COMP 760 - Assignment 1 - Due: Feb 22nd.

You are allowed to collaborate in solving these questions, but each person should write and submit her own solution.

1. Consider the function $\mathrm{DISJ}_n : \{0,1\}^n \times \{0,1\}^n \to \{0,1\}$ defined as

$$\mathrm{DISJ}_n : (S,T) \mapsto \begin{cases} 1 & S \cap T = \emptyset \\ 0 & \text{otherwise} \end{cases}$$

   where the elements of $\{0,1\}^n$ are identified with the subsets of $\{1,\ldots,n\}$ in the natural way. Show that $D(\mathrm{DISJ}_n) = n+1$.

2. Show that there is a randomized protocol with communication complexity $O(\log(n)(\log\log(n) + \log(1/\epsilon)))$ that performs the following task: Given $(x,y) \in \{0,1\}^n \times \{0,1\}^n$, it outputs the first index $i \in [n]$ such that $x_i \neq y_i$, and it has success probability at least $1-\epsilon$ if such an index exists.

3. Prove that if $f : \{0,1\}^n \times \{0,1\}^n \to \{0,1\}$ has at least k distinct "rows", then $D(f) \geq \log\log k$.

4. Identifying $\mathcal{P}(\mathbb{F}_2^n) \equiv \{0,1\}^{2^n}$, define the function $f : \{0,1\}^{2^n} \times \{0,1\}^{2^n} \to \{0,1\}$ as in the following. For subsets $A, B \subseteq \mathbb{F}_2^n$, $f(A,B) = 1$ if and only if A and B are orthogonal linear subspaces (over the field $\mathbb{F}_2$). Prove that $C^1(f) = 2^{\theta(n^2)}$ where $C^1(f)$ is the minimum number of monochromatic rectangles in a *cover* of 1-inputs.

5. Let $f : \{0,1\}^n \times \{0,1\}^n \to \{0,1\}$ and let μ and ν be two distributions on $\{0,1\}^n \times \{0,1\}^n$ such that for every combinatorial rectangle R with $\nu(R) \geq \delta$ we have

$$\mu(R \cap f^{-1}(0)) \geq (\frac{1}{2} - \kappa)\nu(R) \qquad \text{and} \qquad \mu(R \cap f^{-1}(1)) \geq (\frac{1}{2} - \kappa)\nu(R).$$

   (a) Prove that

$$R_\epsilon(f) \geq \log_2 \frac{\frac{1}{2} - \epsilon - \kappa}{\delta}.$$

(b) Show that this is a more general approach than the discrepancy bound. In other words, set ν, κ, δ appropriately so that (a) implies the bound

$$R_\epsilon(f) = \Omega\left(\log_2 \frac{1-2\epsilon}{\mathrm{disc}_\mu(f)}\right).$$