COMP 760 - Fall 2011 - Assignment 2

Due: Nov 2nd, 2011

General rules: In solving this you may consult with each other.

1. This exercise shows that the uniformity is not sufficient to show that a set behaves similar to a random set in terms of the density of all linear structures. We shall construct a set A_n which is very uniform, but the density of certain structure in A_n is different from a random subset with the same density as A_n.

   (a) Show that the set
   $$A_n = \left\{ x \in \mathbb{Z}_2^{2n} \ : \ \sum_{i=1}^{n} x_{2i-1}x_{2i} \equiv 0 \mod 2 \right\}$$
   is $o_{n\to\infty}(1)$-uniform.

   (b) Show that $\lim_{n\to\infty} \frac{|A_n|}{2^{2n}} = \frac{1}{2}$.

   (c) Show that
   $$\lim_{n\to\infty} \left| \mathbb{E}\left[\prod_{1\le u<v<w\le 6} 1_{A_n}(x_u + x_v + x_w) \right] - 2^{-\binom{6}{3}} \right| \neq 0,$$
   where $x_1, \ldots, x_6$ are independent random variables taking values independently in $\mathbb{Z}_2^n$.

2. Consider a graph $H = (V, E)$ and a finite Abelian group G. Suppose that $A \subseteq G$ is δ-uniform and $\alpha := \frac{|A|}{|G|}$. Let $\{x_v : v \in V(H)\}$ be independent random variables taking values in G uniformly at random. Show that
   $$\left| \mathbb{E}\left[\prod_{uv\in E} A(x_u + x_v) \right] - \alpha^{|E(H)|} \right| = o_{\delta\to 0}(1).$$

3. Consider a function $f : \{0,1\}^n \to \{0,1\}$ and a set $T \subseteq [n]$. Prove that for every $1 < p \le 2$, and every k,
   $$\left(\sum_{\substack{S\subseteq T \\ 1\le|S|\le k}} |\widehat{f}(S)|^2 \right)^{1/2} \le 2\left(\frac{1}{\sqrt{p-1}} \right)^k \left(\sum_{S\cap T\neq\emptyset} |\widehat{f}(S)|^2 \right)^{1/p}.$$

4. Consider a function $f : \{0,1\}^n \to \{0,1\}$ and a set $T \subseteq [n]$. For every $x \in \{0,1\}^T$, let $f_x : \{0,1\}^{[n]\setminus T} \to [-1,1]$ denote the function $f_x : y \mapsto f(x,y)$. Prove that if $\deg(f) \le k$, then for every subset $\mathcal{S} \subseteq \mathcal{P}([n] \setminus T)$, we have
   $$\mathbb{E}_x\left[\max_{S\in\mathcal{S}} |\widehat{f_x}(S)| \right] \le 3^{2k} \max_{S\in\mathcal{S}} \left(\sum_{P\supseteq S} |\widehat{f}(P)|^2 \right)^{1/4}.$$

5. Let $C_1, \ldots, C_m$ be $\wedge$-clauses such that $\sum_{i=1}^m C_i \equiv 1$. Let $k > 0$ be an integer, and $f : \{0,1\}^n \to \mathbb{R}$ satisfy $f = f^{=k}$. Pick a uniform $y \in \{0,1\}^n$ and let C_i be the unique clause satisfied by y and T be the set of variables in C_i. Prove

$$\mathbb{E}_y \left(\mathbb{E}_{x_{[n]\setminus T}} [f(y_T, x_{[n]\setminus T})] \right)^2 \leq \sum_{S \subseteq [n]} \Pr\left[S \cap T \neq \emptyset\right] |\widehat{f}(S)|^2.$$