# Sparse halves in dense triangle-free graphs

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**Abstract**

In this paper we study the conjecture that any triangle-free graph $G$ on $n$ vertices should contain a set of $\lfloor n/2 \rfloor$ vertices that spans at most $n^2/50$ edges. This problem was considered by Erdős, Faudree, Rousseau and Schelp in [5]. Krivelevich proved the conjecture for graphs with minimum degree at least $\frac{2}{5}n$ [8]. In [7] Keevash and Sudakov improved this result to graphs with average degree at least $\frac{2}{5}n$. We strengthen these results further by showing that the conjecture holds for graphs with minimum degree at least $\frac{5}{14}n$ and average degree at least $\left(\frac{2}{5} - \epsilon\right) n$ for some absolute $\epsilon > 0$.

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## 1 Introduction

In this paper we consider the edge distribution in triangle-free graphs. A fundamental result in extremal graph theory, Turán's Theorem implies that every graph on $n$ vertices with more than $n^2/4$ edges contains a triangle. One can consider the following generalization of this problem first studied by Erdős, Faudree, Rousseau and Schelp in [5].

Suppose for given $0 < \alpha \le 1$ every set of $\alpha n$ vertices of graph $G$ spans more than $\beta n^2$ edges. A natural question arises - what is the smallest $\beta = \beta(\alpha)$ such that every such graph $G$ necessarily contains a triangle? In particular, one of the Erdős' old and favourite conjectures is on $\beta(\frac{1}{2})$ that he first proposed in [2] and offered a \$250 prize for its solution later in [3].

**Conjecture 1.1** *For a given graph $G$ of order $n$, if every set of $\lfloor n/2 \rfloor$ vertices spans at least $n^2/50$ edges, then $G$ necessarily contains a triangle.*

In [2] the authors also conjectured that $\beta(\alpha)$ is determined by some family of extremal triangle-free graphs. The bound for $\beta(\frac{1}{2})$ is obtained on the uniform *'blow-up'* of $C_5$ which is obtained from the 5-cycle by replacing each vertex $i$ by an independent set $V_i$ of size $n/5$ and each edge $ij$ by a complete bipartite graph joining $V_i$ and $V_j$ (note that the 'blow-up' of Petersen graph also achieves this bound tightly).

In his paper [8] Krivelevich proved that the conjecture holds when $n^2/50$ is replaced by $n^2/36$. He also showed that it is true for triangle-free graphs with minimum degree $\frac{2}{5}n$. Later, Sudakov and Keevash improved this result showing that the conjecture holds for graphs with average degree $\frac{2}{5}n$. We extend their result to graphs with average degree $(\frac{2}{5}-\epsilon)n$ for some computable positive $\epsilon$, thus proving the theorem below.

**Theorem 1.2** *There exists some absolute $\epsilon > 0$ such that in all triangle-free graphs on $n$ vertices with at least $(\frac{1}{5} - \epsilon)n^2$ edges there exists a set of $\lfloor n/2 \rfloor$ vertices that spans at most $n^2/50$ edges.*

We also prove the conjecture for graphs with minimum degree $\frac{5}{14}n$. Our proof is mainly based on the structural characterization of these graphs established by Jin, Chen and Koh in [1,6]. We also use some averaging arguments similar to the ones used in [7,8].

**Theorem 1.3** *In all triangle-free graphs on $n$ vertices with minimum degree at least $\frac{5}{14}n$ there exists a set of $\lfloor n/2 \rfloor$ vertices that spans at most $n^2/50$ edges.*

It is worth mentioning another extremal problem closely related to the one we consider. What is the minimum number of edges that need to be deleted from any triangle-free graph of order $n$ to make it bipartite? One can check that the uniform 'blow-up' of the 5-cycle, defined above, is an example of a graph for which $n^2/25$ edges need to be removed in order to obtain a bipartite graph. In [2] Erdős conjectured that this bound applies to any triangle-free graph.

**Conjecture 1.4** *Every triangle-free graph on n vertices can be made bipartite by the omission of at most $n^2/25$ edges.*

The best result on this problem is $(\frac{1}{18} - \epsilon)n^2$ for some absolute $\epsilon > 0$ by Erdős, Pach, Faudree and Spencer [4]. Krivelevich [8] drew connection between this problem and the local density problem, proving that for regular graphs a bound in the latter problem implies a bound for the problem of making the graph bipartite. More specifically, if in a regular graph $G$ of order $n$ some set $U$ of $n/2$ vertices spans $m$ edges, then by removing at most $2m$ edges $G$ becomes bipartite. However, the converse does not hold (e.g. in the Petersen graph). This shows that the local density problem may be harder than the problem of making the graph bipartite.

In the next section we present the main ideas used in the proofs of Theorem 1.2 and Theorem 1.3.

## 2 Notation and Preliminary results

For a given graph $G$, we denote by $N_v$ the neighborhood of a vertex $v$ and by $d(v)$ the degree of $v$. The maximum and the minimum degrees of the graph are denoted by $\Delta(G)$ and $\delta(G)$, correspondingly.

A graph $G$ is called *triangle-free* if it does not contain a triangle. For a given triangle-free graph of order $n$, we say that there is a *sparse half* if there exists a set of $\lfloor n/2 \rfloor$ vertices that spans at most $n^2/50$ edges. With this definition, the Conjecture 1.1 says that there is a sparse half in every triangle-free graph.

A *weighted* graph is a pair $(G, \omega)$, where $\omega$ is called a *weight function* defined on the vertices, such that $0 < \omega(v) \leq 1$ for each vertex $v$ and the *weight* of the whole graph $G$ is

$$\omega(V(G)) = \sum_{v \in V(G)} \omega(v) = 1.$$

The *degree* of a vertex $v$ in a weighted graph $(G,\omega)$ is defined to be

$$\omega(N_v) = \sum_{u \in N_v} \omega(u).$$

The *minimum degree* of the weighted graph $(G,\omega)$ we simply denote by $\delta(G,\omega)$. The weight of an edge $e=(u,v)$ is defined to be $\omega(e)=\omega(u)\cdot\omega(v)$ and the total weight of any set of edges $F$ is

$$\omega(F) = \sum_{e \in F} \omega(e).$$

We call a real function $s: V(G) \to \mathbb{R}^+$ *a half* of $G$ if for each vertex $v$, we have $0 \leq s(v) \leq \omega(v)$ and

$$s\left(V(G)\right) = \sum_{v \in V(G)} s(v) = \frac{1}{2}.$$

If in addition to these conditions also

$$s\left(E(G)\right) = \sum_{e=\{u,v\} \in E(G)} s(u)s(v) \leq \frac{1}{50},$$

then $s$ is called a *sparse half*. If there exists such an $s$, we shortly say that $(G,\omega)$ has a sparse half.

For every graph $G$ of order $n$, there is a natural way of making it a weighted graph. We assign to each vertex $v$ weight equal to $\frac{1}{n}$. The resulting graph we call a *uniformly weighted* $G$ and denote by $(G,\omega_u)$. The next Lemma shows that if one can define a sparse half on the uniformly weighted graph $(G,\omega_u)$, then there must exist a sparse half in the original graph $G$. Note that the other direction is straightforward. This shows that the notion of sparse halves in weighted and non-weighted graphs are equivalent.

**Lemma 2.1** *For a given graph $G$, if $(G,\omega_u)$ has a sparse half, then so does $G$.*

So in the language of weighed graphs the Conjecture 1.1 says that one can define a sparse half on every uniformly weighted triangle-free graph. We prove this for all weighted triangle-free graphs with minimum degree at least $5/14$. In particular, our proof uses structural characterization of these graphs found by Jin, Chen and Koh in [1,6]. To state their result we need to define graph homomorphism and a special family of graphs.

A graph $G$ is called *homomorphic* to a graph $H$ (or $H$*-type)* if there exists a mapping $\varphi : V(G) \to V(H)$, called a *homomorphism*, such that for any two vertices $u, v$ in $G$, if $\{u, v\}$ is an edge in $G$, then $\{\varphi(u), \varphi(v)\}$ is an edge in $H$. In this paper if $G$ is homomorphic to a graph $H$, we simply say that there is a homomorphism from $G$ to $H$.

Suppose $\varphi$ is a surjective homomorphism from graph $G$ to a graph $H$ and let $\omega$ be any weight function defined on the vertices of the graph $G$, then we define the corresponding weight function $\omega_\varphi$ on $H$ in the following way. For every vertex $v \in V(H)$, let

$$\omega_\varphi(v) = \omega(\varphi^{-1}(v)) = \sum_{u \in \varphi^{-1}(v)} \omega(u).$$

The next Lemma shows that a sparse half in a homomorphic image of the graph $G$ can be lifted to a sparse half in the graph $G$.

**Lemma 2.2** *Let $\varphi$ be a surjective homomorphism from a graph $G$ to some graph $H$. Then for any weight function $\omega$, if $(H, \omega_\varphi)$ has a sparse half, then so does $(G, \omega)$.*

Now let us introduce the family of the graphs that play a key role in our results. Denote by $F_d$, for every integer $d \geq 1$, the graph on vertex set

$$V(F_d) = \{v_1, v_2, ..., v_{3d-1}\},$$

where the vertex $v_j$ has neighbors $v_{j+d}, \ldots, v_{j+2d-1}$, these values taken modulo $3d - 1$.

In his paper [6] Jin showed that every triangle-free graph $G$ of order $n$ with minimum degree $\delta(G) > 10n/29$ is of $F_9$-type and hence 3-colorable. In [1] Chen, Jin and Koh proved that every triangle-free graph of order $n$, with chromatic number $\chi(G) \leq 3$ and minimum degree $\delta > \left\lfloor \frac{(d+1)n}{3d+2} \right\rfloor$ is of $F_d$-type. Therefore, the following Theorem is true.

**Theorem 2.3 (Chen, Jin, Koh [1], [6])** *All triangle-free graphs of order $n$ with minimum degree at least $\frac{5}{14}n$ are of $F_4$-type.*

Now by combining Lemma 2.1, Lemma 2.2 and the Theorem 2.3, one can see that to prove that any graph $G$ of order $n$ and with minimum degree at least $5n/14$ has a sparse half it is enough to prove that the weighted graph $(F_d, \omega_{u\varphi})$ has a sparse half, where $\varphi$ is any homomorphism from graph $G$ to graph $F_d$, $1 \leq d \leq 4$, and $\omega_u$ is the uniform weight defined on graph $G$.

To this end, we show that every weighted graph $(F_d, \omega)$, $1 \leq d \leq 4$, with minimum degree at least $5/14$, has a sparse halve for any weight function $\omega$.

And from that to derive Theorem 1.2 we use the following Lemma.

**Lemma 2.4** *Let $\varphi$ be a homomorphism from graph $G$ to $F_d$, $d \geq 2$. Then either $\varphi$ is surjective or there exists a homomorphism $\varphi'$ from $G$ to $F_{d-1}$.*

Lemma 2.4 simplifies our previous argument, in the sense that for any graph $G$ of order $n$ and with minimum degree at least $5n/14$, we know there exists a surjective homomorphism to some $F_d$, for some $1 \leq d \leq 4$ and even more, since the homomorphism is surjective, this means that the corresponding $F_d$ also has a minimum degree at least $5/14$.

In the next section we present the sketch of the proof that the weighted graph $(F_d, \omega)$, for each $1 \leq d \leq 4$, with minimum degree at least $5/14$, has a sparse halve for any positive weight function $\omega$. In particular $(F_d, \omega_{u_\varphi})$ has a sparse half, where $\varphi$ is any homomorphism from graph $G$ to graph $F_d$, $1 \leq d \leq 4$, and $\omega_u$ is the uniform weight defined on graph $G$. And this by Lemma 2.1 and Lemma 2.2 implies that graph $G$ has a sparse half, hence the Theorem 1.3 follows.

## 3 Triangle-free graphs with minimum degree at least $\frac{5}{14}n$

**Theorem 3.1** *Let $(F_i, \omega)$ be a weighted graph with $\delta_\omega \geq 5/14$ and $1 \leq i \leq 4$. Then there exists a sparse half in $(F_i, \omega)$.*

**Proof.** [Sketch of the proof] The argument is separated into cases based on the value of $i$. Suppose $V(F_1) = \{v_1, v_2\}$ then since $\omega(v_1) + \omega(v_2) = 1$, either $\omega(v_1) \geq \frac{1}{2}$ or $\omega(v_2) \geq \frac{1}{2}$, therefore $v_1$ or $v_2$ supports a sparse half in $(F_1, \omega)$.

The proof for $i = 2$ is also simple. Suppose we are given some positive weight function $\omega$ on the graph $F_2$. If any two consecutive vertices together have total weight at least $1/2$, then they induce an independent set and we are done. Now suppose that none of them do so. Then any three consecutive vertices have total weight more than $1/2$.

We define the following halves $s_i$ for each $i = 1, 2, \ldots, 5$ on the vertices of the graph and prove that there is at least one sparse half among them.

$$\begin{aligned} s_i(v_i) &= \omega(v_i) \\ s_i(v_{i+1}) &= \omega(v_{i+1}) \\ s_i(v_{i+2}) &= \frac{1}{2} - (\omega(v_i) + \omega(v_{i+1})) \\ s_i(v_{i+3}) &= s_i(v_{i+4}) = 0 \end{aligned}$$

Suppose that the claim is false and none of these functions is a sparse half. Then for each $i = 1, 2 \dots, 5$ we have

$$\omega(v_i)\left(1/2 - (\omega(v_i) + \omega(v_{i+1}))\right) > 1/50$$

and by summing up over all $i = 1, \dots, 5$, we get that

$$1/2 - \sum_{i=1}^{5} \omega(v_i)\left(\omega(v_i) + \omega(v_{i+1})\right) > 1/10. \tag{1}$$

To evaluate the minimum value of the expression above, note that

$$\sum_{i=1}^{5} \omega(v_i)\left(\omega(v_i) + \omega(v_{i+1})\right) = \frac{1}{2}\sum_{i=1}^{5}\left(\omega(v_i) + \omega(v_{i+1})\right)^2 \geq \frac{1}{2} \cdot 5 \cdot \frac{4}{25} = \frac{2}{5},$$

by convexity of function $f(t) = t^2$ and because $\sum_{i=1}^{5}\left(\omega(v_i) + \omega(v_{i+1})\right) = 2$. Therefore, the maximum value of the expression in inequality (1) is $1/10$, a contradiction. Hence, one of the functions $s_i$ is a sparse half.

Note that the proof above did not use the minimum degree condition, while we do need that condition in the other two cases.

In case of $i = 3, 4$ the techniques used are similar to the case of $i = 2$, although the calculations are more involved and, while we continue using averaging techniques, direct averaging is not sufficient. In both cases we define halves mainly using some set of consecutive vertices and then averaging over all possible halves, we prove that there is a sparse one. □

## 4 Triangle-free graphs with at least $(1/5 - \epsilon)n^2$ edges

In order to establish the conjecture for the triangle-free graphs with average degree $\left(\frac{2}{5} - \epsilon\right)n$ we separate the cases when the graphs under consideration are not 'close' to the 5-cycle and for those that are. In the first case, we use the result of Sudakov and Keevash [7] that can be rephrased in the following way.

**Theorem 4.1 (P. Keevash, B. Sudakov, [7])** *For every triangle-free graph $G$ of order $n$ if one of these conditions hold*

- $\frac{1}{n}\sum_{v \in V(G)} d^2(v) \geq \left(\frac{2}{5}n\right)^2$ *and* $\Delta(G) < \left(\frac{2}{5} + \frac{1}{135}\right)n$
- $\Delta(G) \geq \left(\frac{2}{5} + \frac{1}{135}\right)n$ *and* $\frac{1}{n}\sum_{v \in V(G)} d(v) \geq \left(\frac{2}{5} - \frac{1}{125}\right)n$,

*then there is a sparse half in it.*

In the second case, we need to introduce the definitions reflecting the notion of being 'close' to a specific graph.

For a weighted graph $(G, \omega)$ we call a distribution $\mathbf{s}$ defined on the set

$$\mathcal{S} = \{s : V(G) \to \mathbb{R}^+, s \text{ is a half of } G\}$$

a *c-uniform sparse half* for some $0 \leq c \leq 1$, if

(i) $\mathbb{E}[\mathbf{s}(e)] \geq c \cdot \omega(e)$ for every edge $e \in E(G)$

(ii) $\mathbb{E}[\mathbf{s}(E(G))] \leq \frac{1}{50}$.

Given a graph $G$ and some subset $F \subseteq E(G)$ of edges, we call $D \subseteq V(G)$ to be $\epsilon$*-controlling* set for $F$, if $|D| \leq \epsilon |V(G)|$ and every edge in $F$ has at least one end in $D$.

We say that the graph $G'$ is an $\epsilon$*-disturbed* graph of $G$ for some $0 < \epsilon < 1$, if the following conditions hold:

(i) $V(G') = V(G)$,

(ii) there exists an $\epsilon$-controlling set for $E(G') - E(G)$,

(iii) $|N_{G'}(v) - N_G(v)| \leq \epsilon |V(G)|$ for every vertex $v \in V(G)$.

Now we are ready for our first technical result that connects $\epsilon$-disturbance and $c$-uniformity with the existence of a sparse half in the graph. We say that the graph $G$ of order $n$ is *c-maximal triangle-free* if it is triangle-free and adding any new edge to $G$ creates at least $cn$ triangles.

**Theorem 4.2** *For every $c > 0$ there exists $\epsilon = \epsilon(c) > 0$ such that if $G$ is a c-maximal triangle-free graph, has a c-uniform sparse half and $G'$ is a triangle-free $\epsilon$-disturbed graph of $G$, then $G'$ has a sparse half.*

Then the Theorem1.2 is implied from the following Theorem.

**Theorem 4.3** *For every $\epsilon > 0$ there exists $\delta > 0$ such that if $G$ is a triangle-free graph which has at least $\left(\frac{1}{5} - \delta\right) n^2$ edges then either*

(i) *$G$ is an $\epsilon$-disturbed graph of a $C_5$-type graph*

(ii) *or at least $\delta n$ vertices of $G$ have degree at least $(\frac{2}{5} + \delta)n$*

(iii) *or there exists an $F \subseteq E(G)$ with $|F| \leq \epsilon n^2$ such that $G \setminus F$ is bipartite.*

This final Theorem above allows us to derive Theorem 1.2 from Theorem 4.1 and Theorem 4.2. Indeed, the proof of Theorem 4.2 can be modifed to show that every $C_5$-type graph with the number of edges 'close' to $n^2/5$ which does not satisfy the outcomes (2) and (3) has a $\frac{1}{20}$-uniform sparse half. Indeed, the following Theorem is true.

**Theorem 4.4** *Every $C_5$-type graph of order $n$ that is not bipartite and has a minimum degree at least $\left(\frac{2}{5} - 2\sqrt{\delta}\right) n$ for $\delta = 18 \cdot 10^{-5}$ has a $1/30$ - uniform sparse half.*

This shows why outcome (1) also implies that Theorem 1.2 holds. Then if outcome (2) holds then we can apply Theorem 4.1. And finally, outcome (3) of Theorem 4.3 easily implies the existence of a sparse half.

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