COMP 760 - Fall 2024 - Assignment 1

Due: Sept 30th, 2024

General rules: In solving these questions, you may consult the lecture notes. You can discuss high-level ideas with each other, but each student must find and write their solution.

1. Let $g : \{0, 1, \ldots, n\} \to \{0, 1\}$, and let $f : \{0, 1\}^n \to \{0, 1\}$ be the corresponding symmetric function defined as $f(x) \coloneqq g(x_1 + \ldots + x_n)$. Prove that for every $S \subseteq [n]$, we have

$$|\widehat{f}(S)| \leq \frac{1}{\sqrt{\binom{n}{|S|}}}.$$

2. Let $k < \frac{N}{2}$, and let $P = \{s, s + d, \ldots, s + (k - 1)d\} \subseteq \mathbb{Z}_N$ be an arithmetic progression of length k in $\mathbb{Z}_N$.

   (a) Compute the Fourier coefficients of P. Your final formula should not involve a sum.

   (b) Prove $\max_{a \neq 0} |\widehat{P}(a)| = \Theta(k/N)$.

3. Let $H = (V, E)$ be a small undirected graph. Let $A \subseteq \mathbb{Z}_2^n$. Consider

$$t_H(A) = \mathbb{E} \prod_{(u,v) \in E} A(\mathbf{x}_u + \mathbf{x}_v),$$

   where $\{\mathbf{x}_u : u \in V\}$ are independent random variables taking values uniformly in $\mathbb{Z}_2^n$.

   In each of the following cases, express $t_H(A)$ in terms of the Fourier coefficients of A. Your formula must be as simple as possible.

   (a) H is a tree.

   (b) H is a cycle on k vertices.

   (c) H is the graph with vertex set $\{1, 2, 3, 4\}$ and edges $\{(1, 2), (1, 3), (1, 4), (3, 2), (4, 2)\}$. In this case, your final formula will involve two sums.

   (d) Similarly, for $A \subseteq \mathbb{Z}_N$, give a Fourier analytic formula for

$$t_{4\text{AP}}(A) \coloneqq \mathbb{E} A(\mathbf{x}) A(\mathbf{x} + \mathbf{y}) A(\mathbf{x} + 2\mathbf{y}) A(\mathbf{x} + 3\mathbf{y}).$$

4. Let $n = 2m$ be an even integer and let

$$A_n = \left\{ x \in \mathbb{Z}_2^n \; : \; \sum_{i=1}^{m} x_{2i-1} x_{2i} \equiv 0 \mod 2 \right\}.$$

   (a) Directly calculate all the Fourier coefficients of A_n.

   (b) What are $\widehat{A_n}(0)$ and $\max_{a \neq 0} |\widehat{A_n}(a)|$?

5. This exercise shows that for some linear patterns, the Fourier uniformity (i.e. having small non-principal Fourier coefficients) is insufficient to guarantee that a set behaves similarly to a random set in terms of the density of the pattern. Let A_n be the same as in the previous exercise.

   Prove that

$$\lim_{n\to\infty} \left| \mathbb{E}\left[\prod_{1\le i<j<k\le 6} A_n(\mathbf{x}_i+\mathbf{x}_j+\mathbf{x}_k) \right] - |\widehat{A_n}(0)|^{\binom{6}{3}} \right| \neq 0,$$

   where $\mathbf{x}_1,\ldots,\mathbf{x}_6$ are independent random variables taking values uniformly in $\mathbb{Z}_2^n$.

6. Let $c>0$ be a constant, and $\epsilon>0$ be a parameter. Let $A\subseteq \mathbb{Z}_2^n$ satisfy $\|\widehat{A}\|_1 := \sum_a |\widehat{A}(a)| \le c$. Prove that there exists an affine subspace $V\subseteq \mathbb{Z}_2^n$ of codimension $d\le c/\epsilon$ such that the following holds. If we identify $V\cong \mathbb{Z}_2^{n-d}$, then $A' := A\cap V\subseteq \mathbb{Z}_2^{n-d}$ satisfies

$$\max_{a\neq 0} |\widehat{A'}(a)| \le \epsilon.$$

7. (Moved to homework 2) Consider a decision tree computing a Boolean function $f:\{0,1\}^n\to\{-1,1\}$. For an $x\in\{0,1\}^n$ and $i\in\{1,\ldots,n\}$ define $R_i(x)=(-1)^{x_i}$ if the variable x_i is queried by the decision tree while computing $f(x)$, and define $R_i(x)=0$ otherwise.

   (a) Prove that for every i, we have $\widehat{f}(\{i\}) = \mathbb{E} f(x)R_i(x)$, and use this to show that if f is monotone, then its total influence satisfies

$$I[f]\le\sqrt{h},$$

   where h is the height of the decision tree.

   (b) (Bonus) Prove the stronger bound that $I[f]\le\sqrt{\log_2 s}$ where s is the number of leaves of the tree.