COMP 760 - Assignment 3 - Due: Mar 9th.

You are allowed to collaborate in solving these questions, but each person should write and submit her own solution.

1. Show that there is a randomized protocol with communication complexity $O(\log(n/\epsilon))$ such that $(x, y) \in \{0,1\}^n \times \{0,1\}^n$ outputs the first index $i \in [n]$ such that $x_i \neq y_i$ with success probability at least $1 - \epsilon$ if such an index exists.

2. Consider a function $f : \{0,1\}^t \to \{-1,1\}$, and let F be (n, t, f)-pattern matrix. Use Theorem 6 from Lecture 8 to prove that for every positive function $g : \{0,1\}^t \to (0, \infty)$,

$$\operatorname{rank}_{\pm}(F) \geq \frac{1}{\Pr[g(x) < 1] + \max_S |\widehat{fg}(S)| \left(\frac{t}{n}\right)^{|S|/2}}$$

3. The following well-known theorem is known as the Bernstein–Markov theorem.

Theorem 1 (Bernstein–Markov) *Let $p \colon [-1,1] \to \mathbb{R}$ be a polynomial of degree d. For every $x \in [-1,1]$,*

$$p'(x) \leq \min\left(d^2, \frac{d}{\sqrt{1-x^2}}\right) \|p\|_\infty.$$

   (a) Use the Bernstein–Markov theorem to prove that if $p : \mathbb{R} \to \mathbb{R}$ is a polynomial with the following properties

   - For every *integer* $i \in [0, n]$, we have $b_1 \leq p(i) \leq b_2$,
   - There exists a real $x \in [0, n]$ with $|p'(x)| \geq c$,

   then

$$\deg(p) \geq \sqrt{\frac{cn}{c + b_2 - b_1}}.$$

(b) Use the previous part to show that if $f : \{0,1\}^n \to \{0,1\}$ satisfies $f(\vec{0}) = 0$ and $f(\mathrm{e}_i) = 1$ for every $i \in [1, n]$, then

$$\deg_{1/3}(f) \geq \sqrt{n/6}.$$

Show that in particular the approximate degrees of the AND function $\wedge_{i=1}^n x_i$ and the OR function $\vee_{i=1}^n x_i$ are $\Omega(\sqrt{n})$.

(c) Recall that the sensitivity of a function $f : \{0,1\}^n \to \{0,1\}$ denoted by $s(f)$ is $\max_x |\{x : f(x \oplus \mathrm{e}_i) \neq f(x)\}|$. Show that

$$\deg_{1/3}(f) \geq \Omega(\sqrt{s(f)}).$$

4. There isn't really a notion of mutual information common to three random variables. Here is one attempt at a definition: Using Venn diagrams, we can see that the mutual information common to three random variables X, Y, and Z can be defined by

$$I(X;Y;Z) = I(X;Y) - I(X;Y|Z).$$

The quantity is symmetric in X, Y, and Z despite the preceding asymmetric definition (this fact will follow from the identities below). Unfortunately, $I(X;Y;Z)$ is not necessarily non-negative. Find X, Y, and Z such that $I(X;Y;Z) < 0$, and prove the following two identities.

(a) $I(X;Y;Z) = H(XYZ) - H(X) - H(Y) - H(Z) + I(X;Y) + I(Y;Z) + I(Z;X)$.

(b) $I(X;Y;Z) = H(XYZ) - H(XY) - H(YZ) - H(ZX) + H(X) + H(Y) + H(Z)$.

5. Let X, Y, Z be three Bernoulli random variables with parameter $\frac{1}{2}$ that are pairwise independent: $I(X;Y) = I(Y;Z) = I(Z;X) = 0$.

(a) Under this constraint, what is the minimum value for $H(XYZ)$?

(b) Give an example achieving this minimum.

(c) What would the minimum value for $H(XYZ)$ be if the constraint above was replaced with $I(X;Y) = I(Y;Z) = I(Z;X) = \alpha$, for some $\alpha \in [0,1]$?

(d) Show (by giving an example, or otherwise) that your bound from the previous part of the question is tight

6. Show that if μ is a product distribution (i.e X and Y are indpendent when $(X,Y) \sim \mu$), then $\mathrm{IC}_{\mathrm{ext}}(\pi, \mu) = \mathrm{IC}(\pi, \mu)$.