COMP 760 - Fall 2024 - Assignment 3

Due: Nov 11th, 2024

General rules: In solving these questions, you may consult the lecture notes. You can discuss high-level ideas with each other, but each student must find and write their solution.

1. Let A be a random subset of $\mathbb{Z}_2^n$ where each element is included independently with probability p. By using the material from the lecture notes, prove that the property of $A + A = \mathbb{Z}_2^n$ exhibits a sharp threshold. (First, determine the $\Theta(\cdot)$ asymptotic of the critical probability).

2. Let V be a vector space over the reals. Given a function $f : \{0,1\}^n \to V$, we can write the fourier expansion of f as $f = \sum \widehat{f}(S)\chi_S(x)$ where $\widehat{f}(S) = \mathbb{E}_{\mathbf{x}} f(\mathbf{x})\chi_S(\mathbf{x}) \in V$ for every $S \subseteq [n]$. Similarly, if $f : (X, \mu)^n \to V$, we can write the Fourier-Walsh expansion of f as $f = \sum F_S$ with $F_S : (X, \mu)^S \to V$.

   (a) For $q \geq 2$, let $\mathcal{M}_k$ be the vector space of $k \times k$ real matrices endowed with the norm

   $$\|M\|_q := \left(\sum_{i=1}^{k} \sigma_i^q \right)^{1/q},$$

   where $\sigma_1 \geq \ldots \geq \sigma_k$ are the singular values of M. Prove that for even numbers q, we have

   $$\|A + B\|_q^q + \|A - B\|_q^q \leq (\|A\|_q + \|B\|_q)^q + (\|A\|_q - \|B\|_q)^q.$$

   You may use the matrix Hölder inequality:

   $$|\operatorname{trace}(A^* B)| \leq \|A\|_p \|B\|_q,$$

   which holds if $\frac{1}{p} + \frac{1}{q} = 1$.

   (b) Use the previous part to prove that for every $f : \mathbb{Z}_2^n \to \mathcal{M}_k$ and even q, we have

   $$\left(\sum_{S \subseteq [n]} \sqrt{q-1}^k \|\widehat{f}(S)\|_q^2 \right)^{1/2} \geq \left(\mathbb{E}_{\mathbf{x}} \|f(\mathbf{x})\|_q^q \right)^{1/q}.$$

3. Let G be a finite Abelian group, and let V_k be the vector space of functions $g : G \to \mathbb{R}$ endowed with the Gowers U^k norm:

   $$\|g\|_{U^k} := \left(\mathbb{E}_{\mathbf{x}_0, \ldots, \mathbf{x}_k \in G} \prod_{S \subseteq [k]} g\left(\mathbf{x}_0 + \sum_{i \in S} \mathbf{x}_i \right) \right)^{1/2^k}$$

Interrelated applications of the Cauchy-Schwarz inequality implies the so-called Gowers-Cauchy-Schwarz inequality:

$$\left|\mathbb{E}_{\mathbf{x}_0,\ldots,\mathbf{x}_k\in G}\prod_{S\subseteq[k]} g_S\left(\mathbf{x}_0+\sum_{i\in S}\mathbf{x}_i\right)\right| \le \prod_{S\subseteq[k]} \|g_S\|_{U^k},$$

for any collection of functions $g_S : G \to \mathbb{R}$. Use

(a) Prove that for $q = 2^k$, and every $h, g : G \to \mathbb{R}$, we have

$$\|g+h\|_{U^k}^q + \|g-h\|_{U^k}^q \le (\|g\|_{U^k} + \|h\|_{U^k})^q + (\|g\|_{U^k} - \|h\|_{U^k})^q .$$

(b) Use the previous part to prove that for every $f : \mathbb{Z}_2^n \to V_k$ and $q = 2^k$, we have

$$\left(\sum_{S\subseteq[n]} \sqrt{q-1}^k \|\widehat{f}(S)\|_{U^k}^2\right)^{1/2} \ge \left(\mathbb{E}_{\mathbf{x}}\|f(\mathbf{x})\|_{U^k}^q\right)^{1/q}.$$

4. Prove the following statements about the hypercube:

(a) Prove that For all $-1 < \rho < 1$ and $0 < \mu \le 1$, there exists

$$\alpha = \alpha(\rho, \mu) > 0$$

such that for large enough n, and any subset $S \subseteq \{-1,1\}^n$ of density at least μ, we have

$$\Pr_{x,y}[x \in S, y \in S] \ge \alpha,$$

where x is chosen uniformly at random from $\{-1,1\}^n$ and y is chosen at random conditioned on $\sum_i x_i y_i = \lfloor \rho n \rfloor$.

(b) Prove that for all $0 < \delta < 1$ and $\mu_1, \mu_2 \in (0,1)$, there exists

$$\alpha = \alpha(\delta, \mu_1, \mu_2) > 0$$

such that for any two subsets $T, U \subseteq \{0,1\}^n$ of densities at least μ_1 and μ_2, we have

$$\Pr_{x,y}[x \in T, y \in U] \ge \alpha - o(1),$$

where x is chosen uniformly at random from $\{0,1\}^n$ and y is chosen at random conditioned on the assumption that $d_H(x,y)$ is either $\lfloor \delta n \rfloor$ or $\lfloor \delta n \rfloor + 1$ with equal probability.