Harmonic Analysis of Boolean Functions

- **Lectures:** MW 10:05-11:25 in MC 103
- **Office Hours:** MC 328, by appointment (hatami@cs.mcgill.ca).
- **Evaluation:**
  - Assignments: 60 %
  - Presenting a paper at the end of the term: 20 %
  - Scribing two lectures: 15 %
  - Attending lectures: 5 %

Tentative plan

- Overview:
  - Basic functional analysis.
  - Basic Fourier analysis of discrete Abelian groups.

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  - Basic Fourier analysis of discrete Abelian groups.
- Mathematical theory:
  - Influences, Noise operator; Discrete Log-Sobolov inequalities, Hyper-contractivity, Threshold Phenomena, Noise sensitivity, etc.
- Applications to computer science:
  - Property testing.
  - Machine Learning.
  - Circuit Complexity.
  - Communication complexity.

What are we going to study?

Boolean Functions

$$f : \{0,1\}^n \to \{0,1\}.$$

What is Harmonic Analysis of Boolean Functions?

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Focuses on the quantitative properties of functions, and how these quantitative properties change when apply various (often quite explicit) operators.

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Fourier analysis

Studies functions by decomposing them into a linear combination of "symmetric" functions. These symmetric functions are usually explicit, and are often associated with physical concepts such as frequency or energy.

Examples I: Circuit Complexity

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Theorem (Linial, Mansour, Nisan 1993)

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- **Corollary**: Parity cannot be computed with a small circuit.

Examples II: Influences

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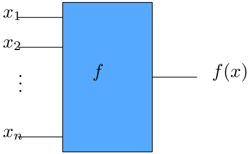

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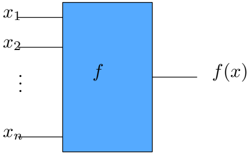
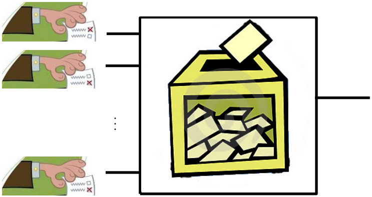


- Think of them as voting systems.



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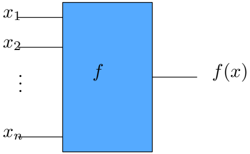

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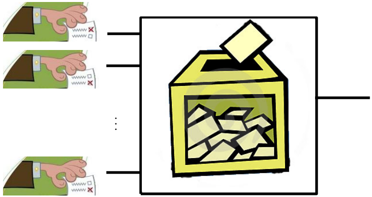

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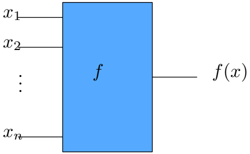

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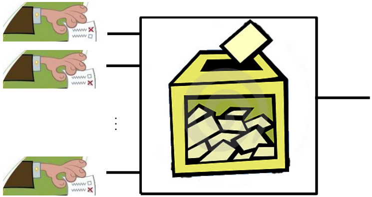

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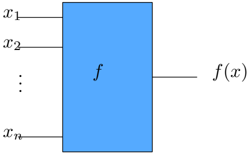

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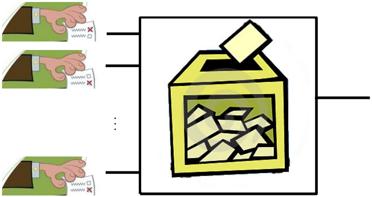

  - Two candidates 0 and 1.
  - Everybody votes 0 or 1.
  - f determines the winner.

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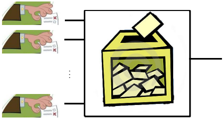

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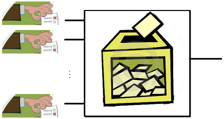

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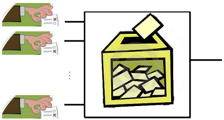

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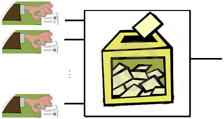

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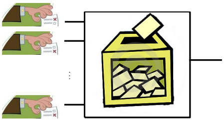

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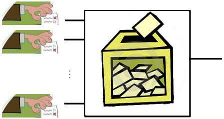

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$$I_i = \frac{\text{scenarios that } x_i \text{ mattered}}{2^n}$$

$$I_1 = \frac{6}{8}$$

$$I_3 = \frac{2}{8}$$

Definition (In Probabilistic Language:)

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- $I_i =$ Pr[i-th voter can change the outcome].

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$f : \{0,1\}^n \to \{0,1\}$ with minimum total influence?

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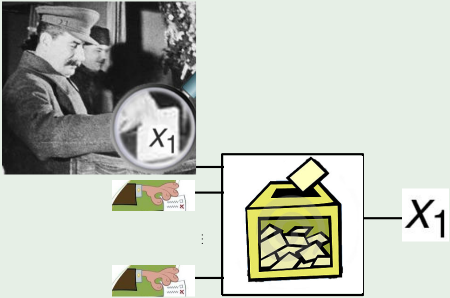
Balanced $f : \{0,1\}^n \to \{0,1\}$ with minimum total influence?

Example (Dictatorship)



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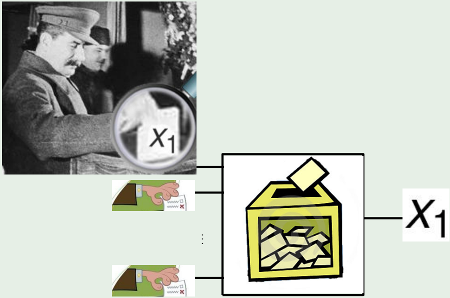

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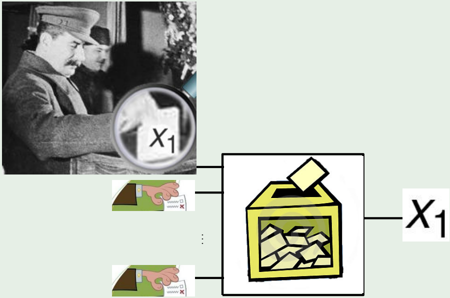

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- $I_1 = 1,\ I_2 = I_3 = \ldots = I_n = 0.$

More examples of Influences

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Question

Which functions have constant $O(1)$ total influence?

Junta session one week after the 1973 coup in Chile.

Definition (Junta)

There is a small set of voters $\{i_1, \ldots, i_k\}$ who decide the election

$$f(x) := g(x_{i_1}, \ldots, x_{i_k}).$$

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- Proof is based on the proof of the KKL inequality.

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- It is based on hyper-contractivity, a phenomenon in harmonic analysis.
- It introduced this tool to the community of computer science and combinatorics.

Example III: Phase Transitions

Erdös-Rényi graph

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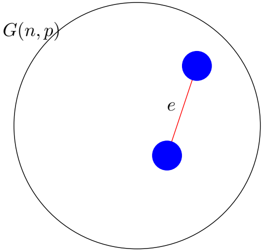
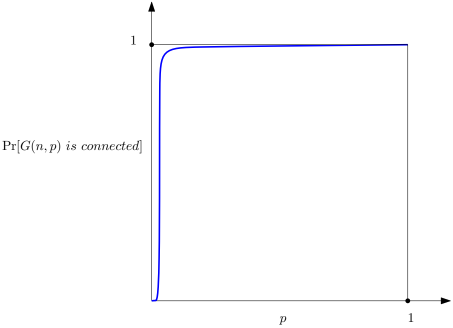
- every edge is present independently with probability p.



$$\Pr[e \in G(n,p)] = p$$

Thresholds

They observed that some fundamental graph properties such as connectivity exhibit a threshold as p increases.



Thresholds

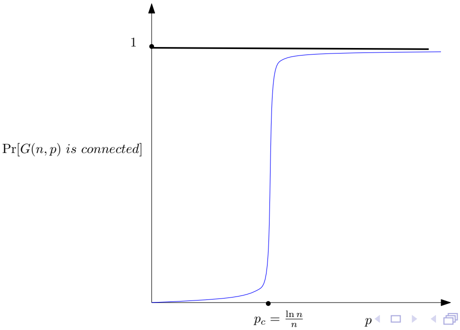
This is an instance of the phenomenon of phase transition in statistical physics which explains the rapid change of behavior in many physical processes.

The critical probability

Definition

Let $f : \{0,1\}^n \to \{0,1\}$ be an increasing function. The critical probability p_c is defined as

$$\Pr_{x \sim \mu_{p_c}} [f(x) = 1] = 1/2.$$



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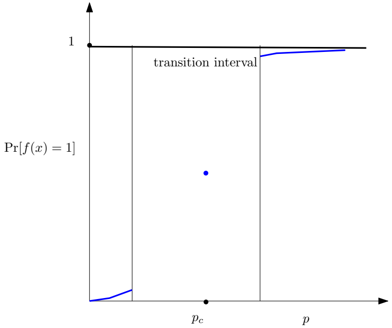
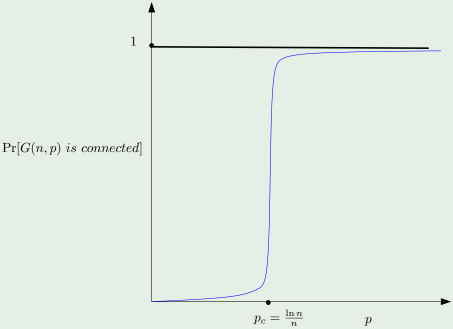


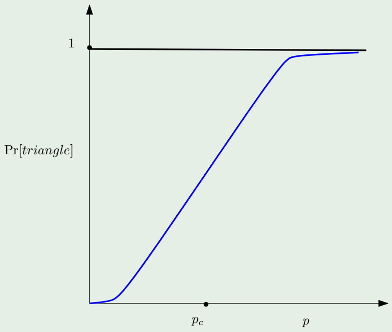
Example

Connectivity:



Example

Containing a triangle:



sharpness of threshold

One of the main questions that arises in studying phase transitions is:

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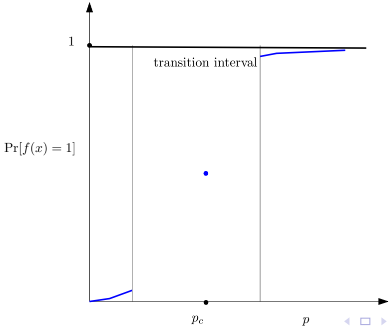
- “How sharp is the threshold?”
- That is how short is the interval in which the transition occurs.



Theorem (Recall Bollobás-Thomason)

Every increasing function $f : \{0,1\}^n \to \{0,1\}$ *exhibits a threshold:*

$$\Pr_{x \sim \mu_p}[f(x) = 1] = \begin{cases} o(1) & p \ll p_c \\ 1 - o(1) & p \gg p_c. \end{cases}$$

Theorem (Recall Bollobás-Thomason)

Every increasing function $f : \{0,1\}^n \to \{0,1\}$ exhibits a threshold:

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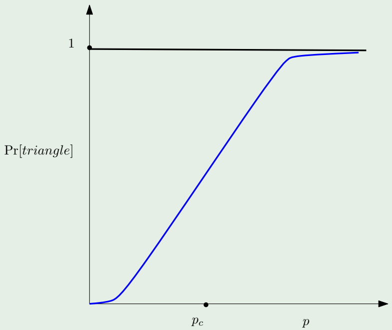
Definition (Sharp threshold)

An increasing function $f : \{0,1\}^n \to \{0,1\}$ exhibits a sharp threshold, if for all $\epsilon > 0$,

$$\Pr_{x \sim \mu_p}[f(x) = 1] = \begin{cases} o(1) & p \leq (1-\epsilon)p_c \\ 1 - o(1) & p \geq (1+\epsilon)p_c. \end{cases}$$

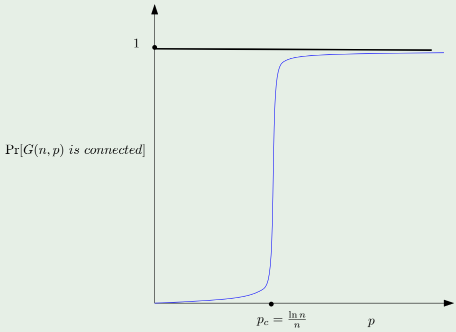
Example

Containing a triangle does not exhibit a sharp threshold.



Example

Connectivity exhibits a sharp threshold.



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Question

What can we say about $f : \{0,1\}^n \to \{0,1\}$ if it does not exhibit a sharp threshold?