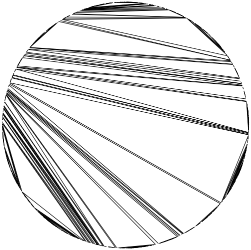
Triangulations, dual trees and fractal dimensions



Henning Sulzbach, McGill Montréal

CMS Winter Meeting, Montréal, December 2015

McGill

joint work with Nicolas Broutin (INRIA Paris, NYU Shanghai)

Outline

1. Recursive triangulations and their background

2. Main results

3. Variations of the scheme

Outline

1. **Recursive triangulations and their background**

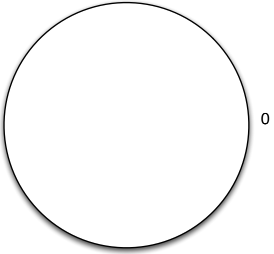
2. Main results

3. Variations of the scheme

Recursive laminations of the disk

Curien and Le Gall 2011:

In each step, connect two uniformly chosen points unless the chord intersects any previously inserted.



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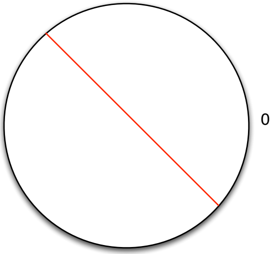

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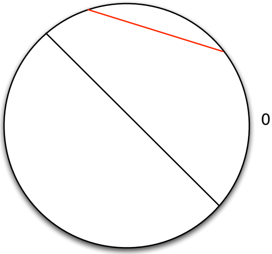

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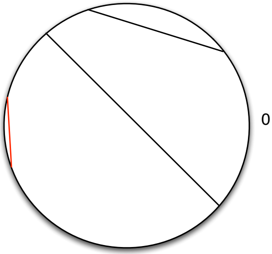

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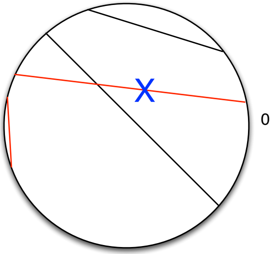

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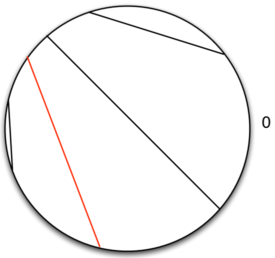

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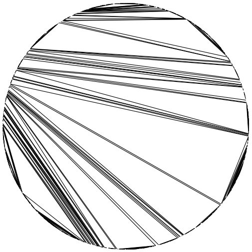
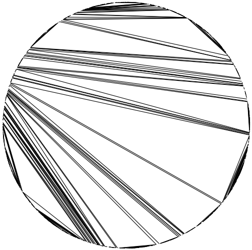


Number of inserted chords at time n is about $\sqrt{\pi n}$.

Lamination: $L_n =$ set of inserted chords at time n.

The limit triangulation

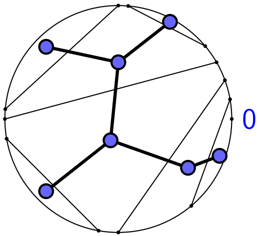


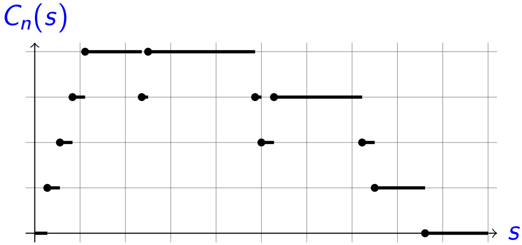
Theorem (Curien, Le Gall)

$\mathcal{L}_\infty := \overline{\bigcup_{n\geq 1} L_n}$ is a triangulation, that is, its complement consists of triangles with vertices on the circle.

Observe: Triangulations are maximal, they cannot be increased by additional chords.

The dual tree





T_n: dual tree, d_{gr}: graph distance on T_n.

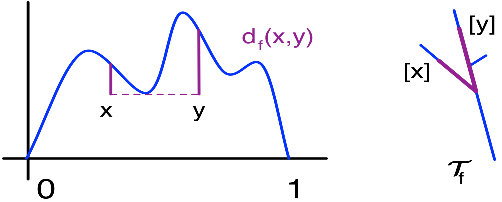
$C_n(s) =$ depth of node at $s \in [0,1]$ in T_n.

Scaling limit of the dual tree T_n ?

Scaling limit of the contour process $C_n(s)$?

Trees encoded by excursions

Let $f : [0,1] \to \mathbb{R}^+$ be a continuous excursion.



$\mathcal{T}_f = {}^{[0,1]}/_{\sim}$ where $s \sim t$ with $s \leq t$ if $d_f(s,t) = 0$ where

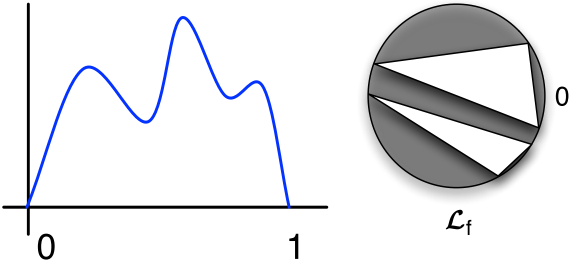
$$d_f(s,t) = f(s) + f(t) - 2\inf\{f(x) : s \leq x \leq t\}.$$

$(\mathcal{T}_f, d_f)$ is a compact tree-like metric space (an $\mathbb{R}$-tree).

Triangulations encoded by excursions

Let $f : [0,1] \to \mathbb{R}^+$ be a continuous excursion with *distinct* local minima.

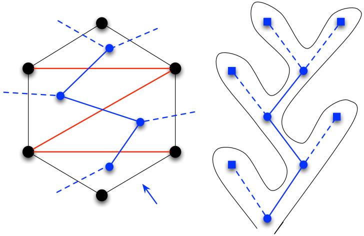
$\mathcal{L}_f$ contains chords connecting $s \leq t$ if and only if $d_f(s,t) = 0$.

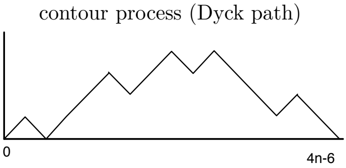


Inner nodes of $\mathcal{T}_f$ correspond to triangles in $\mathcal{L}_f$.

The Brownian world - Aldous '94

Consider uniform triangulations of the n-gon P_n:





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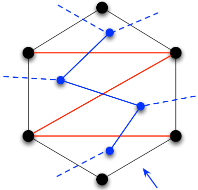
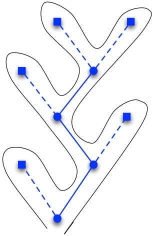
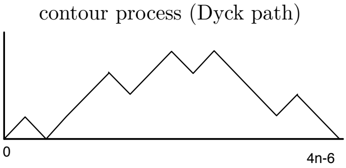
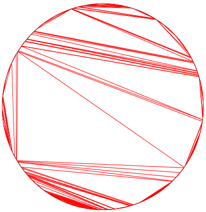
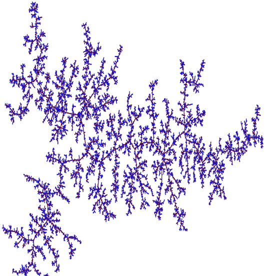


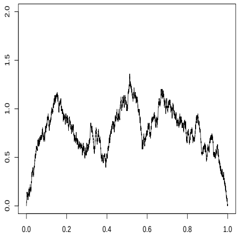




$\downarrow \delta_{\text{Haus}}$ $\downarrow d_{GH}$ $\downarrow \|\cdot\|_\infty$



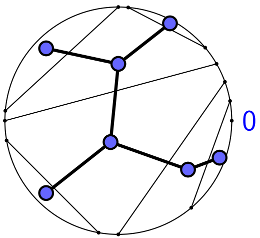


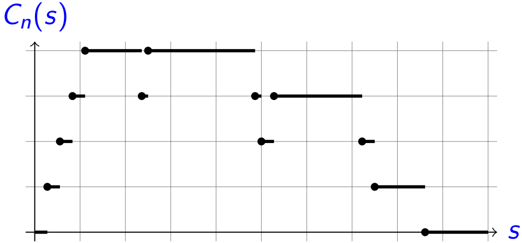


Outline

1. Recursive triangulations and their background
2. **Main results**
3. Variations of the scheme

The dual tree of the lamination





$C_n(s) =$ depth of node at $s \in [0,1]$ in T_n.

Theorem (Broutin, S. '15)

There exists a random continuous process $Z(s), s \in [0,1]$, such that, uniformly in $s \in [0,1]$, almost surely,

$$\frac{C_n(s)}{n^{\beta/2}} \to Z(s), \qquad \beta = \frac{\sqrt{17}-3}{2} = 0.561\dots$$

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$$\frac{C_n(s)}{n^{\beta/2}} \to Z(s), \qquad \beta = \frac{\sqrt{17}-3}{2} = 0.561\ldots$$

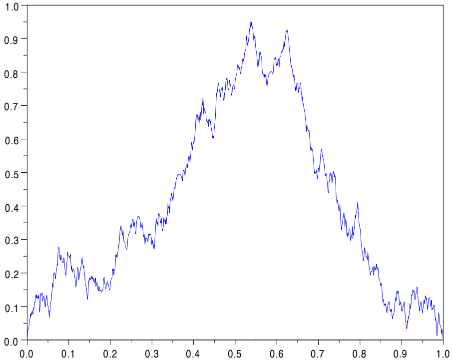
Moreover, $\mathcal{L}_\infty = \mathcal{L}_Z$ *(already proved by Curien and Le Gall).*

Almost surely,

$$(T_n, n^{-\beta/2} d_{gr}) \to (\mathcal{T}_Z, d_Z)$$

in the Gromov-Hausdorff topology on the space of (isometry classes of) compact metric spaces.

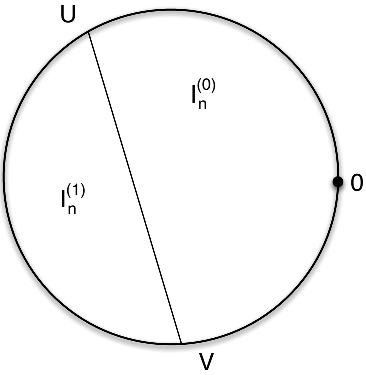
A simulation of the limit



$\mathbb{E}\left[Z(s)\right] \sim (s(1-s))^{\beta}$

Optimal Hölder exponent: $\beta = 0.561\ldots$.

Recursive decomposition

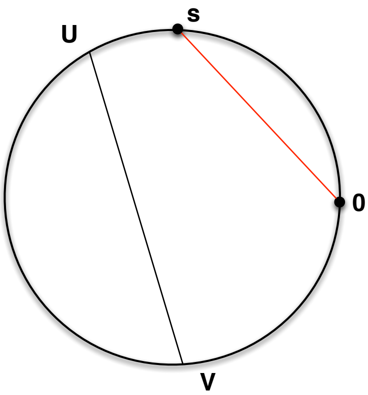


Attempted insertions in subfragments

$$\frac{I_n^{(0)}}{n} \sim (1-(V-U))^2$$

$$\frac{I_n^{(1)}}{n} \sim (V-U)^2$$

Recursive decomposition



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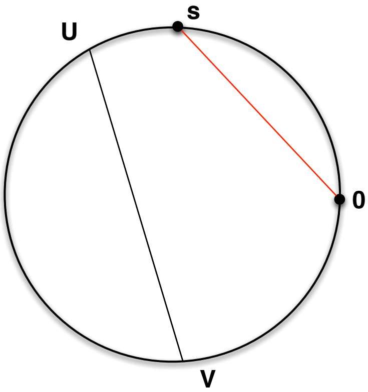
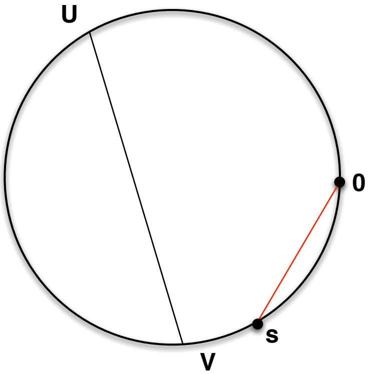


$$C_n(s) \stackrel{d}{=} \mathbf{1}_{[0,U]}(s)\, C^{(0)}_{I_0^{(n)}}\left(\frac{s}{1-(V-U)}\right)$$

Recursive decomposition



$$C_n(s) \stackrel{d}{=} \mathbf{1}_{[0,U]}(s) C^{(0)}_{I_0^{(n)}} \left(\frac{s}{1-(V-U)} \right) + \mathbf{1}_{(V,1]}(s) C^{(0)}_{I_0^{(n)}} \left(\frac{s-(V-U)}{1-(V-U)} \right)$$

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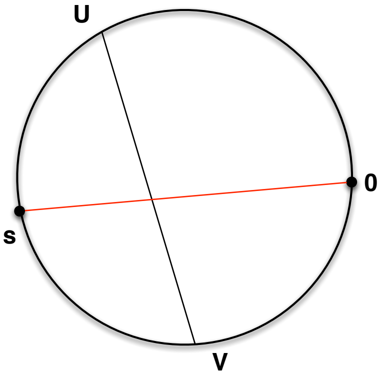

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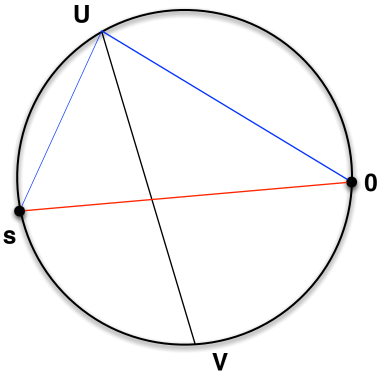
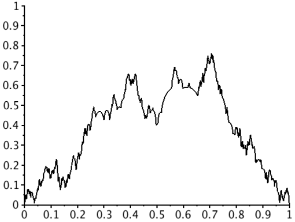


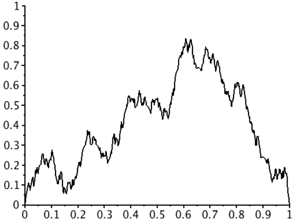
$$C_n(s) \stackrel{d}{=} \mathbf{1}_{[0,U]}(s) C^{(0)}_{I^{(n)}_0}\left(\frac{s}{1-(V-U)}\right) + \mathbf{1}_{(V,1]}(s) C^{(0)}_{I^{(n)}_0}\left(\frac{s-(V-U)}{1-(V-U)}\right)$$
$$+ \mathbf{1}_{(U,V]}(s)\left(1 + C^{(0)}_{I^{(n)}_0}\left(\frac{U}{1-(V-U)}\right) + C^{(1)}_{I^{(n)}_1}\left(\frac{s-U}{V-U}\right)\right)$$

Characterizing Z

(U, V) min and max of two ind. uniforms, here $U = 0.32, V = 0.56$



$Z^{(0)}$



$Z^{(1)}$

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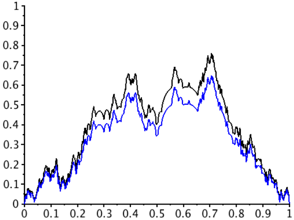
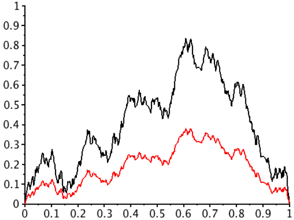


$Z^{(0)}$, $(1-(V-U))^{\beta} Z^{(0)}$



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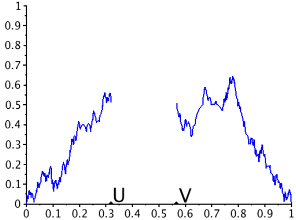

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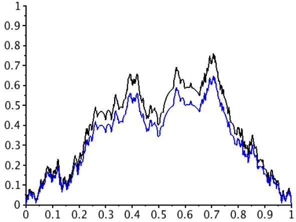

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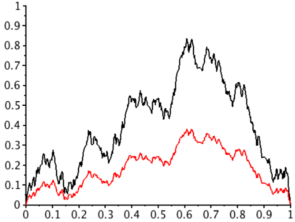
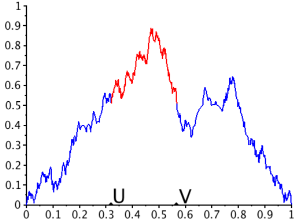


$Z^{(1)}$, $(V-U)^{\beta} Z^{(1)}$



The fractal dimension

Theorem (Broutin, S. '15)

Almost surely, we have

$$\dim(\mathcal{T}_Z) = \frac{1}{\beta} = 1.781\ldots$$

both for Minkowski and Hausdorff dimension.

Compare: $\dim(\mathcal{T}_e) = 2$ for the CRT.

Outline

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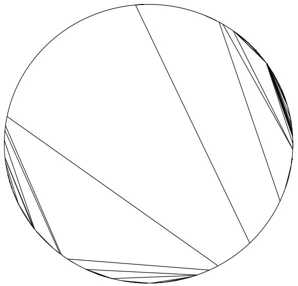
3. **Variations of the scheme**

A homogeneous model

In each step

- choose one fragment uniformly at random
- insert a chord uniformly at random



Observe: $I_0^{(n)}$ is uniformly distributed (Polya urn!), hence

$$\frac{I_0^{(n)}}{n} \to W, \quad n \to \infty,$$

where W is uniform on $[0, 1]$ and independent of (U, V).

A homogeneous model

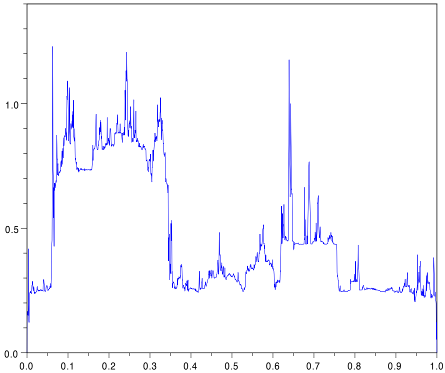
Theorem (Broutin, S. '15)

There exists a random continuous process $H(s), s \in [0,1]$, *such that, uniformly in* $s \in [0,1]$, *almost surely,*

$$\frac{C_n^h(s)}{n^{1/3}} \to H(s).$$

Moreover, $\mathbb{E}[H(s)] \sim (s(1-s))^{1/2}$.

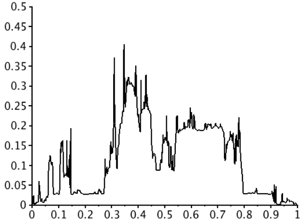
A simulation of H



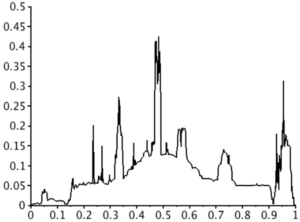
Optimal Hölder exponent: $\frac{3-2\sqrt{2}}{3} = 0.057\ldots$

The characterization of H

(U, V): as before and W another independent uniform.



$H^{(0)}$



$H^{(1)}$

The characterization of H

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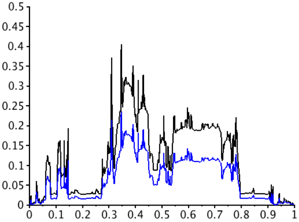
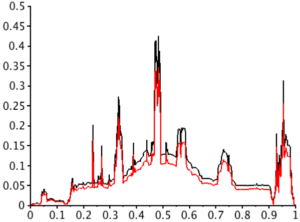


$H^{(0)}$, $(\mathbf{1}-\mathbf{W})^{1/3}H^{(0)}$



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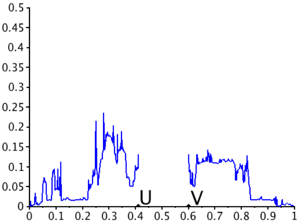

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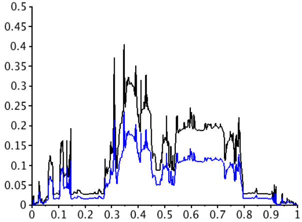

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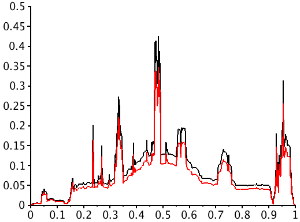
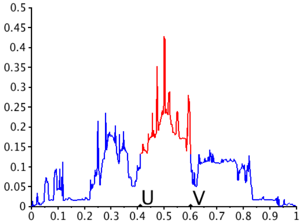


$H^{(1)}$, $\mathbf{W}^{1/3}H^{(1)}$



The fractal dimension

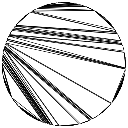
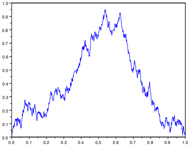
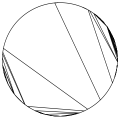
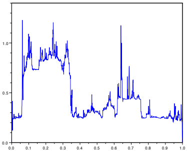
Theorem (Broutin, S.)

Almost surely,

$$\dim(\mathcal{T}_H) = 3$$

both for Minkowski and Hausdorff dimension.

Summary: Triangulations vs. trees vs. excursions

Model	Triangulation	Dual tree	Contour function
Rec.	 $\dim(\mathcal{L}_Z) = 1 + \beta$	$\dim(\mathcal{T}_Z) = 1/\beta$	 o.H.e.: β
Hom.	 $\dim(\mathcal{L}_H) = 1 + \beta$	$\dim(\mathcal{T}_H) = 3$	 o.H.e.: 0.057

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THANK YOU