COMP 760 - Assignment 1 - Due: Jan 30th.

You are allowed to collaborate in solving these questions, but each person should write and submit her own solution.

1. Consider the function $\mathrm{DISJ}_n : \{0,1\}^n \times \{0,1\}^n \to \{0,1\}$ defined as

$$\mathrm{DISJ}_n : (S,T) \mapsto \begin{cases} 1 & S \cap T = \emptyset \\ 0 & \text{otherwise} \end{cases}$$

   where the elements of $\{0,1\}^n$ are identified with the subsets of $\{1,\ldots,n\}$ in the natural way. Show that $D(\mathrm{DISJ}_n) = n+1$.

2. Consider the function $\mathrm{GTE}_n : \{0,1\}^n \times \{0,1\}^n \to \{0,1\}$ defined as

$$\mathrm{GTE}_n : (x,y) \mapsto \begin{cases} 1 & x \geq y \\ 0 & \text{otherwise} \end{cases}$$

   where in $x \geq y$, the two strings x and y are interpreted as binary expansions of numbers in $[0, 2^{n+1}-1]$. Show that $D(\mathrm{GTE}_n) = n+1$. Show that for every $\epsilon > 0$, we have $R_\epsilon(\mathrm{GTE}_n) = O(\log^2 n)$.

3. Prove that if $f : \{0,1\}^n \times \{0,1\}^n \to \{0,1\}$ has at least k distinct "rows", then $D(f) \geq \log\log k$.

4. Prove that for sufficiently large n, for more than 99% of the communication problems $f : \{0,1\}^n \times \{0,1\}^n \to \{0,1\}$, the characteristic matrix has rank at least $2^n - O(1)$ over the reals. In particular, the rank technique gives the tight lower-bound of $n - O(1)$ for the deterministic communication complexity of random functions.

5. Identifying $\mathcal{P}(\mathbb{F}_2^n) \equiv \{0,1\}^{2^n}$, define the function $f : \{0,1\}^{2^n} \times \{0,1\}^{2^n} \to \{0,1\}$ as in the following. For subsets $A, B \subseteq \mathbb{F}_2^n$, $f(A,B) = 1$ if and only if A and B are orthogonal linear subspaces (over the field $\mathbb{F}_2$). Prove that $C^1(f) = 2^{\theta(n^2)}$ where $C^1(f)$ is the minimum number of monochromatic rectangles in a *cover* of 1-inputs.