching in G, i.e., a set of edges that do not share any endpoint. Then we say that a node $y \in Y$ is *covered* by M if y is an end of one of the edges in M.

The Coverage Expansion Problem is as follows. We are given such a bipartite graph G, a matching M of G, and a number k. The problem is to decide whether there is another matching M' so that

- M' has k more edges than M does,
- every $y \in Y$ that is covered by M is also covered by M'.

Give a polynomial time algorithm for this problem.

Linear Programming

Question 1 Write a linear program that, given a bipartite graph $G = (V, E)$, solves the maximum bipartite matching problem.

Question 2 [Minimum-Cost Multicommodity-Flow Problem] Suppose that we are given a directed graph $G = (V, E)$ where each edge $e \in E$ has a nonnegative capacity $c(e) \geq 0$ and a weight $w(e)$. There are k commodities $K_1, K_2, \ldots, K_k$, where commodity K_i has source s_i, sink t_i and demand d_i. For a commodity K_i we want a flow of value d_i which emanates from source s_i and goes to sink

t_i obeying the Conservation Condition. Let $f_i(e)$ denote the flow of commodity K_i on edge e. The General Capacity Condition is that

$$\sum_{i=1}^{k} f_i(e) \leq c(e)$$

Thus, a feasible flow $f = (f_1, \ldots, f_k)$ is such that

- the flow f_i (of commodity K_i) is generated at s_i and consumed at t_i, for $1 \leq i \leq k$;
- the value of the flow f_i (of commodity K_i) is d_i, for $1 \leq i \leq k$;
- the Conservation Condition holds for every flow f_i and every vertex v where $v \neq s_i, t_i$;
- the General Capacity Condition holds.

The weight of a feasible flow f is

$$\sum_{e \in G} w(e) f(e)$$

The Minimum-Cost Multicommodity-Flow Problem is to find a feasible flow with minimum weight. For simplicity, we may assume that there are no edge entering the sources s_i, and no edge leaving the sinks t_i. Express this problem as a linear program. [You are *not* required to solve this problem.]

Question 3 A film producer wants to make a motion picture. For this, she needs to choose among n available actors. Actor i demands a payment of s_i dollars to participate in the picture.

The funding for the picture will come from m investors. The kth investor will pay the producer p_k dollars, but only under the following condition. The investor has a list of actors $L_k \subseteq \{1, 2, \ldots, n\}$, and he will only invest if *all* the actors on his list appear in the picture.

The profit of the producer is the sum of the payments from the investors that she agrees to take funding from, minus the sum of payments she makes to the actors that appear in the picture. The goal is to maximize the producer's profit.

Formulate this problem as a 0-1 Integer Programming problem so that the optimal value of the objective function is the maximum profit of the producer. Precisely specify the variables, the objective function, and the constraints. (You are *not* required to solve the 0-1 Integer Programming problem.)