COMP 760 - Fall 2024 - Assignment 2

Due: Oct 19th, 2024

General rules: In solving these questions, you may consult the lecture notes. You can discuss high-level ideas with each other, but each student must find and write their solution.

1. Consider a decision tree computing a Boolean function $f : \{0,1\}^n \to \{-1,1\}$. For an $x \in \{0,1\}^n$ and $i \in \{1,\dots,n\}$ define $R_i(x) = (-1)^{x_i}$ if the variable x_i is queried by the decision tree while computing $f(x)$, and define $R_i(x) = 0$ otherwise.
   (a) Prove that for every i, we have $\widehat{f}(\{i\}) = \mathbb{E} f(x) R_i(x)$, and use this to show that if f is monotone, then its total influence satisfies
   $$I_f \le \sqrt{h},$$
   where h is the height of the decision tree.
   (b) (Bonus) Prove the stronger bound that $I_f \le \sqrt{\log_2 s}$ where s is the number of leaves of the tree.

2. Given $f : \{0,1\}^n \to \{0,1\}$ and $g : \{0,1\}^m \to \{0,1\}$, define $f \circ g : \{0,1\}^{nm} \to \{0,1\}$ as
   $$f \circ g(\{x_{ij}\}_{i\in[n],j\in[m]}) := f(g(\{x_{1,j}\}_{j\in[m]}), \dots, g(\{x_{n,j}\}_{j\in[m]})).$$
   (a) Given $f : \{0,1\}^n \to \{0,1\}$ and $g : \{0,1\}^m \to \{0,1\}$ prove $\deg(f \circ g) = \deg(f)\deg(g)$ and $\mathrm{bs}(f \circ g) \ge \mathrm{bs}(f)\mathrm{bs}(g)$.
   (b) Recall that we proved $\deg(f) \ge \sqrt{\mathrm{bs}(f)/2}$. By applying this inequality to $g = f^{\circ k} = f \circ f \circ \dots \circ f$ and combining it with Part (a) conclude
   $$\deg(f) \ge \sqrt{\mathrm{bs}(f)}.$$
   Similarly argue
   $$\deg(f) \ge \sqrt{\mathrm{s}(f)}.$$

3. Let $f : \{-1,1\}^n \to [-1,1]$ be a function of real degree d. Let $\partial_i f(x) = \frac{f(x)-f(x^{(i)})}{2}$ and $s_f(x) = \sum_{i=1}^n |\partial_i f(x)|$, where $x^{(i)}$ is obtained from x by negating its i-th coordinate.
   We wish to prove that $s(f) := \max_x s_f(x) \le 2d^2$.
   (a) Explain, why it suffices to prove $s_f(\vec{1}) \le 2d^2$ for every degree d function $f : \{-1,1\}^n \to [-1,1]$.
   (b) Prove that $s_f(\vec{1}) \le 2 \max_S |\sum_{i\in S} \partial_i f(\vec{1})|$. Note that the absolute value is outside the sum.
   (c) Use Markov's inequality (about the degree and the derivative of univariate polynomials) to prove that for every $S \subseteq [n]$,
   $$|\sum_{i\in S} \partial_i f(\vec{1})| \le d^2$$

4. For $k \le n/2$, let $f : \{0,1\}^n \to \{0,1\}$ be defined as $f(x) = 1$ iff $\sum x_i \le k$. Prove that

$$e^{-k}\sqrt{\binom{n}{\le k}} \le \sum_S |\widehat{f}(S)| \le \sqrt{\binom{n}{\le k}},$$

where $\binom{n}{\le k} = \sum_{i=0}^{k} \binom{n}{i}$.

5. (a) Prove that there exists a universal constant K such that for every function $g : \{0,1\}^n \to \mathbb{R}$, we have

$$\sum_{S \neq \emptyset} \frac{1}{|S|} \widehat{g}(S)^2 \le \frac{K\|g\|_2^2}{1 + \log(\|g\|_2/\|g\|_1)}.$$

   (b) Conclude that for $f : \{0,1\}^n \to \{0,1\}$, we have

$$\mathrm{Var[f]} \le K \sum_i \frac{I_i(f)}{1 + \frac{1}{2}\log I_i(f)}.$$

   (c) Conclude the KKL inequality from the above inequality.