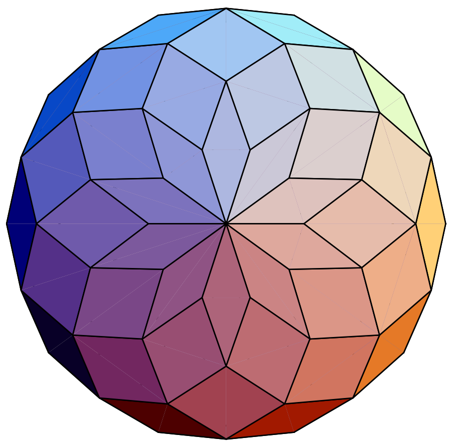
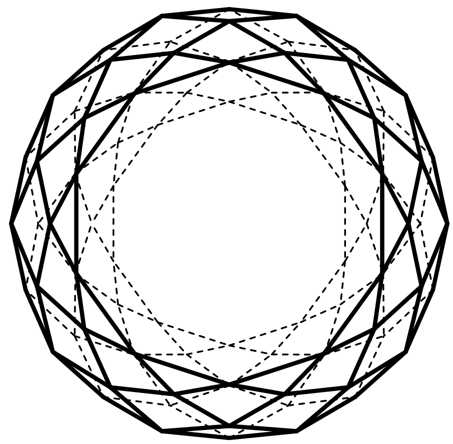
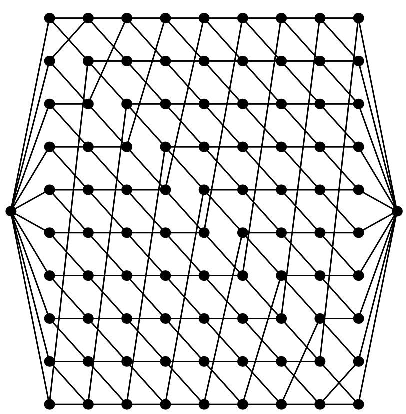
Lecture Notes on
Oriented Matroids and Geometric Computation







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Contents

0 Convex Polyhedra and Arrangements: Quick Tour

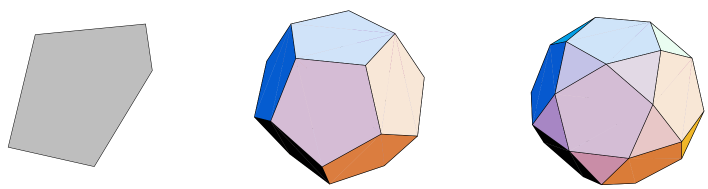
The main purpose of this section is to present some basic terminology and theorems on two important objects for our study: convex polyhedron and arrangement of hyperplanes. Those who are familiar with these objects should be able to scan through this section quickly, or even skip it and visit when statements in later sections demand clarifications of the associated terminology.

0.1 Convex Polyhedra and Faces

The intersection of a finite number of closed halfspaces in R^d is called a *convex polyhedron* or simply a *polyhedron*. Equivalently, a polyhedron is a subset P of R^d which can be represented as the solution set to a system of linear inequalities:

$$P = \{x \in R^d : Ax \leq b\}$$

where A is an $m \times d$ real matrix and b is a real m dimensional vector. Clealy a polyhedron is not necessarily bounded. A bounded polyhedron will be called a *polytope*. Here are some polytopes in the plane R^2 and the 3-space R^3.



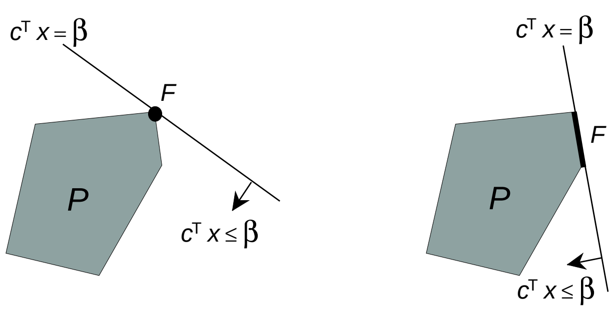
One can easily show that every polyhedron is *convex*, i.e., if x^1 and x^2 are points in a polyhedron P, then any *convex combination* of x^1 and x^2, $x = \lambda x^1 + (1-\lambda)x^2$ ($0 \leq \lambda \leq 1$), is also in P, or equivalently the whole line segment $[x^1, x^2]$ is contained in P.

There are natural substructures of a polyhedron, called faces, which are themselves polyhedra. The notion of faces plays an essential role in the combinatorial study of polyhedra. We shall now define faces and study some basic properties.

Let P be a polyhedron. For a real d-vector c and a real number β, a linear inequality $c^T x \leq \beta$ is called *valid* for P if $c^T x \leq \beta$ holds for all $x \in P$. A subset F of a polyhedron P is called a *face* of P if it is represented as

$$F = P \cap \{x : c^T x = \beta\}$$

for some valid inequality $c^T x \leq \beta$.



By this definition, the polyhedron itself and the empty set are faces of any polyhedron. But from the definition, it is not clear how many faces a polyhedron can have. It is not even clear whether the number of faces of a polyhedron is finite. The first theorem states that it is in fact finite and the faces are generated by any system of linear inequalities to represent the polytope.

We say a linear inequality system $Ax \leq b$ *represents* P if $P = \{x : Ax \leq b\}$.

Theorem 0.1. *Let $Ax \leq b$ be a linear inequality system representing a polyhedron P. Then a nonempty subset F of P is a face of P if and only if F is represented as the set of solutions to an inequality system obtained from $Ax \leq b$ by setting some of the inequalities to equalities, i.e.,*

$$F = \{x : A^1x = b^1 \text{ and } A^2x \leq b^2\},$$

where $A = \begin{bmatrix} A^1 \\ A^2 \end{bmatrix}$ *and* $b = \begin{bmatrix} b^1 \\ b^2 \end{bmatrix}$.

For our proof of Theorem 0.1, the following lemma is useful.

Lemma 0.2. *Let $Ax \leq b$ be any linear inequality system representing a polyhedron P, and let $c^Tx \leq \beta$ be a valid inequality for P such that there exists a point $x \in P$ with $c^Tx = \beta$. Define*

$$S = \{i \in \{1, 2, \ldots, m\} : x \in P \text{ and } c^Tx = \beta \text{ imply } A_ix = b_i\}.$$

Then the following statements hold.

(a) *There exists a point $x' \in P$ such that $c^Tx' = \beta$, $A_ix' = b_i$ for all $i \in S$, and $A_jx' < b_j$ for all $j \in \{1, 2, \ldots, m\} \setminus S$.*

(b) *There exists no point $x'' \in P$ such that $c^Tx'' < \beta$ and $A_ix'' = b_i$ for all $i \in S$.*

Proof. Exercise. □

Proof. (**of Theorem 0.1**) Let $Ax \leq b$ be a linear inequality system representing a polyhedron P, and let F be a nonempty subset of P.

To prove the "only if" part, we assume that F is represented by $F = \{x : A^1x = b^1 \text{ and } A^2x \leq b^2\}$ where $A = \begin{bmatrix} A^1 \\ A^2 \end{bmatrix}$ and the vector $b = \begin{bmatrix} b^1 \\ b^2 \end{bmatrix}$. Define $c = \mathbf{1}^TA^1$ and $\beta = \mathbf{1}^Tb^1$, where $\mathbf{1}$ denotes the vector of all 1's (of an appropriate size). Then one can easily verify that $F = P \cap \{x : c^Tx = \beta\}$, implying F is a face of P.

For the "if" part, we assume F is a face of P and

$$F = P \cap \{x : c^Tx = \beta\}$$

for a valid inequality $c^Tx \leq \beta$. Let

$$S = \{i \in \{1, 2, \ldots, m\} : x \in P \text{ and } c^Tx = \beta \text{ imply } A_ix = b_i\}.$$

By Lemma 0.2 (b), we have the equality

$$P \cap \{x : c^Tx = \beta\} = P \cap \{x : A_ix = b_i \quad \forall i \in S\}.$$

This implies that the face F is the set of solutions to the inequality system obtained from $Ax \leq b$ by setting the inequalities $A_ix \leq b_i$, $i \in S$ to equalities. This completes the proof. □

0.2 Theorem of Minkowski-Weyl

By definition, every convex polyhedron is represented as a finite number of linear inequalities. The well-known theorem of Minkowski and Weyl states that a convex polyhedron has another "finite" representation.

In order to state the theorem, we need some definitions. For vectors $v_1, v_2, \ldots, v_k \in R^d$, we define $conv\{v_1, v_2, \ldots, v_k\}$ as their *convex hull*:

$$conv\{v_1, v_2, \ldots, v_k\} \;=\; \{x : x = \sum_{j=1}^{k} \lambda_j v_j \;,\; \sum_{j=1}^{k} \lambda_j = 1 \text{ and } \lambda_j \geq 0 \;\forall j = 1, 2, \ldots, k\},$$

and define $nonneg\{v_1, v_2, \ldots, v_k\}$ as their *nonnegative hull*:

$$nonneg\{v_1, v_2, \ldots, v_k\} \;=\; \{x : x = \sum_{j=1}^{k} \lambda_j v_j \text{ and } \lambda_j \geq 0 \;\forall j = 1, 2, \ldots, k\}.$$

For two subsets P and Q of R^d, $P + Q$ denotes the *Minkowski sum* of P and Q:

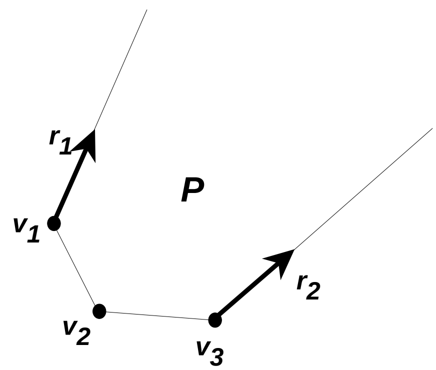
$$P + Q \quad = \quad \{p + q : p \in P \text{ and } q \in Q\}.$$

Theorem 0.3 (Minkowski-Weyl's Theorem). *For a subset P of R^d, the following statements are equivalent:*

(a) *P is a polyhedron, i.e., for some matrix A and vector b, $P = \{x : Ax \leq b\}$;*

(b) *There are vectors $v_1, v_2, \ldots, v_n$ and $r_1, r_2, \ldots, r_s$ such that $P = conv\{v_1, v_2, \ldots, v_n\} + nonneg\{r_1, r_2, \ldots, r_s\}$.*

The second representation (b) is refered as a (finite) *generator representation* or *V-representation*, while the first representation is (finite) *inequality representation* or *H-representation*. Of course, V stands for vertices and H stands for halfspaces. The essence of the theorem lies in the finiteness of generators $v_1, v_2, \ldots, v_n$ and $r_1, r_2, \ldots, r_s$: every polyhedron is finitely generated. Clearly if we allow the number of generators to be infinite in (b), the equivalence does not hold. Here is a simple illustration of the theorem:



This theorem is one of the most important theorems on convex polyhedra. The conversions of one representation, (a) or (b), to the other are fundamental transformations with many applications, while they can be considered as a constructive proof.

We shall discuss these transformations with various different approaches in detail in Section 2.1. In order to analyze the complexity of these transformations, it is essential to understand the combinatorial structure of convex polytopes.

0.3 Face Lattice of Polytopes

Let P be a polytope in R^d. The set of all faces of P, denoted by $\mathcal{F}(P)$, ordered by set inclusion is called the *face lattice* of the polytope. Clearly the least element and the greatest element of $\mathcal{F}(P)$ are $\emptyset$ and P, respectively. This set is finite by Theorem 0.1. A trivial bound for the cardinality of $\mathcal{F}(P)$ is 2^m where m is the number of inequalities to represent P. A large part of convex polytope theory has been devoted to study the set $\mathcal{F}(P)$.

In this section, we state some of the fundamental results on $\mathcal{F}(P)$ without proofs, since all the results given here can be extended to the broader context of oriented matroids, and we shall present these extensions with proofs later.

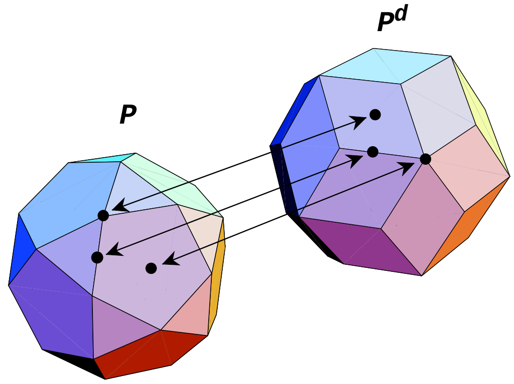
Proposition 0.4. *Let P be a convex polytope. Then its face lattice $\mathcal{F}(P)$ satisfies the* Jordan-Dedekind chain property*, i.e., every maximal chain: $F = F_1 \subset F_2 \subset \cdots \subset F_k = F'$ between any two ordered elements $F \subset F'$ in $\mathcal{F}(P)$ has the same length.*

This proposition is a corollary of a more general statement in oriented matroids, Proposition 1.2. This property implies that the *dimension* of each face F can be defined combinatorially. The *dimension* $dim(F)$ of a face F in a polytope P is one less than the length of any maximal chain from the smallest face to F. Consequently, the empty face has dimension -1. The dimension of a polytope P is the dimension of P as a face. A face of dimension i will be called i-face, and a polytope of dimension i is called a i-polytope. We shall denote by $\mathcal{F}_i(P)$ the set of all i-faces of P.

The standard definition of dimension of a face F is the maximum number of affine independent vectors in F. This is not convenient for our purpose because this cannot be extended to abstract combinatorial polytopes.

There are convenient names for faces of particular dimensions. The faces of dimension 0 are called the *vertices*, those of dimension 1 the *edges*, and those of dimension $dim(P) - 1$ the *facets*.

Theorem 0.5 (Duality of Polytopes). *For any polytope P, there exists a polytope P^d whose face lattice is isomorphic to the polar of the face lattice of P.*



Any polytope P^d satisfying the property of the theorem above is called a *dual polytope* of P. There is a natural construction of a dual polytope. Let P be a d-dimensional polytope in R^d, and assume that P contains the origin 0 in its interior. (If not, one can translate P to satisfy this property.) Then it can be proved that the following set P^*, called the *polar* of P,

$$P^* = \{y \in R^d : x^T\ y \leq 1 \text{ for all } x \in P\}$$

is a polytope, and a dual polytope of P. In fact, by using the Minkowski-Weyl's Theorem, Theorem 0.3, we know that the polytope P can be represented by the convex hull of a finite set of points as: $P = conv\{v_1, v_2, \ldots, v_n\}$. Then, it is easy to see that P^* is a polytope because it has a finite description, namely,

$$P^* = \{y \in R^d : v_i^T\ y \leq 1 \text{ for all } i = 1, 2, \ldots, n\}.$$

What remained to be proved is the existence of a one-to-one correspondence between $\mathcal{F}(P)$ and $\mathcal{F}(P^*)$ which reverses the set inclusion. We leave this as an exercise for the reader.

For a d-polytope P, we denote by $f_i(P)$ the number of i-faces of P, for each $i = -1, 0, 1, \ldots, d$, i.e., $f_i(P) = |\mathcal{F}_i(P)|$. The f-vector $f(P)$ of a d-polytope P is $(f_0(P), f_1(P), \ldots, f_{d-1}(P))$.

The following is a classical theorem:

Theorem 0.6 (The Euler-Poincaré Relation). *For any d-polytope P, the following equality holds:*

$$\sum_{i=-1}^{d} (-1)^i f_i(P) = 0.$$

There are different proofs for the theorem. For example, the book of Grünbaum [Grü67] (page 141) contains a geometric proof using induction of d. It is sometimes hard to verify geometric proofs in higher dimensions as correct statements in low dimensions may not generalize as one hopes. For a proof not depending on geometric reasoning, the axiomatic framework of oriented matroids is useful, see [CVM82]. This theorem is also a direct consequence of the shellability theorem of convex polytopes, which was first stated by L. Schläfli in 1852 and proved only recently by [BM71]. The shellability theorem has been generalized in oriented matroids by Edmonds-Fukuda-Mandel, see [Fuk82, Man82].

Another fundamental result on f-vectors of polytopes is the upper bound theorem. Consider the moment curve in R^d given by $x(t) = (t, t^2, t^3, \ldots, t^d)$. For n real numbers, $t_1 < t_2 < t_3 < \cdots < t_n$ with $n > d$, we define $C(n, d)$ be the convex hull of the n points $\{x(t_1), x(t_2), \ldots, x(t_n)\}$. This polytope, known to be the *cyclic polytope*, has been shown to have the largest number of i-faces among all d-polytopes with n vertices. This theorem, known to be the upper bound theorem for convex polytopes, was first formulated as a conjecture by Motzkin [Mot57] and proved by McMullen [McM70].

Theorem 0.7 (Upper Bound Theorem). *For any d-polytope P with n-vertices, the following inequality holds for each $-1 \le i \le d$:*

$$f_i(P) \le f_i(C(n,d)).$$

For each i, the number $f_i(C(n,d))$ can be written as a function of n and d explicitly (see, e.g. [MS71]). Among them, the most important one is for $i = d-1$,

$$f_{d-1}(C(n,d)) = \binom{n - \lfloor (d+1)/2 \rfloor}{n-d} + \binom{n - \lfloor (d+2)/2 \rfloor}{n-d},$$

which is the maximum number of facets in a d-polytope with n-vertices. Using the polarity, Theorem 0.5, this number is also the maximal number of vertices in a d-polytope with n-facets.

This number is of course an exponential function of n and d. To see how fast it can grow, we can use a symbolic computational system such as *Maple* or *Mathematica* to compute this number exactly. Here is the table of $f_{d-1}(C(n,d))$ for some "small" polytopes.

$n \setminus d$	1	2	3	4	5	6	7	8	9	10
10	2	10	16	35	42	50	40	25	10	–
20	2	20	36	170	272	800	1120	2275	2730	4004
30	2	30	56	405	702	3250	5200	17250	25300	63756
100	2	100	196	4850	9312	152000	285760	3460375	6367090	60990020

Finally we mention a couple of nice results on the graph structure of the polytope.

Proposition 0.8 (Diamond Property). *Let P be a polytope. For any two ordered faces $F \subset F'$ of P with $dim(F') = dim(F) + 2$, there are exactly two faces of P between F and F'.*

This proposition implies that in any polytope P each edge contains exactly two vertices. This defines the graph $G(P)$ of a polytope P. A graph is called *(d-)polytopal* if it is isomorphic to the graph of a (d-)polytope.

The most well-known theorem on the graph of polytopes is perhaps Steinitz' Theorem on 3-polytopes.

Theorem 0.9 (Steinitz' Theorem). *A graph G is 3-polytopal if and only if it is simple, planar and 3-connected.*

There is no known characterization of polytopal graphs in general. The following theorem of Balinski [Bal61] gives a necessary condition for a graph to be d-polytopal.

Theorem 0.10 (d-Connectivity). *The graph $G(P)$ of a d-polytope P is d-connected.*

There are recent results (by Mnëv and Richter-Gebert) implying that it is very unlikely that a simple characterization of d-polytopal graphs exists for any fixed $d \ge 4$. These suggest that we should look for a larger class of graphs that can be nicely characterized. The graphs of oriented matroid polytopes (also known as tope lattices) might be an excellent one for future investigation.

A d- polytope P is called *simple* if each vertex of P is adjacent to exactly d-vertices, and *simplicial* if its dual polytope P is simple.

Blind and Mani-Levitska [BML87] has recently shown that the graph structure of a simple polytope contains the complete information of its combinatorial structure.

Theorem 0.11. *The face lattice of any simple d-polytope is uniquely determined by its graph $G(P)$.*

See Section 0.6 for an elegant proof by Kalai [Kal88]. Unfortunately, the proof does not yield an efficient way to reconstruct the face lattice of a simple polytope from its graph. In particular, it would be very interesting if one finds an algorithm which runs in time polynomial in the size of the face lattice. The complexity of the problem is still unknown and it remains to be a challenging open problem.

0.4 Arrangements of Hyperplanes and Spheres

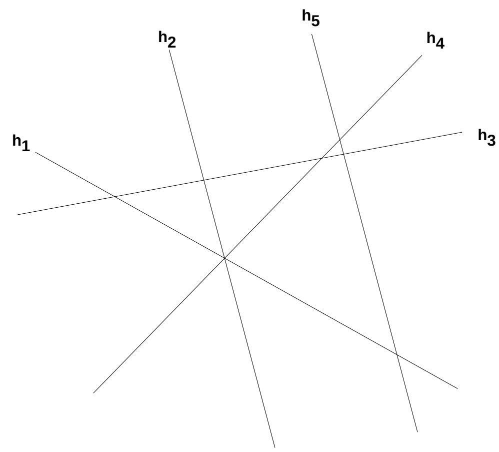
Arrangemet of Hyperplanes

A finite family $\mathcal{A} = \{h_i : i = 1, 2, \ldots, m\}$ of hyperplanes in R^d is called an *arrangement of hyperplanes*. To eliminate nonessential generalities, we always assume the following regularity assumption.

Assumption 0.12. *There is a subset of $\mathcal{A}$ whose intersection is a single point.*

If this assumption is not satisfied, the intersection of all hyperplanes is an affine space of dimension at least 1, and there is a hyperplane orthogonal to this affine space which can serve as the ground Euclidean space R^d for smaller d. The combinatorial structure of such nonregular arrangement, the notion to be defined precisely later in this section, will be isomorphic to that of a lower dimensional regular arrangement.

The following drawing illustrates an arrangement of hyperplanes in R^2, which satisfies the assumption.



A representation of an arrangement $\mathcal{A}$ is the pair (A, b) where A is a real $m \times d$ matrix, b is a real m-vector and $h_i = \{x : A_i\, x = b_i\}$ for $i = 1, 2, \ldots, m$. The arrangement represented by (A, b) is denoted by $\mathcal{A}(A, b)$.

Let $\mathcal{A}$ be an arrangement of hyperplanes. Associated with each hyperplane h_i in the arrangement, there are two open halfspaces bounded by the hyperplane, which will be denoted by h_i^+ and h_i^-. We call h_i^+ (h_i^-) the *plus side* (*minus side*) of the hyperplane. Here it is not important which side is the plus side. The only importance is to fix the assignment of $+$ and $-$. Accordingly, we shall use h_i^0 for the hyperplane itself h_i, which can be called the *zero side* of the hyperplane. When the arrangement is represented by (A, b), there is a canonical way to assign the sides as

$$h_i^+ = \{x : A_i\, x < b_i\}, \qquad h_i^- = \{x : A_i\, x > b_i\}$$

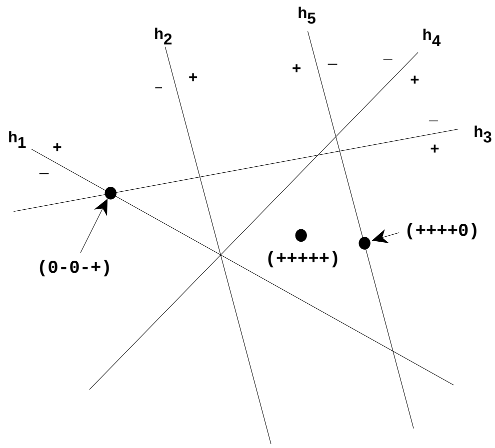
For each point $x \in R^d$, we shall associate with the sign vector $\delta(x)$ on $\{1, 2, \ldots, m\}$ defined by

$$\delta(x)_i = \begin{cases} + & \text{if } x \in h_i^+ \\ 0 & \text{if } x \in h_i^0 \\ - & \text{if } x \in h_i^- \end{cases} \qquad i \in \{1, 2, \ldots, m\}.$$

Since the vector $\delta(x)$ encodes the position of a point x with respect to each hyperplanes, it is called the *position vector* of x. We denote by $\mathcal{F}(\mathcal{A})$ the set of all position vectors, i.e.,

$$\mathcal{F}(\mathcal{A}) = \{\delta(x) : x \in R^d\}.$$

A *face* of the arrangement is a subset F of R^d which can be represented as $F = \cap_{i=1}^m h_i^{Y_i}$ for some sign vector $Y \in \mathcal{F}(\mathcal{A})$. It is easily seen that two different vectors in $\mathcal{F}(\mathcal{A})$ determine different faces. Thus we shall identify the vectors in $\mathcal{F}(\mathcal{A})$ with the faces of the arrangement. Also, it is important to note that the definition of faces does not depend on any particular assignment of "+" and "−" to the halfspaces of each hyperplane.

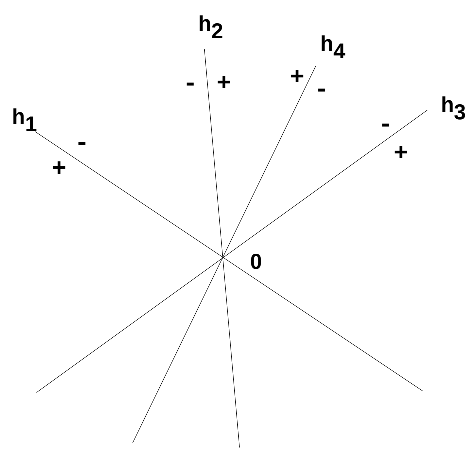


We are most interested in the geometric incidence relationship between faces of the arrangement. For this purpose, we define *facial incidence relation* on sign vectors: for two sign vectors Y and Y' in $\{+, 0, -\}^m$, Y is said to be a *face* of Y' (denoted by $Y \preceq Y'$; we also say Y *conforms to* Y') if $Y_i = Y'_i$ for all i with $Y_i \neq 0$. The poset $\mathcal{F}(\mathcal{A})$ ordered by facial relation $\preceq$ is what we call the *combinatorial structure* of the arrangement.

A better mathematical model to study the combinatorial structure of a general arrangement of hyperplanes is the central arrangement which we introduce now.

Central Arrangement of Hyperplanes and Arrangement of Spheres

An arrangement of hyperplanes in which all its hyperplanes contain the origin 0 is called a *central arrangement of hyperplanes*. By Assumption 0.12, this condition is equivalent to $\cap_{i=1}^m h_i = \{0\}$.



A central arrangement is a special type of hyperplane arrangements. However, any arrangement of hyperplanes can be considered as the intersection of a central arrangement with another hyperplane. Furthermore, central arrangements are much easier to deal with mathematically because the faces are all homogeneous cones, unlike noncentral arrangements where two different types of faces exist, bounded and unbounded. Indeed, the set of position vectors of any central arrangement $\mathcal{A}$:

$$\mathcal{F}(\mathcal{A}) = \{\delta(x) : x \in R^d\}$$

is closely related to a vector subspace. To see this, let $(A, 0)$ be a representation of a central arrangement $\mathcal{A}$, and let V be the column space of A, i.e.,

$$V = \{y \in R^m : y = Ax, x \in R^d\}$$

Note that V is a vector subspace of R^m. Now let $\sigma(V)$ be the set of the sign vectors of vectors in V. Recall that the sign vector $\sigma(y)$ of a real m-vector y is simply the vector in $\{+, 0, -\}^m$ with $\sigma(y)_j = sign(y_j)$.

Observing that $\delta(x) = \sigma(Ax)$ for $x \in R^d$, we have

$$\mathcal{F}(\mathcal{A}) = \sigma(V).$$

Consequently the combinatorial structure of a central arrangement is "merely" the sign patterns of vectors in a vector subspace, and vice versa.

We shall see that oriented matroid is an abstraction of sets of form $\sigma(V)$. Therefore, any general statement on oriented matroids can be interpreted as a general statement on the combinatorial structure of vector subspaces and of central arrangements. Furthermore, it appears that the axioms of oriented matroids captures much of the combinatorial structure of arrangements if it is not most of it. This fact will be verified later in the lecture notes.

Finally, we define an *arrangement of spheres* on the unit d-sphere S^d (in R^{d+1}) is a set $\mathcal{A} = \{s_i : i = 1, 2, \ldots, m\}$ of unit $(d-1)$-spheres on S^d. An arrangement of spheres can be considered as the cut section of a central arrangement in R^{d+1} with the unit sphere S^d. A *representation* of an arrangement of spheres is simply a representation $(A, 0)$ of the associated central arrangement. With a representation $(A, 0)$, each sphere s_i is merely the sets

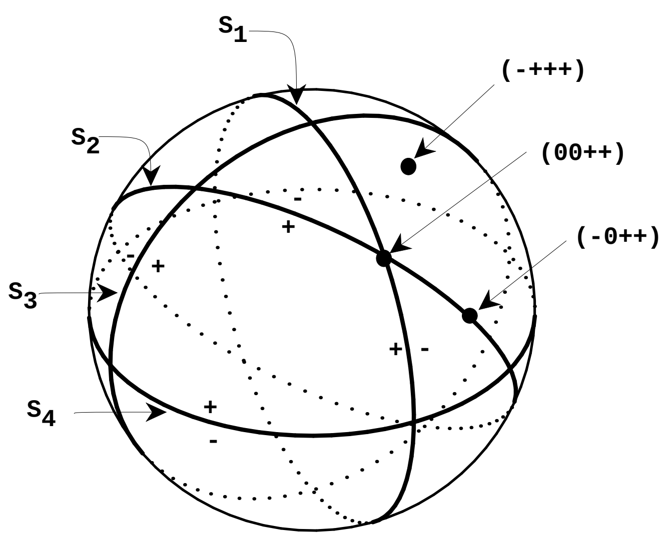
$$s_i = \{x : A_i\, x = 0\} \cap S^d$$

for each $i = 1, 2, \ldots, m$.

The regularity assumption for arrangements of spheres, corresponding to Assumption 0.12, is

Assumption 0.13. *The intersection of all spheres s_i in $\mathcal{A}$ is empty, or equivalently in terms of representation $(A, 0)$, the matrix A is row full rank.*

The faces of an arrangement of spheres are defined accordingly, i.e., a subset F of S^d is a face if it is the intersection of S^d and some face of the corresponding central arrangement. Thus, as combinatorial objects, an arrangement of spheres is equivalent to some central arrangement.



There are two reasons that we prefer arrangements of spheres to central arrangements of hyperplanes. The first reason is practical. A central arrangement of hyperplanes in R^3 is very difficult to visualize, but the corresponding arrangement of spheres in S^2 can be visualized easily as we see above. The second reason is more important: as we see in the Section 1.2, every oriented matroid can be represented by an arrangement of topological spheres, which is a generalization of arrangements of spheres. Therefore, arrangements of spheres are a more natural structure to deal with in the study of oriented matroids. We shall define this general structure later.

0.5 Polyhedral Realization of Arrangements

In this section we show that every arrangement of spheres (and hence every central arrangement of hyperplanes) is combinatorially equivalent to some convex polytope. The following figure is a simple example to illustrate this claim.

Figure 0.1: Sphere System and Its Polyhedral Realization

Let $\mathcal{A}$ be an arrangement of spheres $\{s_i : i = 1, 2, \ldots, m\}$ in S^{d-1}. Since the face poset $\mathcal{F}(\mathcal{A})$ of the arrangement does not contain the greatest element, it is convenient to use $\hat{\mathcal{F}}(\mathcal{A})$ to denote the set $\mathcal{F}(\mathcal{A})$ with the artificial largest element $\mathbf{1}$ added. We shall call this set $\hat{\mathcal{F}}(\mathcal{A})$ the *face lattice* of the arrangement A.

Now suppose that the arrangement $\mathcal{A}$ has a representation $(A, 0)$, and $s_i = \{x : A_i\, x = 0\}$. Consider the following polytope:

$$P_A = \{x : y^T\ A\ x \leq 1, \forall\, y \in \{-1, +1\}^m\}$$

Theorem 0.14. *The face lattice $\hat{\mathcal{F}}(\mathcal{A})$ is isomorphic to the face lattice of the polytope P_A.*

Observing that the face lattice of a d-dimensional polytope in R^d is isomorphic to a substructure (lower ideal) of the face lattice of an arrangement of spheres in S^d, the polyhedral representation of an arrangement is rather surprising. Each of two structures is "specializations" of the other. This two way relationship is important in the study of these fundamental structures. One simple application of the theorem is that every combinatorial theorem on general convex polytopes is true for arrangements.

We call an inequality $c^T x \leq \beta$ *facet-defining* for a polytope P if the set $P \cap \{x : c^T x = \beta\}$ is a facet of P. The key lemma for proving the theorem above is:

Lemma 0.15. *Let y be any vector in $\{-1, +1\}^m$. Then, the sign vector $\sigma(y)$ of y is in $\mathcal{F}(\mathcal{A})$ if and only if the inequality $y^T\ A\ x \leq 1$ is facet-defining for P_A.*

The polar of the polytope P_A is the polytope

$$\begin{aligned} (P_A)^* &= conv\{y^T A \in R^d : y \in \{-1, +1\}^m\} \\ &= \{y^T A \in R^d : y \in [-1, +1]^m\} \\ &= L_1 + L_2 + \cdots + L_m, \end{aligned}$$

where L_i is the line segment $[-A_i, A_i]$. This set, which is the Minkowski sum of a finite set of line segments, is known to be a *zonotope*. By the polarity of polytopes, Theorem 0.5, we have the following correspondence:

Lemma 0.16. *Let y be any vector in $\{-1, +1\}^m$. Then, the sign vector $\sigma(y)$ of y is in $\mathcal{F}(\mathcal{A})$ if and only if the vector yA is a vertex of the zonotope $(P_A)^*$.*

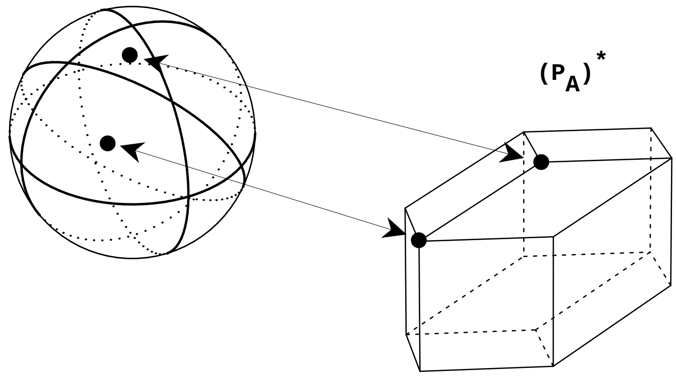


Figure 0.2: Face Correspondence: an Arrangement and its Zonotope

0.6 Construction of the Face Lattice from the Graph

For any d-polytope P, we denote by $\mathcal{F}(P)$ the face lattice of P and by $G(P)$ the graph of P. It is known that $G(P)$ does not determine $\mathcal{F}(P)$ uniquely (up to isomorphism). In fact, the complete graph K_n with n vertices with $n \geq 5$ can be realized as the graph of 4-polytopes (or more explicitely 4-dimensional cyclic polytopes), as well as the graph of $(n-1)$-dimensional simplex.

It is a quite surprising result of Blind and Mani-Levistska [BML87] that for simplex polytopes the graph contains the complete information. We recall the theorem:

Theorem 0.11. The graph of a simplex d-polytope determines its face lattice uniquely up to isomorphism.

Here we present the elegant constructive proof of this theorem due to Kalai [Kal88]. For this proof, the notion of good orientation is important.

Let P be any d-polytope. An orientation O of $G(P)$ is said to be *good* if it is acyclic and the subgraph $G(F)$ induced by each face F of P has a unique sink. Here we represent each face as the set of its vertices. Such an orientation always exists: take any linear function $c^T x$ which is not constant on any edge of P, and orient each edge so that the function increases along the direction. Such an orientation can be called an LP orientation. Furthermore, it is easy to show that

(*) there is an LP orientation with a property that a prescribed face F is a *source set*, meaning every edge between F and its complement is directed away from F.

Now we present the main lemmas.

Lemma 0.17. *Let P be a simple d-polytope with f faces and let O be any acyclic orientation of $G(P)$. Denote by h^O_k the number of vertices of indegree k, and define*

$$f^O := \sum_{k=0}^{d} 2^k \, h^O_k.$$

Then the folloing statements hold:

(a) $f \leq f^O$*;*

(b) $f = f^O$ *if and only if O is good.*

Lemma 0.18. *Let P be a simple d-polytope and let O be any good orientation of $G(P)$. If a subset F of vertices is a source set of O and $G(F)$ is k-regular then F is a k-face of P.*

Proof. (of Theorem 0.11) Let G be the graph of a simple d-polytope P. We describe how to construct all faces of P. First of all, we can generate all good orientations of G by Lemma 0.17 just with the knowlegdge of G. For each good orientation O, we list all source sets F which induce some regular subgraph. By Lemma 0.18 and (*), the generated family of sets F must coincides with the set of all faces of P. □

1 Oriented Matroids: Overview

In this section, we shall introduce the notion of oriented matroids and present many fundamental results without proofs. The reader will find proofs and detailed discussions in later sections.

1.1 Face Axioms

There are many different ways to define oriented matroids, but for our purpose the following definition is most natural.

Let E be a finite set. A *sign vector* on E is a vector in $\{+, 0, -\}^E$. We denote by $\mathbf{0}$ the sign vectors of all 0's. The negative $-Y$ of a sign vector Y is the sign vector Z on E with $Z_j = -Yj$ for all j. An index $j \in E$ is said to *separate* two sign vectors Y and Y' on E if Y and Y' have opposite signs on j, i.e. $Y_j = -Y'_j \neq 0$. The set of all elements separating Y and Y' is denoted by $D(Y, Y')$. The *composition* $Y \circ Y'$ of two sign vectors Y and Y' on E is the sign vector Z on E with

$$Z_j = \begin{cases} Y_j & \text{if } Y_j \neq 0 \\ Y'_j & \text{otherwise} \end{cases}$$

for all $j \in E$.

An *oriented matroid on* E is a pair $M = (E, \mathcal{F})$, where E is a finite set and $\mathcal{F}$ is a set of sign vectors on E satisfying the following axioms:

(F1) $\mathbf{0} \in \mathcal{F}$;

(F2) $Y \in \mathcal{F} \Longrightarrow -Y \in \mathcal{F}$; (symmetry)

(F3) $Y, Y' \in \mathcal{F} \Longrightarrow Y \circ Y' \in \mathcal{F}$; (composition)

(F4) $Y, Y' \in \mathcal{F}$ and $f \in D(Y, Y') \Longrightarrow$ there exists $Z \in \mathcal{F}$ such that $Z_f = 0$ and $Z_j = (Y \circ Y')_j$ for all $j \notin D(Y, Y')$. (strong elimination)

Axiom 1.1: Face Axioms (or Covector Axioms)

Here we call each member of $\mathcal{F}$ a *face* of the oriented matroid. The faces are sometimes called *covectors*.

The most straighforward example of an oriented matroid arises from a vector subspace V of R^E. Namely, let $\mathcal{F}$ be $\sigma(V)$, the set of sign vectors of vectors in V. Then $M = (E, \mathcal{F})$ satisfies the axioms (F1)∼(F4). Such an oriented matroid is called *representable* or *linear*. As we remarked in Section 0.4, the set $\mathcal{F}(\mathcal{A})$ of positions vectors of any central arrangement of hyperplanes satisfies the axioms as well, since it is the linear oriented matroid of the column space of A.

There are infinitely many nonlinear oriented matroids. We will see soon in this section one simple way of constructing nonlinear oriented matroids, using classical theorems on geometry such as Pappus' Theorem or Desargue's Theorem. Later in the lecture notes, we will see completely different constructions which use the idea of linear programming. The latter construction leads us to define non-Euclidean oriented matroids which constitute a "strongly non-linear" class. It appears that this class identify those non-linear oriented matroids that are badly behaving. In other words, a deeper understanding of non-Euclidean oriented matroids is necessary to resolve many open problems (to be discussed as we go along).

Abstraction is a very important element of mathematics, but in my personal opinion, should not be abused. There are good abstractions and bad abstractions. Good abstractions allows us to see more about the concrete objects we are interested in, and allows us to simplify the existing proofs, to clarify what is essential for the validity of existing theorems and eventually to discover new theorems, new constructions and new algorithms.

For us, the concrete objects of interests are convex polyhedra, arrangements of hyperplanes (or spheres), vector subspaces of R^d and optimization problems over convex polyhedra. Oriented matroids has been proven to be an extremely good abstraction for these. First of all, it has been shown over the last twenty years that oriented matroids retain much of the combinatorial properties of the concrete objects. In fact, it is very difficult to recognize whether a given oriented matroid is linear. It has been proved by Mnëv that the recognition problem is NP-hard. Secondly using oriented matroid axioms, we can prove many fundamental theorems on the concrete objects in higher dimensions rigorously, while our history witnessed many wrong intuitive proofs for higher geometry. Finally many new theorems and algorithms have been found via oriented matroids. The main purpose of the rest of the section is to present these new discoveries.

1.2 OM and Topological Representation Theorem

One of the most important theorems in oriented matroid theory is the topological representation theorem [FL78, Man82]. By this theorem, every oriented matroid can be considered as a simple topological object as well as a combinatorial object. This fact is extremely important when one tries to formulate a conjecture on oriented matroids, because the face axioms 1.1 do no give much imagination to guess what can be true for the structure.

Let S^d be the unit d-sphere in R^{d+1} and denote by $s_{(d-1)}$ the $(d-1)$-sphere $\{x \in S^d : x_1 = 0\}$ embedded in S^d. The sphere $s_{(d-1)}$ partitions S^d into three sets $\{s^+_{(d-1)}, s^0_{(d-1)}, s^-_{(d-1)}\}$ where $s^0_{(d-1)} = s_{(d-1)}$, $s^+_{(d-1)} = \{x \in S^d : x_1 > 0\}$ and $s^-_{(d-1)} = \{x \in S^d : x_1 < 0\}$.

A *d-sphere* is a topological space homeomorphic to S^d, and a *sphere* is a d-sphere for some d. A subset s of a sphere S is called a *equator* of S if there is a homeomorphism $f : S^d \to S$ such that $f(s_{(d-1)}) = s$. Every equator s of S partitions S into three sets s^+, s^0, s^- which are the images of, respectively, $s^+_{(d-1)}, s^0_{(d-1)}, s^-_{(d-1)}$ under f. The sets s^+ and s^- are called the *sides* of s.

A *sphere system* of dimenstion d is a triple $(E, S, \mathcal{A})$ where E is a finite set, S a d-sphere, and $\mathcal{A}$ is a family $\{\{s^+_i, s^0_i, s^-_i\} : i \in E\}$ satisfying the following axioms:

(S1) Each s^0_i is an equator of S whose sides are s^+_i and s^-_i, for each $i \in E$;

(S2) For each subset F of E, $\cap_{i\in F} s^0_i$ is a sphere, called a *flat*;

(S3) For each flat t and any $i \in E$, either $t \subset s^0_i$ or $t \cap s^0_i$ is an equator of t whose sides are $t \cap s^+_i$ and $t \cap s^-_i$.

Axiom 1.2: Sphere System Axioms

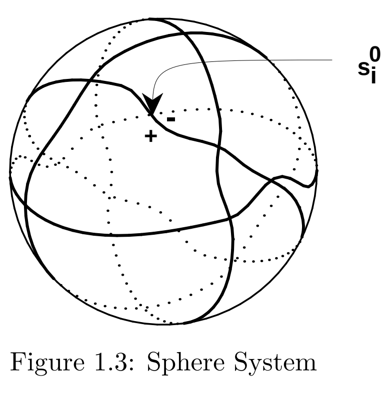


Figure 1.3: Sphere System

For a sphere system $(E, S, \mathcal{A})$, there is a natural way to define the position vector $\delta(x)$ of each point $x \in S$:

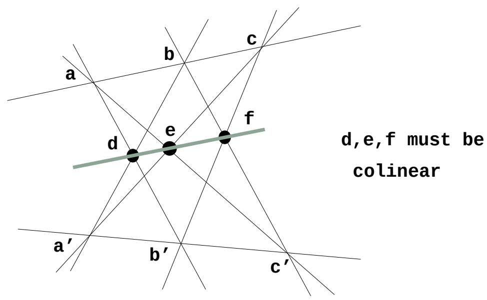
$$\delta(x)_i = \begin{cases} + & \text{if } x \in s^+_i \\ 0 & \text{if } x \in s^0_i \\ - & \text{if } x \in s^-_i \end{cases} \qquad i \in E.$$

Theorem 1.1. *Let $M = (E, \mathcal{F})$ be an oriented matroid. Then there exists a sphere system $(E, S, \mathcal{A})$ such that $\delta(S) = \mathcal{F}$. Conversely, if $(E, S, \mathcal{A})$ is a sphere system, then $(E, \delta(S))$ is an oriented matroid.*

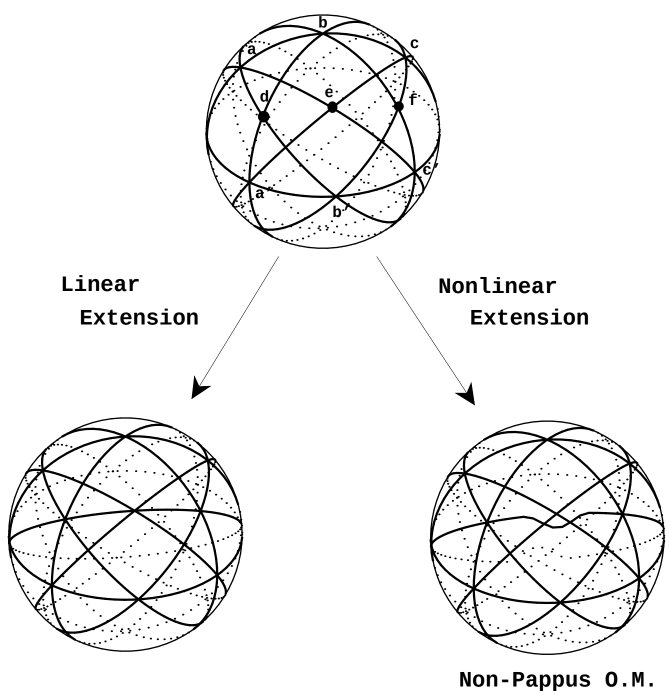
1.3 Construction of Nonlinear OMs

As we mentioned, there are infinitely many oriented matroids that are nonlinear. Several different ways to construct nonlinear oriented matroids are known. The easiest way is perhaps to use one of the well known theorems on projective geometry such as Pappus Theorem and Desargues Theorem. Another construction is to use the abstraction of linear programming: oriented matroids of rank 4 or more can be constructed so that the simplex algorithm applied to the associated program will produce a cycle of nondegenerate pivots, see [Fuk82, BVS+93]. This construction yields nonlinear OMs of special interests which are called non-Euclidean OMs. We shall study the latter construction later in the lecture.

Here we explain the main idea to construct a non-Pappus oriented matroid which has rank 3. (It is quite easy to see that every oriented matroid of rank 2 or less is always linear.) The following drawing illustrates the Pappus Theorem, which states that if three points a, b and c in the plane are colinear and other three points a', b' and c' are also colinear, then the intersection points d, e and f in the figure must be colinear.



In order to construct a nonlinear oriented matroid, we first construct a linear arrangement of 8 spheres in S^2. Then, we add one more topological 1-sphere so that it passes through the points d and f but e. The resulting sphere arrangement and the associated set of signed vectors cannot be linear because of the Pappus theorem.



1.4 Faces, Vertices and Topes

We introduce a partial order among the faces of an oriented matroid which abstracts the geometrical facial incidence relationship in arrangements of spheres.

For two sign vectors Y and Y' in $\{+,0,-\}^E$, Y is said to be a *face* of Y' (denoted by $Y \preceq Y'$) if $i \in E$ and $Y_i \neq 0$ imply $Y_i = Y'_i$. This relation is also called conformal relation and $Y \preceq Y'$ reads Y conforms to Y'. This partial order coincides with the facial incidence relation in arrangements of hyperplanes and spheres.

Let $M = (E, \mathcal{F})$ be an oriented matroid, and consider the set $\mathcal{F}$ of faces as a poset ordered by $\preceq$. Since $\mathcal{F}$ does not contain the greatest element, it is convenient to define $\hat{\mathcal{F}}$ the set $\mathcal{F} \cup \{\mathbf{1}\}$, where $\mathbf{1}$ is the artificial greatest element. Then the set $\hat{\mathcal{F}}$ is a lattice.

The following theorem allows us to define "dimension" of each face nicely, and also to define some associated graph structures.

Proposition 1.2. *Let $M = (E, \mathcal{F})$ be an oriented matroid. Then, the following statements hold:*

(a) *The lattice $\hat{\mathcal{F}}$ has the Jordan-Dedekind chain property, i.e. every maximal chain between two ordered faces Y and Y' in $\mathcal{F}$ has the same length.*

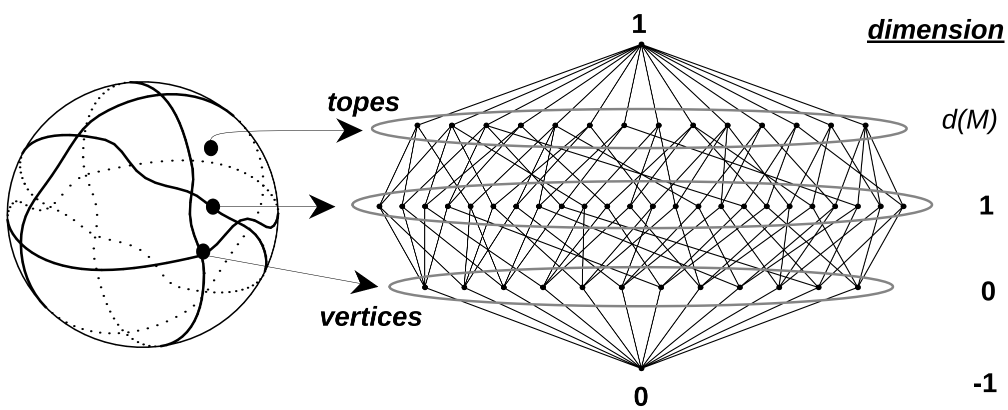
(b) *For any ordered faces Y, Y' in $\hat{\mathcal{F}}$ with distance two, there are exactly two faces between them.*

The *rank* $r(Y)$ of a face Y in an oriented matroid M is defined as the length of any maximal chain from $\mathbf{0}$ to Y. More intuitive notion is the *dimension* $d(Y)$ of a face Y which is simply one less than its rank, $d(Y) = r(Y) - 1$. The *dimension* $d(M)$ (*rank* $r(M)$) *of an oriented matroid* is defined as the dimension (rank, respectively) of any maximal face.

For a set of sign vectors $\mathcal{S}$, the set of all minimal (maximal) members of $\mathcal{S}$ is denoted by $Min(\mathcal{S})$ ($Max(\mathcal{S})$). For an oriented matroid $M = (E, \mathcal{F})$, the maximal faces of M are called the *topes* and the nonzero minimal faces are called the *vertices*. The set of topes and the set of vertices are denoted by $\mathcal{T}(M)$ and $\mathcal{V}(M)$, respectively, i.e.,

$$\begin{aligned} \mathcal{T}(M) &= Max(\mathcal{F}) \\ \mathcal{V}(M) &= Min(\mathcal{F} \setminus \{\mathbf{0}\}). \end{aligned}$$

The geometrical meaning of vertices and topes should be quite clear in an arrangement of spheres. These two subsets of $\mathcal{F}$ are both very important since they actually contain complete information of the oriented matroid.



A sphere system (= oriented matroid) and its face lattice $\hat{\mathcal{F}}$

Proposition 1.3. *Let $M = (E, \mathcal{F})$ be an oriented matroid. Then the following statements hold:*

(a) *The set $\mathcal{T} = \mathcal{T}(M)$ uniquely determines the oriented matroid by*

$$\mathcal{F} = \{Y \in \{+,0,-\}^E : Y \circ T \in \mathcal{T} \quad \forall T \in \mathcal{T}\}.$$

(b) *The set $\mathcal{V} = \mathcal{V}(M)$ uniquely determines the oriented matroid by*

$$\mathcal{F} = < \mathcal{V}, \circ >$$

where $< \mathcal{V}, \circ >$ denotes the composition closure of $\mathcal{V}$.

Now, a natural question is: can we characterize the set of vertices (or topes)? It turns out that it is easy for vertices, but we have no "simple" way to characterize the sets of topes. Here "simple" means intuitively "easy to write and check". We shall explain this more rigorously later.

For a sign vector Y on E, we denote by $\underline{Y}$ its nonzero support, i.e. $\{j \in E : Y_j \neq 0\}$.

Proposition 1.4. *Let $\mathcal{V}$ be a set of sign vectors on E. Then $\mathcal{V}$ is the set of vertices of an oriented matroid if and only if:*

(V1)	$V, V' \in \mathcal{V}$ and $\underline{V} \subseteq \underline{V} \Longrightarrow$ either $V = V'$ or $V = -V'$;	(minimality)
(V2)	$V, V' \in \mathcal{V}$, $f \in D(V, V')$ and $g \in \underline{V} \setminus D(V, V') \Longrightarrow$ there exists $W \in \mathcal{V}$ such that $W_f = 0$, $W_g = V_g$ and $W_j = V_j$, V'_j or $\mathbf{0}$ for all $j \in E$.	(vertex elimination)

Axiom 1.3: Vertex (Cocircuit, Circuit) Axioms

Those who are familiar with the notion of matroids might notice that the set of supports of vertices satisfy matroid circuit axioms. More specifically, setting

$$\mathcal{C} = \underline{\mathcal{V}} := \{\underline{V} : V \in \mathcal{V}\}.$$

Then we can easily see that the set C satisfies the matroid circuit axioms:

(M1)	$C, C' \in \mathcal{C}$ and $C \subseteq C' \Longrightarrow C = C'$;	(minimality)
(M2)	$C, C' \in \mathcal{C}$, $f \in V \cap C'$ and $g \in C \setminus C' \Longrightarrow$ there exists $D \in \mathcal{C}$ such that $f \notin D$, $g \in D$ and $D \subset C \cup C'$	(circuit elimination)

Axiom 1.4: Matroid Circuit Axioms

We shall call the pair $\underline{M} = (E, \underline{\mathcal{V}})$ the *underling matroid* of an oriented matroid $M = (E, \mathcal{F})$, and call each member a *cocircuit* of the matroid. The underlying matroid contains only a partial information of the associated oriented matroid. Yet, as we see soon, it contains essential information for determining the dimensions of faces, the number of k-dimensional faces, etc.

Now we observe some basic properties of the set of topes. For this we need one definition. Two elements f and g in E are called *parallel* in M if either $Y_f = Y_g$ for all faces $Y \in \mathcal{F}$ or $Y_f = -Y_g$ for all faces $Y \in \mathcal{F}$. The parallelness is an equivalence relation, and the set of elements parallel to an element f is denoted by $[f]$.

Proposition 1.5. *Let $\mathcal{T}$ be the set of topes of an oriented matroid $M = (E, \mathcal{F})$. Then it satisfies the following properties, known as the face axioms of* acycloids*:*

(T1)	$T \in \mathcal{T} \Longrightarrow -T \in \mathcal{T}$;	(symmetry)
(T2)	$T, T' \in \mathcal{T} \Longrightarrow \underline{T} = \underline{T'}$;	(unique support)
(T3)	$T, T' \in \mathcal{T}$ and $T \neq T' \Longrightarrow$ there exist $f \in D(T, T')$ and $U \in \mathcal{T}$ such that $U_{[f]} = -T_{[f]}$ and $U_j = T_j$ for all $j \in E \setminus [f]$.	(shelling property)

Axiom 1.5: Face Axioms of Acycloids

The first two conditions (T1) and (T2) are clearly satisfied by the set of topes. The third condion (T3) is the most important property of topes. For instance, to prove the shellabililty of tope lattices, which generalizes the shallablity of polytope face lattices due to Bruggesser and Mani [BM71], the condition (T3) plays a very important role. Here, a *tope lattice* is a lattice isomorphic to the interval $[0, T]$ for some tope T of an oriented matroid.

One natural question is whether the conditions (T1), (T2) and (T3) are sufficient to characterize the set of topes. The answer is negative. There are counterexamples on a ground set of five elements which we will present later. There are some known characterizations of topes, e.g., [BC87, Han91, dS91], but none of them yields an efficient way for verification. One open question is

Open Problem 1.1 Tope Characterization

Is there any axiomatization for the topes of an oriented matroid that can be verified in time polynomial in the size of input?

On the other hand, there is a polynomial time algorithm [FST91] to verify whether a set $\mathcal{T}$ of sign vectors on E is the set of topes of an oriented matroid. It is interesting to note that to construct a polynomial time algorithm, a new theorem on complexity of oriented matroids had to be proved. As a corollary, this theorem yields that the number of faces of any oriented matroid is bounded by a small polynomial of the number of topes.

1.5 Geometric Computation and Oriented Matroids

We have already seen that the notion of oriented matroids is a simple and straighforward abstraction of vector subspaces, arrangements and polytopes. It has been remarked that many theorems on these "concrete" objects can be generalized in the setting of oriented matroids.

In this section, we shall formulate some classical problems for geometric computation using our oriented matroid language. By doing this, it will be clear how oriented matroid theory can help to solve some geometric computation problems. Also we see how to decompose a geometric computation into numerical computation and combinatorial computation.

For a $E \times d$ matrix A, we denote by V the column space $\{Ax : x \in R^d\}$ of A. Let $\mathcal{F} = \sigma(V)$ and let M be the linear oriented matroid $(E, \mathcal{F})$ respresented by A. The sets of topes and vertices are denoted by $\mathcal{T}$ and $\mathcal{V}$.

Geometric Computation 1.1 Face Enumeration for Arrangements

Input : $A \in R^{m \times d} \Longrightarrow$ Output: $\mathcal{F}$.

Geometric Computation 1.2 Tope (or Cell) Enumeration for Arrangements

Input : $A \in R^{m \times d} \Longrightarrow$ Output: $\mathcal{T}$.

Geometric Computation 1.3 Vertex Enumeration for Arrangements

Input : $A \in R^{m \times d} \Longrightarrow$ Output: $\mathcal{V}$.

Now we list fundamental problems for polytopes. Let T be any fixed tope with a prescribed sign pattern. Without loss of generality, let $T = (+, +, \ldots, +)$. The associated *tope cell* is the interval (lower ideal) $[\mathbf{0}, T]$, which the face lattice of the polyhedral cone $P = \{x : A\,x \geq 0\}$.

Geometric Computation 1.4 Face Enumeration for Polytopes

Input : $A \in R^{m \times d} \Longrightarrow$ Output: $[\mathbf{0}, T]$.

Geometric Computation 1.5 Facet Enumeration for Polytopes

Input : $A \in R^{m \times d} \Longrightarrow$ Output: all coatoms of $[\mathbf{0}, T]$ (all faces covered by T).

Geometric Computation 1.6 Vertex Enumeration for Polytopes

Input : $A \in R^{m \times d} \Longrightarrow$ Output: all atoms (vertices) of $[\mathbf{0}, T]$.

The six problems listed above are similar in nature, that is, the size of output might be very large, meaning, might be exponential in the size of input A. In order to evaluate algorithms for these type of problems, we must introduce a different measure from the usual polynomial time algorithm. Namely, a *polynomial-time algorithm for enumeration problems* runs in polynomial in both input size and output size. An ideal algorithm is *linear-time algorithm* which is a polynomial-time algorithm that runs linear in output size.

For enumeration problems, the space requirement is also important. Clearly an algorithm which requires space polynomial in both input and output sizes is (much) less preferable to one which requires space polynomial only in input size and independent of output size. An algorithm of the latter type is called *compact*.

Consequently, it is best if one could find a compact linear-time algorithm. Among the six problems above, the problems for which a compact linear-time algorithm is known are Problem 1.2 (see [AF92b]), Problem 1.4 (see [FLM94]) and Problem1.5 (see [OSS95]). Only (non-compact) polynomial-time algorithms are known for Problem 1.1 (see [FST91]) and Problem 1.3 (see [Sey94]). No polynomial algorithm is known for 1.6. Finally, when the input is nondegenerate (i.e., the vector subspace V contains no nonzero vector with at least d zero components), both Problems 1.3 and 1.6 admit a compact linear-time algorithm, see [AF92a].

We shall present the basic ideas of the algorithms mentioned above in Section 3.

1.6 The Underlying Matroid

Let $M = (E, \mathcal{F})$ be an oriented matroid and let $\mathcal{V}$ be the set of vertices. Recall that $\underline{\mathcal{V}}$ is the set of cocircuits of the underlying matroid $\underline{M}$. An element j contained in no cocircuit is called a *loop*.

The complement $E \setminus C$ of a cocircuit C is called a *hyperplane*, and any subset L of E which is the intersection of some hyperplanes is called a *flat* of the matroid. The set of all hyperplanes (all flats) are denoted by $\mathcal{H}$ ($\mathcal{L}$, respectively). The flats (hyperplanes) are nothing but the the *zero supports* $E\setminus\underline{Y}$ of faces (vertices, respectively) Y of M.

The following terminology applies to any matroid on E, but for our purpose we fix our attention to the underlying matroid $\underline{M}$. Let S be any subset of E. The *span of a* S is the smallest flat that contains it, or equivalently the intersection of all flats containing S. A set S is said to *span* L if L is the span of S. A subset J of E is called *independent* if it is a minimal set whose span is the span of J. A *basis* of S is defined as an independent subset of S that spans S. It is important to note that every basis of S has the same cardinality, called the rank $r(S)$. A *basis of the matroid* $\underline{M}$ is a basis of the ground set E. The rank $r(M) = r(\underline{M})$ of the matroid is simply the cardinality of any basis.

The sets of hyperplanes, flats, independent sets and bases all uniquely define the matroid, and there are simple matroid axioms for each of these objects, see e.g. [Wel76, (ed87].

We shall use these terms for oriented matroids to mean the same objects for the underlying matroids, e.g, a basis of an oriented matroid M means a basis of the underlying matroid $\underline{M}$.

There are many statements on an oriented matroid that only depends on the underlying matroid. First of all, the dimension of each face Y depends on $\underline{M}$.

Proposition 1.6. *The dimension $d(Y)$ $(= r(Y)-1)$ of each face Y in an oriented matroid M is determined by the rank function r of the underlying matroid by*

$$d(Y) = r(M) - r(E\setminus\underline{Y}) - 1.$$

Define $f_k = f_k(M)$ to be the number of k-faces of an oriented matroid M, for $k = -1, 0, \ldots, d(M)$. The following is an extension, due to Las Vergnas [Ver80], of Zaslavsky's theorem for arrangements of hyperplanes [Zas75] to oriented matroids.

Theorem 1.7. *The face numbers $f_k(M)$ are uniquely determined by the underlying matroid.*

1.7 Shellability of Tope Lattices and OM lattices

A (finite) lattice isomorphic to the face lattice of an oriented matroid is called *OM* lattice. A lattice isomorphic to some tope cell $[0, T]$ in an oriented matroid is called a *tope* lattice. The face lattice of every central arrangement of hyperplanes is an OM lattice, and the face lattice of every convex polytope is a tope lattice. Each constitute the "linear" subclass of the corresponding lattices.

Both lattices enjoy many of nice properties of the "linear" counterpart. In particular, both OM and tope lattices are shellable, which intuitively means they can be constructed by pasting together faces in such a way that each pasting part is "nice". This niceness can be best phrased topologically as ball or sphere of dimension one less than the pasted face. For example, the boundary of the dodecahedron can be constructed by pasting the twelve (pentagon) facets in a sequence $F_1, F_2, \dots, F_{12}$ is such a way that each pasting part $(\cup_{i=1}^{j-1} F_i) \cap F_j$ ($j = 2, 3, \dots, 12$) is topologically a 1-ball except for the last one that is a 1-sphere, see Figure 1.4.

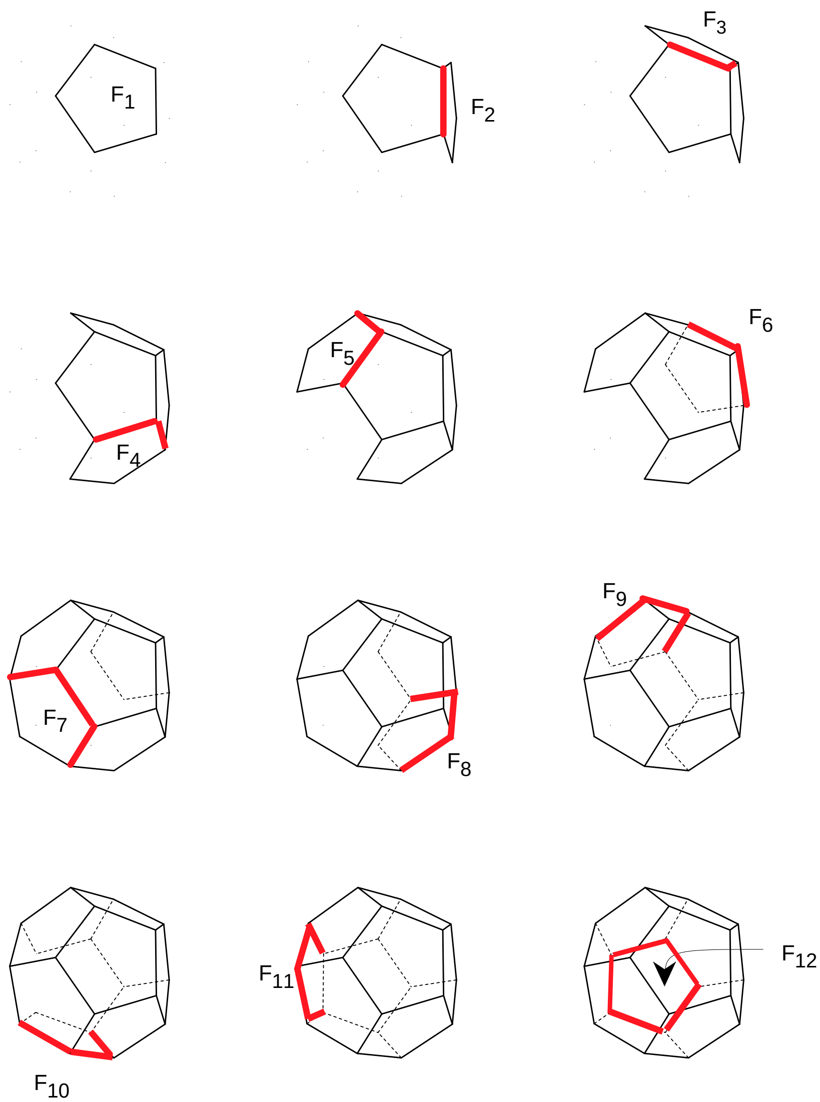


Figure 1.4: Shelling of a Dodecahedron

The actual definition must pose similar conditions recursively to each pasting part so that we can apply inductive arguments in proofs.

The notion of shellability goes back to the classical paper of Schläfli, where the shellability of convex polytopes is used without proof. It was only recently that this nice constructive property of convex polytopes is proved by Bruggesser and Mani [BM71].

In order to present the main results, we employ a combinatorial definition of shellability. A poset is *JD* if every maximal chains between any two ordered elements have the same height. When a JD poset has the least element, the dimension $d(X)$ of an element is the height of X minus one. A *pure poset P of dimension d* is a JD poset with the least element in which every maximal elements have the same dimension d. The *boundary ∂P* of a pure poset P is the set of all elements of dimension $d(P)-1$ contained in exactly one maximal element together with all elements below.

A poset P is called *shellable d-ball* if it is a pure poset of dimension d (with $d \geq 0$) satisfying either

(SB1) P has the greatest element 1 and $\partial P = P - 1$ is a shellable $(d-1)$ sphere; or

(SB2) P has multiple maximal elements and they can be ordered $p_1, p_2, \ldots, p_s$ in such a way that

(a) The interval $F_j := [0, p_j]$ is a shellable d-ball for each $j = 1, 2, \ldots, s$; and

(b) $(\cup_{i=1}^{j-1} F_i) \cap F_j$ is contained in $\partial(\cup_{i=1}^{j-1} F_i)$ and it is a shellable $(d-1)$-ball.

A poset P is called *shellable d-sphere* if it is a pure poset of dimension d (with $d \geq -1$)satisfying either

(SS1) $d = -1$ (i.e., $P = \{0 = 1\}$) ; or

(SS2) $d \geq 0$ and P has a maximal element q such that both $P - q$ and $Q := [0, q]$ are shellable d-balls and $Q \cap (P - q) = \partial Q$.

Finally a poset P is called *shellable* if it is either a shellable d-ball or shellable d-sphere. Let us denote by $f_k(P)$ the number of k dimensional elements of a pure poset P. The following proposition is easily verified from the definition above.

Proposition 1.8 (The Euler-Poincaré Relation for Shellable Posets).

(a) *If P is a shellable d-ball, then* $\sum_{i=-1}^{d} (-1)^i f_i(P) = 0$.

(b) *If P is a shellable d-sphere, then* $\sum_{i=-1}^{d} (-1)^i f_i(P) = (-1)^d$.

A generalization, due to Edmonds-Fukuda-Mandel [Fuk82, Man82], of the shellability theorem [BM71] of convex polytopes is valid, implying the Euler-Poincaré relation for tope lattices.

Theorem 1.9 (Shellability of Tope Lattices). *Every tope lattice of dimension d is a shellable d-ball.*

The proof is a natural extension of the elegant and constructive proof for the convex polytope case by Bruggesser and Mani. The idea is very simple and best understood graphically. Let L be an oriented line which intersects with a given convex polytope P. (Here P should be considered as the set of faces.) We assume L is in general position so that it intersects with the affine hulls of each facets of P at distinct points.

Figure 1.5 (a) illustrates such a line L for the dodecahedron example. Now, we traverse this line, starting from an interior point, following the given orientation. By considering the two points at infinity are "joined", we return to the starting point after crossing every facet hyperplanes. This traverse generates a natural ordering of the crossing points, say z_1, z_2, ..., z_m, and thus the associated ordering of the facets, F_1, F_2, ..., F_m. An ordering of facets arising this way is called a *line shelling* of P, and turns out to be a right shelling order.

In order to prove that a polytope P is shellable, we prove a stronger statement that every visible and invisible parts of P are shellable. Here, for a given point (eye location) $z \in R^d$, let

$$\partial P^+(z) = \{F : F \text{ is a visible face from } z\} \quad \text{(a visible part)}$$

$$\partial P^-(z) = \{F : F \text{ is a hidden face from } z\}. \quad \text{(a hidden part)}$$

(Note that a face lying on both visible and hidden facets are considered both visible and hidden.) This crucial part of the proof is illustrated in Figure 1.5 (b). Namely, to prove the part $\cup_{i=1}^{4} F_i$ (which is visible from any point beyond z_4 and before z_5) shellable, we must prove the pasting part $(\cup_{i=1}^{3} F_i) \cap F_4$ is shellable. But this part is in fact a visible part of the facet F_4 from the point z_4. This is the key idea to make an inductive argument work.

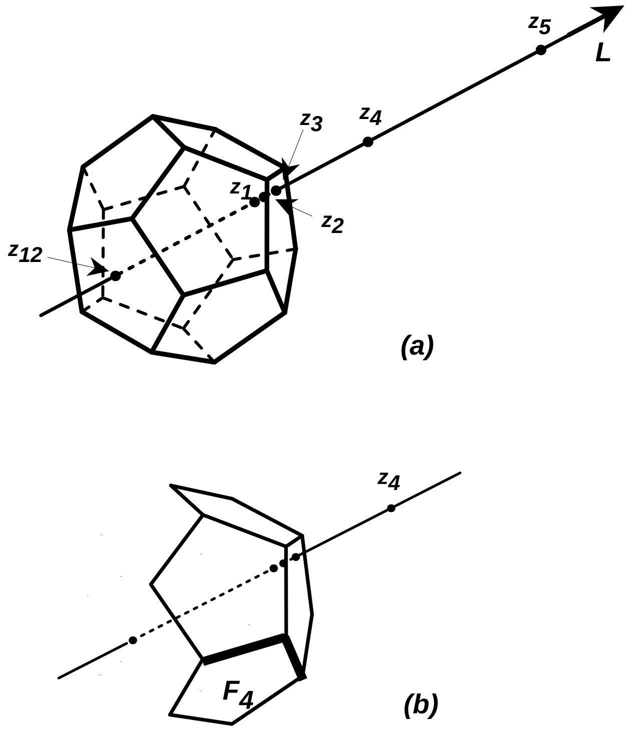


Figure 1.5: A General Line Through Dodecahedron

A correct abstraction of this line shelling is the notion of tope-graph shelling. In the prove Theorem 1.9 we use a directed path from the tope of all +'s to its negative to induce an order of elements of E. It is not difficult to foresee that the notion of visible and hidden parts of a polytope is naturally extended to tope cells. A formal proof will be give in a later section.

After proving the shellability of tope lattices, Theorem 1.9, it is quite simple to prove the shellability of OM lattices. This was first proved by Lawrence.

Theorem 1.10 (Shellability of OM Lattices). *Every OM lattice of dimension d is a shellable d-ball.*

The key idea of Lawrence is very simple. In order to prove the shellability of an OM lattice, it is essential to construct a shelling ordering of all topes. A Lawrence sequence is a linear ordering of all topes T_0, T_1, T_2, $\ldots$, T_t such that the distance between each tope T_j from the initial T_0 is nondecreasing with j. Here the distance between two topes T and T' is $|D(T,T')|$.

Figure 1.6 illustrates distances of each topes from a tope numbered 0. Starting with the initial 0 tope, then adding any sequence of distance 1 topes, any sequence of distance 2 topes, 3 topes, etc., we obtain a Lawrence shelling of the OM lattice. A formal proof is given in a later section, but it is a good exercise to show that this ordering is a shelling ordering.

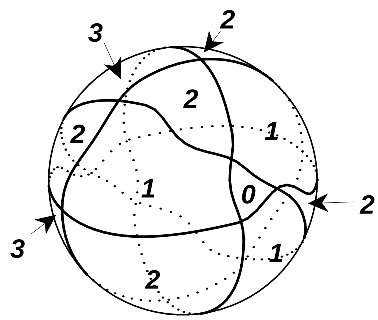


Figure 1.6: Partition of Topes by Distance From a Fixed Tope

1.8 Complexity of Oriented Matroids and Arrangements

For an oriented matroid $M = (E, \mathcal{F})$ of dimension d, $f_k(M)$ denotes the number of k-faces in M, for each $-1 \le k \le d$. We shall use m to denote the cardinality of E. The f-vector $f(M)$ of M is defined as $(f_0(M), f_1(M), \ldots, f_{d-1}(M))$. Note that f_0 is the number of vertices and f_d is the number of topes.

The first result is the exact upper bound for each f_k, due to [Buc43, Zas75, Ver77].

Proposition 1.11. *Let $M = (E, \mathcal{F})$ be an oriented matroid of dimension d and let $m = |E|$. Then*

$$f_k(M) \le 2 \sum_{i=0}^{\lfloor d/2 \rfloor} \binom{m}{d-2i} \binom{d-2i}{d-k} \qquad \forall 0 \le k \le d. \tag{1.1}$$

Furthermore, the equality holds if and only if M is nondegenerate, *i.e., every vertex has exactly d zero components.*

The next theorem states how the face number f_k can be bounded by the number f_d of topes.

Theorem 1.12 ([FST91]). *Let $M = (E, \mathcal{F})$ be an oriented matroid of dimension d. Then*

$$f_k(M) \le \binom{d}{k} f_d(M) \qquad \forall 0 \le k \le d. \tag{1.2}$$

Some special cases of these inequalities include:

$$f_{d-1}(M) \le d\, f_d(M) \tag{1.3}$$

$$f_0(M) \le f_d(M). \tag{1.4}$$

The last inequality immediately implies that the f-vectors of oriented matroids constitute a considerably different class from the f-vectors of convex polytopes, since the number of vertices can be much larger than that of facets in convex polytopes, e.g. hypercubes.

These inequalities are useful when we design efficient algorithms for various problems in arrangements and oriented matroids, discussed in Chapter 2. For example, we obtain a polynomial algorithm for generating all faces from the set of topes. This algorithm can be used to efficiently check whether a given set of sign vectors is the set of topes of an oriented matroid.

Also, this face enumeration algorithm serves as one of the two key components to design a polynomial algorithm for the face enumeration problem in arrangements of hyperplanes, Geometric Computation 1.1.

In order to prove the main theorem, Theorem 1.12, we use inductive arguments on d and m. We shall present the key lemma without proof here.

For this, let us define two basic operations on sign vectors.

For a set $\mathcal{F}$ of sign vectors on E, and for any subset R of E, we define two sets:

$$\mathcal{F} \backslash R = \{Y_{E \setminus R} : Y \in \mathcal{F}\} \tag{1.5}$$

$$\mathcal{F} / R = \{Y_{E \setminus R} : Y \in \mathcal{F} \text{ and } Y_R = \mathbf{0}\} \tag{1.6}$$

It is straightforward to verify that if $\mathcal{F}$ is the set of faces of an oriented matroid M, then both pairs

$$M\backslash R := (E\backslash R, \mathcal{F}\backslash R) \tag{1.7}$$

$$M/R := (E\backslash R, \mathcal{F}/R) \tag{1.8}$$

are oriented matroids. $M\backslash R$ is called the *minor of M obtained by deleting R*, and M/R the *minor of M obtained by contracting R*.

Lemma 1.13 ([FST91]). *Let $M = (E, \mathcal{F})$ be an oriented matroid of dimension d with $m \geq d + 2$. Then for any element $e \in E$ such that $dim(M\backslash e) = d$,*

$$f_k(M) \leq f_k(M\backslash e) + f_k(M/e) + f_{k-1}(M/e) \qquad \forall 0 \leq k \leq d.$$

Furthermore, if M is simple (i.e., has no loops and no parallel elements),

$$f_d(M) = f_d(M\backslash e) + f_{d-1}(M/e).$$

These recursive relations are intuitively obvious by looking at arrangements of spheres in the 2-sphere S^2. But proving the lemma for the linear case by intuition in general dimension is quite erroneous. Using the oriented matroid axioms, we do not (and in fact cannot) depend on particular figures or intuition when we prove this kind of geometric theorems.

Theorem 1.12 can be strengthened in various ways. One such extension is the following, which appears to be considerably harder to prove.

Theorem 1.14 ([FTT93]). *Let $M = (E, \mathcal{F})$ be an oriented matroid of dimension d. Then for any $0 \leq j < k \leq d$, the average number of j-subfaces of a k-face is less than or equal to $2^{k-j}\binom{k}{j}$. Moreover, equality holds if and only if $j = 0$ and $k = 1$.*

Using the fact that each $(k-1)$-face is a subface of at least $2(d-k+1)$ k-faces, we have the following corollary.

Corollary 1.15. *Let $M = (E, \mathcal{F})$ be an oriented matroid of dimension d. Then*

$$f_{k-1}(M) \leq \frac{k}{d-k+1} f_k(M) \qquad \forall 0 < k \leq d. \tag{1.9}$$

Simple special cases include the following inequalities:

$$f_k(M) \leq f_{d-k}(M) \qquad \forall 0 \leq k \leq \lfloor d/2 \rfloor, \tag{1.10}$$

which shows that the upper half of the face lattice $\hat{\mathcal{F}}$ is always larger than the lower half. For example, Figure 1.7 shows a face lattice of a degenerate 3-dimensional oriented matroid with 6 elements. More precisely this is the linear oriented matroid of the column space of the matrix

$$A = \begin{bmatrix} 2 & -1 & 1 & 0 \\ -2 & -2 & 2 & 1 \\ -2 & -1 & -1 & -1 \\ 2 & 1 & 2 & 2 \\ 0 & -3 & 3 & 1 \\ 2 & -1 & 2 & 1 \end{bmatrix}.$$

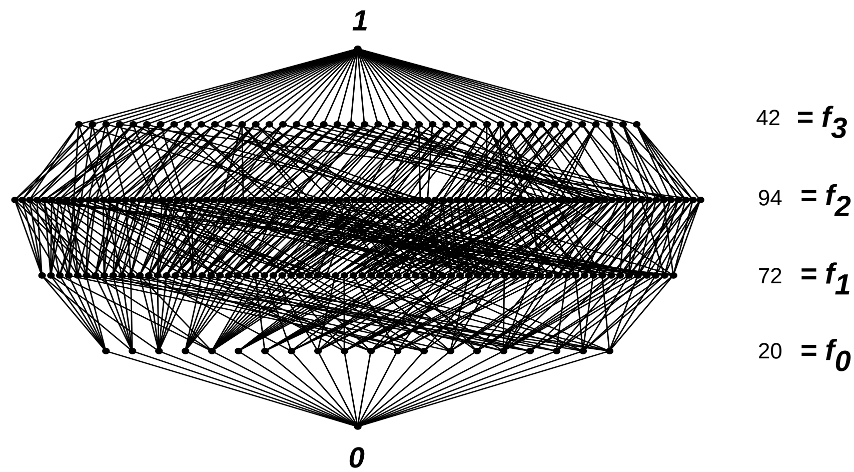


Figure 1.7: Face Lattice of a 3-dimensional OM

There are relatively few results for general hyperplane arrangements, since the classical mathematics focused mostly on problems in nondegenerate (simple) arrangements. Certainly the theory of matroids and oriented matroids has been useful in finding and proving new results in the general degenerate cases.

1.9 Duality

There is a natural notion of duality in vector subspaces. For a given vector subspace V of R^E, the orthogonal dual space $V^\perp$ of V is the set of vectors orthogonal to all vectors in V. The basic duality says: $V^\perp$ is again a vector subspace and its dual $V^{\perp\perp}$ is V (as long as $|E|$ is finite).

This section introduces the notion of duality in oriented matroids that generalizes the duality of vector subspaces. The first key fact is the set $\sigma(V^\perp)$ of sign vectors of the dual space $V^\perp$ is uniquely determined by the set $\sigma(V)$ of sign vectors of the original space V. To show this, we define orthogonal relation for sign vectors.

For two sign vectors X and Y on a finite set E are said to be *orthogonal* (denoted by $X * Y$) if either

(1.11) $\underline{X} \cap \underline{Y} = \emptyset$; or

(1.12) there exist two elements $f, g \in \underline{X} \cap \underline{Y}$ such that $X_f = Y_f$ and $X_g = -Y_g$.

For any set $\mathcal{F}$ of sign vectors on E, its *dual space* $\mathcal{F}^*$ is defined as the set of vectors orthogonal to all vectors in $\mathcal{F}$:

$$\mathcal{F}^* = \{X \in \{+,0,-\}^E : X * Y \quad \forall Y \in \mathcal{F}\}. \tag{1.13}$$

If two real vectors x and y of R^E are orthogonal, so are their sign vectors $\sigma(x)$ and $\sigma(y)$. This implies that if V is a vector subspace of R^E, $\sigma(V^\perp) \subseteq \sigma(V)^*$. The following theorem says they are equal.

Theorem 1.16. *For any vector subspace V of R^E, $\sigma(V)^* = \sigma(V^\perp)$.*

The most important message of this theorem is: for any vector subspace V, its oriented matroid $\sigma(V)$ contains the complete information of the oriented matroid of the orthogonal dual space. As a generalization of the property $V^{\perp\perp} = V$, we have:

Theorem 1.17 (OM Duality). *Let $M = (E, \mathcal{F})$ be an oriented matroid on E. Then the following statements hold.*

(a) *the pair $M^* := (E, \mathcal{F}^*)$ is an oriented matroid, called the* **dual of** *M;*

(b) *the dual M^{**} of the dual is the original M, that is, $\mathcal{F}^{**} = \mathcal{F}$.*

The following claims that deletion and contraction are dual operations.

Theorem 1.18. *Let $M = (E, \mathcal{F})$ be an oriented matroid on E. For any subsets R, S of E the following statements hold.*

(a) $(M \backslash R)^* = M^*/R$;

(b) $(M/S)^* = M^* \backslash S$.

For a system of linear inequalities, there are so-called "theorems of alternatives". These theorems state necessary and sufficient conditions for inconsistency that can be verified efficiently. Farkas' Lemma is perhaps most well known among them. We shall write below a disguised version of Farkas' Lemma.

Theorem 1.19. *For any vector subspace V of R^E, and for any fixed index $g \in E$, exactly one of the following statements holds:*

(a) $\exists y \in V$ *such that* $y \geq \mathbf{0}$ *and* $y_g > 0$;

(b) $\exists x \in V^\perp$ *such that* $x \geq \mathbf{0}$ *and* $x_g > 0$.

One can derive more familiar forms of alternative theorems by setting V to be of a particular matrix representation. For instance, if $V^\perp$ is represented by

$$V^\perp = \{x \in R^E : [\overset{E-g}{A} \quad -\overset{g}{b}]\, x = \mathbf{0}\},$$

then the theorem yields Farkas' Lemma:

Theorem 1.20 (Farkas' Lemma). *For any matrix $A \in R^{m \times d}$ and any vector $b \in R^m$, exactly one of the following statements hold:*

(a) $\exists z \in R^m$ *such that* $z^T A \geq \mathbf{0}$ *and* $z^T b < 0$.

(b) $\exists x \in R^d$ *such that* $x \geq \mathbf{0}$ *and* $A x = b$;

It turns out that a combinatorial generalization of this theorem needs very little assumptions.

Theorem 1.21 (Farkas' Lemma for OM's). *Let $\mathcal{F}$ be a set of sign vectors on a finite set E satisfying (F2) and (F3). Then for any fixed index $g \in E$, exactly one of the following statements holds:*

(a) $\exists Y \in \mathcal{F}$ *such that* $Y \geq \mathbf{0}$ *and* $Y_g > 0$;

(b) $\exists X \in \mathcal{F}^*$ *such that* $X \geq \mathbf{0}$ *and* $X_g > 0$.

For a subset R of E and for a sign vector Y on E, we denote by ${}_RY$ the sign vector on E obtained from Y by reversing signs on R. Similarly, for a set $\mathcal{F}$ of sign vectors on E, we denote by ${}_R\mathcal{F}$ the family

(1.14) $$ {}_R\mathcal{F} = \{ {}_RY : Y \in \mathcal{F} \}. $$

One can easily verify:

(1.15) $$ ({}_R\mathcal{F})^* = {}_R(\mathcal{F}^*). $$

Also, if $M = (E, \mathcal{F})$ is an oriented matroid,

(1.16) $$ {}_RM := (E, {}_R\mathcal{F}) $$

is also an oriented matroid. We say ${}_RM$ is the *oriented matroid obtained from M by reversing signs on R.*

When $\mathcal{F}$ is an oriented matroid, OM Farkas' Lemma can be strengthened by Theorem 1.18 and the remarks above to yield the following theorem, known as the *Coloring Lemma*.

Corollary 1.22 (Coloring Lemma for OM's). *Let $\mathcal{F}$ be the set of faces (covectors) of an oriented matroid on a finite set E. Then for any partition ("coloring") of E into disjoint subsets R, G, B, W and for any fixed index $r \in R \cup G$, exactly one of the following statements holds:*

(a) $\exists Y \in \mathcal{F}$ *such that* $Y_R \geq \mathbf{0}$, $Y_G \leq \mathbf{0}$, $Y_B = \mathbf{0}$ *and* $Y_r \neq 0$;

(b) $\exists X \in \mathcal{F}^*$ *such that* $X_R \geq \mathbf{0}$, $X_G \leq \mathbf{0}$, $X_W = \mathbf{0}$ *and* $X_r \neq 0$.

The Coloring Lemma was first formulated for digraphs and certain combinatorial structure (called digraphoids) more general than digraphs by Minty [Min66]. The theorem above can be considered as a natural extension of Minty's results. Let $D = (V, E)$ be a digraph. For any nonempty subset S of V, define $\delta^+(S)$ ($\delta^-(S)$) as the set of edges leaving (entering, respectively) S. The *cut determined by S* is the union $\delta(S) := \delta^+(S) \cup \delta^-(S)$ The *cut vector $cut(S)$ of S* is simply the $(+1, -1, 0)$ incidence vector on E whose $+1$ and -1 components are exactly $\delta^+(S)^+$ and $\delta^-(S)$. The linear space generated by all cut vectors in D is called the *cut space* $V(D)$ of D. The orthogonal dual space $V(D)^\perp$ is the cycle space of D, the space generated by the incidence vectors of cycles of D.

Considering M as the linear oriented matroid of $V(G)$, the Coloring Lemma yields Minty's theorem for digraphs:

Corollary 1.23 (Minty's Coloring Lemma for Digraphs). *Let $D = (V, E)$ be a digraph. For any partition ("coloring") of edges E into disjoint subsets R, G, B, W and for any fixed edge $r \in R$, exactly one of the following statements holds:*

(a) *there exists a cut $K = \delta(S)$ in D containing r such that all edges in $K \cap R$ ("red" edges) are leaving S, all edges in $K \cap G$ ("green" edges) are entering S, and it contains no edges of W (contains no "white" edges);*

(b) *there exists a cycle C in D containing r such that all edges in $C \cap R$ ("red" edges) are directed equally, all edges in $C \cap G$ ("green" edges) are directed against r, and it contains no edges in B (contains no "black" edges).*

1.10 Tope Graphs and Cocircuit Graphs

By Proposition 1.2, there are two graphs naturally associated with any oriented matroid. The first one is the *tope graph* $TG(M) = (\mathcal{T} \equiv \mathcal{F}_d, \mathcal{F}_{d-1})$ where the nodes are the topes and two topes T, T' are adjacent if they have a common $(d-1)$-face. The second one is the *cocircuit graph* $CG(M) = (\mathcal{V} \equiv \mathcal{F}_0, \mathcal{F}_1)$ where the nodes are the vertices (cocircuits) of M and two vertices are adjacent if they are the two 0-faces of some 1-face.

Two oriented matroids $M = (E, \mathcal{F})$ and $M' = (E', \mathcal{F}')$ are called *isomorphic* if the two posets $\mathcal{F}$ and $\mathcal{F}'$ are isomorphic.

We shall list below some fundamental theorems on tope graphs and cocircuit graphs.

Theorem 1.24 ([BEZ90, CFdO00, BFF01]). *Let $M = (E, \mathcal{F})$ be an oriented matroid of dimension d. Then the following statements hold:*

(a) *The tope graph $TG(M)$ determines the oriented matroid up to isomorphism.*

(b) *The cocircuit graph $CG(M)$ determines the oriented matroid up to isomorphism if the oriented matroid is nondegenerate (uniform). Furthermore, there exists a degenerate (non-uniform) oriented matroid whose cocircuit graph does not determine the oriented matroid.*

The fact (a) above is the motivation of the following open problem which is closely related to Open Problem 1.1.

Open Problem 1.2 Tope Graph Characterization
Is there any simple (graph theoretical) characterization of the tope graphs of oriented matroids?

While the characterization problem is open in general, there exists a polynomial-time algorithm [FST91] to check whether a given graph is the tope graph of some oriented matroid. The algorithm exploits the fact that the faces of an oriented matroid can be generated by the set of topes in polynomial time. This suggests that it is likely that there is a good characterization of tope graphs.

Recently, a similar result was obtained for the cocircuit graphs of nondegenerate (uniform) oriented matroids. The paper [BFF01] contains a polynomial-time algorithm to check whether a given graph is the cocircuit graph of some uniform oriented matroid.

A natural graph theoretical question one might ask is: what are the connectivities of the tope graph and the cocircuit graphs?

Theorem 1.25 ([CF93]). *Let $M = (E, \mathcal{F})$ be an oriented matroid of dimension d. Then the following statements hold:*

(a) *The tope graph $TG(M)$ is $(d+1)$-connected.*

(b) *The cocircuit graph $CG(M)$ is $2d$-connected.*

The *m-cube* Q_m is the graph with vertex set $\{1, -1\}^m$ where two vertices are adjacent if they are different in exactly one component. When the dimension is greater than two, the tope graphs and the cocircuit graphs constitute disjoint classes of graphs by the following.

Proposition 1.26 ([FH93]). *Let $M = (E, \mathcal{F})$ be an oriented matroid of dimension $d \geq 3$. Let $m = |E|$. Then the following statements hold:*

(a) *The tope graph $TG(M)$ is bipartite, and isometrically embeddable in the m-cube Q_m.*

(b) *The cocircuit graph $CG(M)$ contains a K_3 (triangle) as a vertex-induced subgraph, and thus is not bipartite.*

A much stronger statement can be proved for the statement (b) if the oriented matroid is linear: the cocircuit graph $CG(M)$ contains a complete graph K_{d+1}, see [Sha79]. Las Vergnas [Ver80] conjectured that this is true for nonlinear oriented matroids as well.

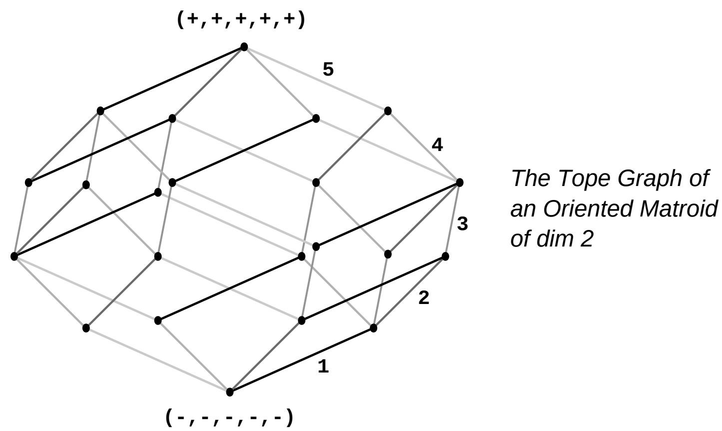
A graph G is called *antipodal* if for each vertex v there is a unique vertex $\overline{v}$ such that $distance(v, u) < distance(v, \overline{v})$ for all neighbors u of $\overline{v}$.

Proposition 1.27 ([FH93]). *The tope graph of an oriented matroid is antipodal.*

Now we present a good characterization of tope graphs of oriented matroids of dimension at most 2.

Theorem 1.28 ([FH93]). *A graph G is isomorphic to the tope graph of an oriented matroid of dimension $d \leq 2$ if and only if G is isometrically embeddable in an m-cube, antipodal and planar.*

Figure 1.8 shows the tope graph of an oriented matroid of dimension 2 and the tope graph of a non-matroidal acycloid. The second graph has similar structures as the first one in the sense that it is isometrically embeddable in the 5-cube and antipodal. However the crucial difference is: the first one is planar but the second one is not.



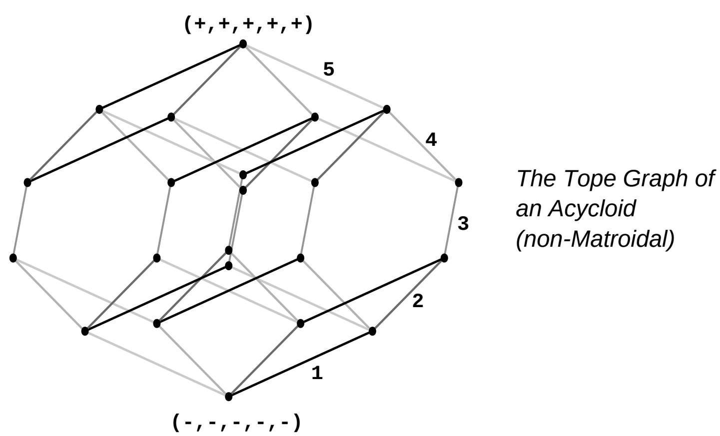


Figure 1.8: Tope graphs of an OM and an Acycloid

Observe that the tope graph of a linear oriented matroid of dimension d is isomorphic to the graph of a $(d+1)$-dimensional zonotope. The characterization above is a sort of the zonotope analogue of Steinitz' Theorem (Theorem 0.9) for 3-polytopes. But it is not exactly, since there are nonlinear 2-dimensional oriented matroids, as we constructed in Section 1.3. Thus one might be tempted to ask:

Is there any good characterization of the graphs of 3-zonotopes?

The question is still open but it is extremely unlikely that such a characterization exists. In fact, the problem of determining whether a given graph is the graph of a 3-zonotope is NP-hard. This result is due to Mnëv who proved a stronger result that the stretchability of pseudolines is equivalent to the existential theory of reals, see [Mnë88, Sho91].

1.11 Constructing Nonlinear Polytopes: Lawrence's Construction

A lattice P is called *polytopal* if it is isomorphic to the face lattice of some polytope.

For an oriented matroid $M = (E, \mathcal{F})$, a cell in M is the interval $[\mathbf{0}, Y]$ some face $Y \in \mathcal{F}$. The *canonical cell* $\mathcal{F}^+(M)$ is the set of nonnegative faces of M.

When M is linear and represented by matrix A by $\mathcal{F} = \sigma(C(A))$ with $C(A)$ being the column space of A, the canonical cell P_M is polytopal since it is isomorphic to the face lattice of the polyhedral cone $P(A) = \{x : A\,x \geq \mathbf{0}\}$ with the empty face excluded (which is the face lattice of some polytope). Thus every canonical cell (and all cells) of a linear oriented matroid is polytopal.

The main purpose of this section is to answer the fundamental question: does there exist any non-polytopal OM cell? Clearly such a cell, if exists, must be a cell of a nonlinear OM.

In Section 1.3 we presented a certain construction of nonlinear oriented matroids. The construction makes use of any known arrangement of pseudolines in the plane that are not stretchable. Each cell in such a construction is very primitive and easily seen to be polytopal.

The idea of the construction of nonlinear OM's was in a sense not quite new. We could simply use some old known technique. When the existence of a nonlinear cell was still open in the late seventies, Lawrence created a very simple and geneous idea, with which he answered the question positively. With his construction, for every OM M on m elements, there is the associated OM $\Lambda(M)$ (called a Lawrence matroid) on $2m$ elements whose canonical cell $\mathcal{F}^+(\Lambda(M))$ encodes the complete information of M. In particular, the importance and the beauty of the construction can be explained by the fact that M is nonlinear if and only if $\mathcal{F}^+(\Lambda(M))$ is non-polytopal.

In order to present the construction, let $M = (E, \mathcal{F})$ be an oriented matroid with $|E| = m$. For each element e in E, we denote by e' the *copy of* e, and let E' be the copy $\{e' : e \in E\}$ of the set E. For each sign vector Y on E, let $\tilde{Y}$ denote the sign vector on $E \cup E'$ whose E part is exactly Y and whose E' part is the "negative of Y". More precisely, $\tilde{Y}$ is the sign vector on $E \cup E'$ such that

$$\tilde{Y}_e = Y_e \text{ for each } e \in E \tag{1.17}$$

$$\tilde{Y}_{e'} = -Y_e \text{ for each } e \in E. \tag{1.18}$$

Then, we define the oriented matroid $\tilde{M} = (E \cup E', \tilde{\mathcal{F}})$ by

$$\tilde{\mathcal{F}} := \{\tilde{Y} : Y \in \mathcal{F}\}. \tag{1.19}$$

Thus the matroid $\tilde{M}$ is the double copy of M with opposite signs for the second part. We shall call this matroid $\tilde{M}$, the *symmetric double of* M.

Finally, the *Lawrence matroid* $\Lambda(M)$ *of* M is defined as the oriented matroid $\left(\widetilde{(M^*)}\right)^*$, the dual of the symmetric double of M^*.

Since the dual and the symmetric doubling operations are clearly linearity preserving operations, the Lawrence matroid is linear if M is linear. In fact one can nicely present the Lawrence construction in terms of matrix representation for the linear case.

Proposition 1.29 (Lawrence Matrix [BS90]). *Let M be a linear oriented matroid on a finite set E with an $m \times r$ representation matrix A. Then the Lawrence matroid $\Lambda(M)$ is the linear oriented matroid representable by the $(2m) \times (m + r)$ matrix $\Lambda(A) = \begin{bmatrix} I & A \\ I & 0 \end{bmatrix}$.*

Proof. Let A be an $m \times r$ representation matrix of M. It is sufficient to prove the result for the case that A is column full rank, since the column spaces of A and $\Lambda(A)$ are invariant under the removal of any redundant columns of A. Then there is an $m \times (m - r)$ representation matrix A^* of M^*, and the matrix of $(m - r)$ independent columns, $B = \begin{bmatrix} A^* \\ -A^* \end{bmatrix}$ is a representation matrix of $\widetilde{(M^*)}$. Any matrix consisting of $(m + r)$ independent column vectors that are orthogonal to every column of B is a representation matrix of the Lawrence matroid $\Lambda(M)$. It is easy to see that the given matrix satisfies the condition. □

For an oriented matroid M, the canonical cell $\mathcal{F}^+(\Lambda(M))$ of the Lawrence matroid $\Lambda(M)$ is called the *Lawrence cell* associated with M. The main theorem of this section is the following.

Theorem 1.30 (Lawrence Construction, see [BS90]). *An oriented matroid M is nonlinear if and only if the Lawrence cell $\mathcal{F}^+(\Lambda(M))$ is non-polytopal.*

The theorem is a direct consequence of the following two lemmas:

Lemma 1.31. *If M is an oriented matroid and the Lawrence cell $\mathcal{F}^+(\Lambda(M))$ is polytopal, then there exists a real matrix A such that the matrix $\Lambda(A)$ represents $\Lambda(M)$.*

Lemma 1.32. *An oriented matroid M is uniquely determined by the Lawrence cell $\mathcal{F}^+(\Lambda(M))$.*

Lawrence's Theorem 1.30 is a very powerful theorem. By this, the construction of a new nonlinear oriented matroid yields automatically a new nonpolytopal cell. Since each OM cell is known to be a sphere (by the shellability theorem, Theorem 1.9) , Lawrence's Λ construction can be used as a tool to construct and study "almost polytopal" but nonpolytopal spheres.

One might feel that the Lawrence construction is difficult to visualize geometrically. The dimension of a Lawrence cell is quite large. When an oriented matroid M has m elements and rank r (i.e. dimension $r-1$), the Lawrence matroid has $2m$ elements and rank $m+r$. This means that for the non-Pappus oriented matroid M with $r = 3$ and $m = 8$, constructed in Section 1.3, the associated Lawrence matroid has rank 11 and 16 elements, and the non-polytopal cell has dimension 10 and 16 facets.

Nevertheless, there is a nice way to see the relation between M and $\mathcal{F}^+(\Lambda(M))$ algebraically. In a sense, the OM lattice $\mathcal{F}$ is completely embedded in the Lawrence cell. In order to state this relation more rigorously, we need some definitions.

Following [BS90], the *cover poset* $\mathcal{U}(M)$ of an oriented matroid $M = (E, \mathcal{F})$ is the set of pairs (Y, I) where $Y \in \mathcal{F}\backslash I$, together with the partial order $\preceq$

$$(Y, I) \preceq (Z, J) \text{ if and only if } I \subseteq J \text{ and } Y_J \preceq Z_J. \tag{1.20}$$

Theorem 1.33 ([BS90]). *For any oriented matroid M, the map*

$$\phi : (Y, I) \longmapsto ((Y^+ \cup I) \cup (Y^- \cup I)', \emptyset) \tag{1.21}$$

is an order-preserving isomorphism from $\mathcal{U}(M)$ to $\mathcal{F}^+(\Lambda(M))$. Consequently, the conver poset $\mathcal{U}(M)$ is isomorphic to the Lawrence cell associated with M.

1.12 Oriented Matroid Programming

One of the strongest motivation to create the notion of oriented matroid stems from Rockafeller's insights [Roc69] to foresee a combinatorial extension of the strong duality theorem in linear programming. Folkman-Lawrence [FL78] first proved the OM extension of the strong duality theorem, and Bland [Bla77] gave a constructive proof of the theorem using a finite pivot algorithm to solve the OM programming problem, a combinatorial abstraction of the linear programming problem. Edmonds-Fukuda [Fuk82] studied OM programming and found some strange phenomenon of cycling of non-degenerate simplex pivots. This leads us to define non-Euclidean oriented matroids, a class of "extremely nonlinear" oriented matroids, which will be presented later in this chapter.

Despite its purely mathematical motivation of extending linear programming theory, the oriented matroid programming theory is responsible for many discoveries of new finite algorithms for linear programming. The famous smallest index rule for the simplex method was a byproduct of the pioneering Bland's work [Bla77]. Several different pivot algorithms were proposed by Edmonds-Fukuda [Fuk82], Todd [Tod85], Terlaky [Ter87] and Wang [Wan87]. These methods, except for Todd's algorithm, are completely different from Dantzig's simplex method in the sense that they generate neither a sequence of feasible points nor a sequence increasing (or decreasing) the objective function monotonically. The key idea to ensure the finiteness of these algorithms is an inductive lemma. It should be also noted that Seidel's randomized algorithm for linear programming makes use of basically the same lemma, and it can be considered as a specialization of Bland's recursive algorithm [Bla77]. We shall discuss some of these algorithms in Chapter 7.

The OM programming has been extended to generalize the quadratic programming (QP) theory and the linear complementarity (LCP) theory. For this, Todd [Tod84, Tod85] proved that many of the fundamental theorems in QP and LCP can be generalized in the OM setting. Then, the results have been extended (with simpler arguments) by Lüthi-Lemke [LL86], Klafszky-Terlaky [KT89] and Fukuda-Terlaky [FT92].

In this section, we shall present two of the most important theorems in OM programming, the duality theorem in OM programming and the duality theorem in OM complementarity.

LP Duality and OM Extension

Linear programming problem can be defined in various way. The form we choose here is quite different from any of the well-known canonical forms but it is perhaps the simplest and most symmetric.

Let E be a finite set and let V be a vector subspace of R^E. For any two fixed elements f and g of E, the *linear programming problem* is find x to

$$\text{LP}(V,g,f) \qquad \begin{aligned} &\text{maximize } x_f \\ &\text{such that } x \in V \\ &\qquad\qquad x_j \geq 0 \quad \forall j \in E\backslash\{f,g\} \\ &\qquad\qquad x_g = 1. \end{aligned}$$

We call f the *objective element* and g the *infinity element*. A vector x satisfying all the constraints is said to be *feasible*. A vector $z \in V$ is a *direction* if $z_g = 0$. An *unbounded* direction is a direction z such that $z_j \geq 0$ for all $j \in E\backslash\{f,g\}$. For a feasible solution x, an direction z is said to be *augmenting* at x if $z_f > 0$ and $z_j \geq 0$ for all $j \in E\backslash\{f,g\}$ with $x_j = 0$. Finally, LP is said to be *feasible* if it admits a feasible solution, and *unbounded* if it is feasible and admits an unbounded direction.

In order to get a familiar form of LP, one can simply consider V as the null space of some real $d \times E$ matrix

$$\overline{A} = \begin{bmatrix} -b & A & I & \mathbf{0} \\ 0 & -c^T & \mathbf{0} & 1 \end{bmatrix}$$

where the first and the last columns are indexed by g and f, respectively. This LP is equivalent to maximize $c^T x$ subject to $A\,x \leq b$ and $x \geq \mathbf{0}$. The following theorem is fundamental.

Theorem 1.34 (LP Fundamental Theorem). *Every linear program LP(V,g,f) either is infeasible, unbounded or has an optimal solution.*

The *dual LP problem* of the LP(V, g, f) is defined as $(V^\perp, f, g)$, or more explicitly

$$\text{LP}(V^\perp, f, g) \qquad \begin{aligned} &\text{maximize } y_g \\ &\text{such that } y \in V^\perp \\ &\qquad\qquad y_j \geq 0 \quad \forall j \in E \backslash \{f, g\} \\ &\qquad\qquad y_f = 1. \end{aligned}$$

Clearly the dual of the dual LP is the primal problem LP (V, g, f). Also, if the primal (dual, respectively) problem admits an unbounded direction, then the dual (primal) problem is infeasible, because of the orthogonality of V and $V^\perp$.

For the case that V is given as the null space of the matrix $\overline{A}$ above, the dual problem is to minimize $b^T \lambda$ subject to $A^T\ \lambda \geq c$ and $\lambda \geq \mathbf{0}$.

Two vectors x and y in R^E are said to be *complementary* (with respect to LP(V, g, f)) if $x_j = 0$ or $y_j = 0$ for all $j \in E \backslash \{f, g\}$. It is well known and easily shown that if a primal feasible x and a dual feasible y are complementary then both x and y are optimal solutions.

Theorem 1.35 (LP Duality Theorem). *For any dual pair of linear programs LP(V, g, f) and LP$(V^\perp, f, g)$, exactly one of the following two statements holds:*

(a) *There exists a primal feasible x and a dual feasible y that are complementary;*

(b) *There exists either a primal unbounded direction z or a dual unbounded direction w (thus either the primal or the dual problem is infeasible).*

This cerebrated theorem can be extended very naturally to oriented matroids, simply by replacing V by an oriented matroid. Given, an oriented matroid $M = (E, \mathcal{F})$ and two distinguished elements g and f, an *oriented matroid programming* (OMP) is to find X :

$$\text{OMP}(M, g, f) \qquad \begin{aligned} &\text{maximize } f \\ &\text{such that } X \in \mathcal{F} \\ &\qquad\qquad X_j \geq 0 \quad \forall j \in E \backslash \{f, g\} \\ &\qquad\qquad X_g = +. \end{aligned}$$

Here we say X *maximizes* f if there is no *augmenting direction* Z at X, i.e., $Z \in \mathcal{F}$ such that $Z_g = 0$, $Z_f = +$ and $Z_j \geq 0$ for all $j \in E \backslash \{f, g\}$ with $X_j = 0$. An *unbounded direction* is a vector $Z \in \mathcal{F}$ such that $Z_g = 0$ and $Z_f = +$ and $Z_j \geq 0$ for all $j \in E \backslash \{f, g\}$. An OMP is said to be *feasible* if it admits a feasible solution, and *unbounded* if it is feasible and admits an unbounded direction.

Following the definition of the dual LP, we define the *dual problem* P^* of $P =$OMP(M, g, f) as OMP(M^*, f, g). That is, the dual problem is obtained from the primal problem P by dualizing the oriented matroid and by exchanging the objective and the infinity elements. Two sign vectors X and Y on E are said to be *complementary* if $X_j = 0$ or $Y_j = 0$ for all $j \in E \backslash \{f, g\}$. It follows from the coloring lemma, Theorem 1.22, that if a primal feasible X and a dual feasible Y are complementary then both X and Y are optimal solutions.

Now we can state the two most important theorems for OMP's.

Theorem 1.36 (OMP Fundamental Theorem). *Every OMP either is infeasible, unbounded or has an optimal solution.*

Theorem 1.37 (OMP Duality Theorem). *For any dual pair of OMP's P and P^*, exactly one of the following two statements holds:*

(a) *There exists a primal feasible X and a dual feasible Y that are complementary;*

(b) *There exists either a primal unbounded direction Z or a dual unbounded direction W.*

Note the this duality theorem follows directly from the first theorem, Theorem 1.36, and the coloring lemma. We give two proofs for the first theorem, one inductive and the other algorithmic in Chapter 7.

For both proofs, the following induction lemma is essential. For $P =$OMP(M, g, f), and for any element $e \in E \backslash \{f, g\}$, we define two subproblems $P \backslash e$ and P/e as OMP$(M \backslash e, g, f)$ and OMP$(M/e, g, f)$, respectively.

Lemma 1.38 (OMP Induction Lemma). *Let P be any OMP(M, g, f), and let e be any element in $E \backslash \{f, g\}$. Then the following statements hold.*

(a) *If both $P \backslash e$ and P/e have optimal solutions, then P has an optimal solution;*

(b) *If $P \setminus e$ has an optimal solution and P/e is infeasible, then P either has an optimal solution or is infeasible;*

(c) *If $P \backslash e$ is unbounded and P/e has optimal solution, then P either is infeasible or has an optimal solution;*

(d) *If $P \backslash e$ is unbounded and P/e is infeasible, then P is either infeasible or unbounded.*

Linear Complementarity and Extensions to OM

One natural way to generalize the OMP duality theorem is to extend the duality theorems in convex quadratic programming and linear complementarity problem to oriented matroids. We shall present only the duality theorem of oriented matroid complementarity without showing that it is a generalization of the OMP duality theorem. For the detail, the reader must read Chapter 7.

A linear complementarity problem (LCP) is to find vectors w and z in R^n satisfying

LCP (A, b)
$$w = Az + b, z \geq 0, w \geq 0$$
$$z^T w = 0.$$

It is known that LP can be reduced to an LCP with a positive semidefinite (PSD-) matrix A. A fundamental theorem in LCP [CD68, Lem68] says that if A is positive semidefinite or P-matrix, then either the LCP(A, b) has a solution or the LCP without the complementarity conditions $z^T w = 0$ is infeasible. Cottle, Pang and Venkateswaran [CPV89] extended this theorem to the class of sufficient matrices. For the definition of these matrix classes, see Chapter 7. We shall present here a further generalization of the theorem.

In order to write the problem LCP (A, b) in a form that will be convenient for extending it to the setting of oriented matroids, we set

$$V(A, b) = \{x \in R^{2n+1} \mid [-b \;\; -A \; I] \, x = 0\}.$$

We denote the components of a vector $x \in R^{2n+1}$ by $x = (x_0, x_1, x_2, \ldots, x_{2n})^T$ instead of the usual $x = (x_1, x_2, \ldots, x_{2n+1})^T$, while we denote a vector $x \in R^{2n}$ by $x = (x_1, \ldots, x_{2n})^T$. Then, one can rewrite LCP (A, b) as

$$\text{find } x \in V(A, b) \text{ satisfying} \tag{1.22}$$
$$x \geq 0, \; x_0 = 1 \tag{1.23}$$
$$x_i \cdot x_{n+i} = 0 \text{ for all } i = 1, \ldots, n. \tag{1.24}$$

A vector x in R^{2n} or R^{2n+1} is called *complementary* if the complementarity condition (1.24) is satisfied. A solution to the LCP is a nonnegative complementary vector x in $V(A, b)$ with $x_0 = 1$.

For a nonnegative integer n, let E_{2n} denote a set of $2n$ elements partitioned into prescribed n pairs; for each $e \in E_{2n}$ there is a unique element $\overline{e}$ in E_{2n}, called the complement of e such that $\overline{e} \neq e$ and $\overline{\overline{e}} = e$. For a subset S of E_{2n} let $\overline{S} = \{\overline{e} \mid e \in S\}$. A subset S is called *complementary* if $S \cap \overline{S} = \emptyset$. We use $\hat{E}_{2n}$ to denote $E_{2n} \cup \{g\}$ for a given E_{2n} and a distinguished element g which is assumed to be not in E_{2n}.

One canonical choice of E_{2n} and $\hat{E}_{2n}$ is $\{1, 2, \ldots, 2n\}$ and $\{0, 1, 2, \ldots, 2n\}$ with $g = 0$, respectively, where $\overline{i} = i + n$ if $1 \leq i \leq n$, and $\overline{i} = i - n$ if $n < i \leq 2n$.

For a given oriented matroid $M = (\hat{E}_{2n}, \mathcal{F})$ with $n \geq 0$, the OM *complementarity problem* is to

OMCP(M)
$$\text{find } X \in \mathcal{F} \text{ satisfying}$$
$$X \geq 0, \; X_g = +$$
$$X_e \cdot X_{\overline{e}} = 0 \text{ for all } e \in E_{2n}.$$

A signed vector X on E_{2n} or on $\hat{E}_{2n}$ is called *complementary* if the last condition is satisfied, called *strictly sign-preserving* (s.s.p.) if

$$X_e \cdot X_{\overline{e}} \geq 0 \text{ for all } e \in E_{2n} \text{ and} \tag{1.25}$$
$$X_f \cdot X_{\overline{f}} > 0 \text{ for some } f, \tag{1.26}$$

and called *strictly sign-reversing* (s.s.r.) if

(1.27) $X_e \cdot X_{\overline{e}} \leq 0$ for all $e \in E_{2n}$ and

(1.28) $X_f \cdot X_{\overline{f}} < 0$ for some f.

Theorem 1.39 (OMCP Duality Theorem). *Let $M = (\hat{E}_{2n}, \mathcal{F})$ be an oriented matroid satisfying the following conditions:*

(1.29) *either* $\mathcal{F}$ *contains no s.s.r. vector* X *with* $X_g = 0$
or $\mathcal{F}$ *contains no s.s.p. vector* X *with* $X_g \neq 0$ *and,*

(1.30) *either* $\mathcal{F}^*$ *contains no s.s.r. vector* Y *with* $Y_g = 0$
or $\mathcal{F}^*$ *contains no s.s.p. vector* Y *with* $Y_g \neq 0$.

Then exactly one of the following statements holds:

(a) *there exists a nonnegative complementary vector X in $\mathcal{F}$ with $X_g = +$;*

(b) *there exists a nonnegative complementary vector Y in $\mathcal{F}^*$ with $Y_g = +$.*

We call an oriented matroid $M' = (E_{2n}, \mathcal{F}')$ *sufficient* if $\mathcal{F}'$ contains no s.s.r. vector and its dual $(\mathcal{F}')^*$ contains no s.s.p. vector. If $M = (\hat{E}_{2n}, \mathcal{F})$ is an single-element extension of a sufficient oriented matroid $M' = (E_{2n}, \mathcal{F}')$ by g, i.e., $M \backslash g = M'$, then clearly M satisfies the conditions (1.29) and (1.30) of Theorem 1.39. Therefore we have the following corollary of the duality theorem.

Corollary 1.40. *Let $M = (\hat{E}_{2n}, \mathcal{F})$ be any single-element extension of a sufficient oriented matroid M' by g. Then exactly one of the following statements holds:*

(a) *there exists a nonnegative complementary vector X in $\mathcal{F}$ with $X_g = +$;*

(b) *there exists a nonnegative complementary vector Y in $\mathcal{F}^*$ with $Y_g = +$.*

For the elementary proof of Theorem 1.39, what we need is the notion of minors of OMCP, which was introduced by Todd [Tod84]. Let $M = (\hat{E}_{2n}, \mathcal{F})$ be an oriented matroid with $n > 0$. For each complementary element e of E_{2n}, let $M(e) = (\hat{E}_{2n} \backslash (\{e, \overline{e}\}), \mathcal{F}(e))$ denote the oriented matroid $M \backslash e / \overline{e}$, which is called a *complementary minor* of M. The OMCP duality theorem follows immediately from the following two lemmas.

Lemma 1.41. *Let $M = (\hat{E}_{2n}, \mathcal{F})$ be an oriented matroid satisfying the conditions (1.29) and (1.30) of Theorem 1.39. Then the complementary minor $M(e)$ of M satisfies the same conditions for any element e of E_{2n}.*

Lemma 1.42. *Let $M = (\hat{E}_{2n}, \mathcal{F})$ be an oriented matroid satisfying the conditions (1.29) and (1.30) of Theorem 1.39. Then for each $e \in E_{2n}$, at most one of the following four statements holds:*

(a1) $\exists$ *a complementary vector X^1 in $\mathcal{F}$ with $X^1 \backslash \{e, \overline{e}\} \geq 0, X^1_{\overline{e}} = -$ and $X^1_g = +$;*

(a2) $\exists$ *a complementary vector X^2 in $\mathcal{F}$ with $X^2 \backslash \{e, \overline{e}\} \geq 0, X^2_e = -$ and $X^2_g = +$;*

(b1) $\exists$ *a complementary vector Y^1 in $\mathcal{F}^*$ with $Y^1 \backslash \{e, \overline{e}\} \geq 0, Y^1_e = -$ and $Y^1_g = +$;*

(b2) $\exists$ *a complementary vector Y^2 in $\mathcal{F}^*$ with $Y^2 \backslash \{e, \overline{e}\} \geq 0, Y^2_{\overline{e}} = -$ and $Y^2_g = +$.*

1.13 Topological Interpretation of Oriented Matroid Programming

The definitions of OMP and OMCP in the previous section were a natural abstraction of LP and LCP, and it enable us to present a generalization of the LP and LCP duality theorems quickly.

Yet, these definitions do not give much intuitive ideas about the generalized optimization problems. In this section, we shall interprete OMP topologically, using the topological representation given in Section 1.2.

First we give few definitions which will be useful for the interpretation. For the sequel of the section, let $M = (E, \mathcal{F})$ be an oriented matroid and let g and f be two fixed elements of E. We consider the problem P=OMP(M, g, f). We define two sets

(1.31) $$\mathcal{A} = \{X \in \mathcal{F} : X_g = +\} \text{ and}$$

(1.32) $$\mathcal{A}^\infty = \{X \in \mathcal{F} : X_g = 0\},$$

where $\mathcal{A}$ is called the *affine space* and $\mathcal{A}^\infty$ is called the *infinity space*. We shall call a vector in $\mathcal{A}^\infty$ a *direction*. The *feasible region* of the OMP is the subset of the affine space defined by

(1.33) $$\mathcal{P} = \{X \in \mathcal{A} : X_j \geq 0 \quad \forall j \in E \backslash \{f, g\}\}.$$

For a feasible vector $X \in \mathcal{P}$, a direction $Z\mathcal{A}^\infty$ is said to be *feasible* at X if $X \circ Z \in \mathcal{P}$. Recall that a direction Z is called augmenting at X if $Z_f = +$ and $Z_j \geq 0$ for all $j \in E \backslash \{f, g\}$ with $X_j = 0$. It follows that a direction Z is augmenting at X if and only if it is a feasible direction and $Z_f = +$.

Now look at a topological representation of M. We assume that M has dimension 2 and thus can be represented by a sphere system $(E, S, \mathcal{A})$ where S a 2-sphere, and $\mathcal{A}$ is a family $\{\{s_i^+, s_i^0, s_i^-\} : i \in E\}$ of partitions satisfying Axioms 1.2. In order to illustrate the OMP, we only need to see the nonnegative part of the infinity equator s_g, which corresponds to the union of $\mathcal{A}$ and $\mathcal{A}^\infty$. The figure below shows the affine space $\mathcal{A}$ as an open disk and the infinity space $\mathcal{A}^\infty$ as the bounding circle.

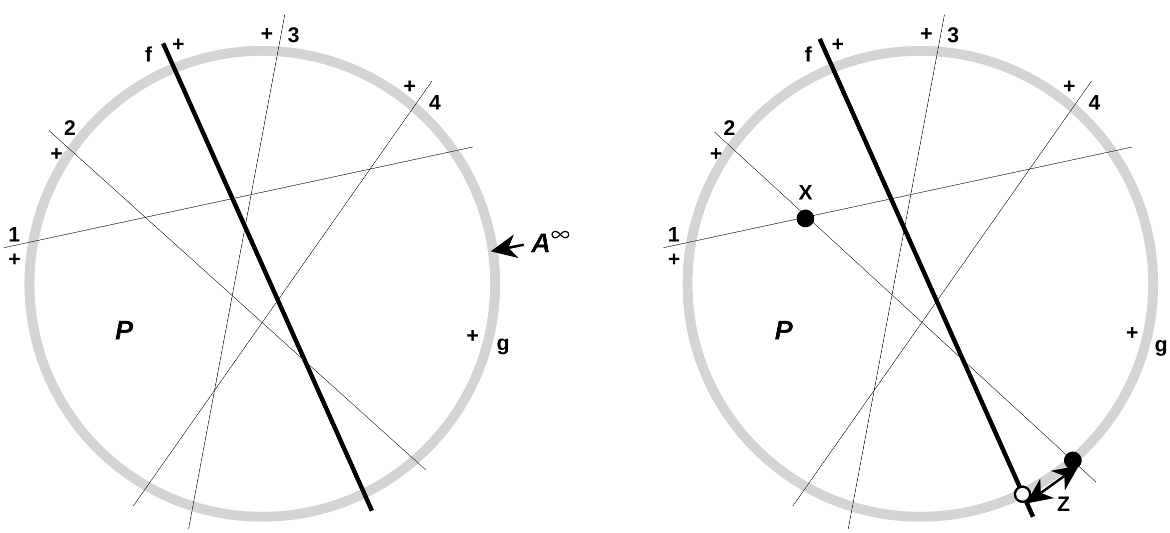
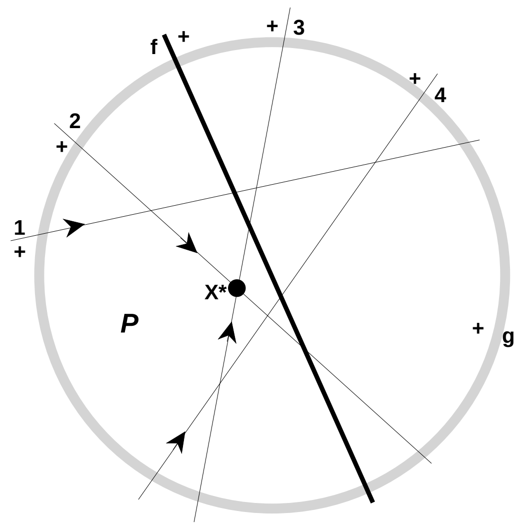


Figure 1.9: The Affine Space, the Infinity Space and Augmenting Directions

The feasible region $\mathcal{P}$ is simply the intersection of the nonegative sides of each constraints $1, 2, 3, 4$ within the affince space. What about augmenting directions? The following figure shows the set of augmenting directions Z for some feasible point X. By definition, this set is the intersection of the nonegative sides of 1 and 2, the positive side of f and the infinity space.

A simple way to characterize an optimal solution is by orienting one dimensional affine flats of the oriented matroid. Here, an affine flat is the intersection of some equators with the affine space. The empty set is the only -1 flat, the vertices in the affine space are the 0 flats, and the lines are the minimal affine flats containing the vertices properly. In our small examples, the lines coincide with the *hyperplanes* of $1, 2, 3, 4, f$, the intersection of each equators with the affine space.

Now we can orient each line generated by the constraint hyperplanes $1,2,3,4$. Each affine line either crosses the objective hyperplane or does not cross it. For the latter case (parallel case), we do not give any orientation to the line. Whenever a line crosses f, we orient the line from the $-$ side of f to the $+$ side of f. This orientation is merely the abstraction of the incremental direction for linear programming. In our example, all hyperplanes $1,2,3,4$ crosses f and all affine lines are oriented as follows:



It is intuitively obvious that a feasible vertex X^* is optimal if it is a sink of the directed graph of the feasible region. But it is not that trivial and a rigorous proof is necessary. Given this, the foundamental theorem of OMP states that if the feasible region is nonempty, then there exists either a sink (i.e. an optimal solution) or a feasible line oriented toward infinity (i.e., an unbounded ray).

It is quite natural to extend the simplex method for LP to OMP. The simplex method for OMP produces a sequence of feasible vertices of the feasible region. In order to make the simplex method finite, it is important not to produce a cycle of pivots, and thus a cycle of vertices. For this, Bland [Bla77] asked a question whether there exists a directed cycle of adjacent vertices for some OMP, where two vertices are adjacent if there is a line containing both. The answer is yes, and an oriented matroid which admits a directed cycle for some selections of f and g is known to be *non-Euclidean*.

In the next section, we shall present a construction of non-Euclidean oriented matroids, which was discovered thanks to the topological interpretation of OMP's.

1.14 Construction of non-Euclidean OM's

In order to construct a non-Euclidean OM, we need to work in 3-dimensional space. The simplest construction starts with the following arrangement of 7 hyperplanes including the infinity plane g, see Figure 1.10. Here, we have a feasible region in the middle bounded by 6 hyperplanes $1,2,3,4,5,6$. This region $\mathcal{P}$ is a bipyramid and there is no hyperplane going through the middle horizontal triangle $\{a,b,c\}$. Also, we have a infinity sphere g containing the whole arrangement, corresponding to the infinity space $\mathcal{A}^\infty$. In the figure, we draw only a small part of this infinity sphere where the intersections with hyperplane 2 and 5 are shown.

Next we add the objective hyperplane f which is parallel to the triangle $\{a,b,c\}$ as shown in Figure 1.11. The optimal solution of the OMP is the top vertex of the bipyramid, and its nonhorizontal edges are oriented upward and the horizontal edges of the triangle are not oriented. Note that this OMP is an LP since every hyperplane is linear.

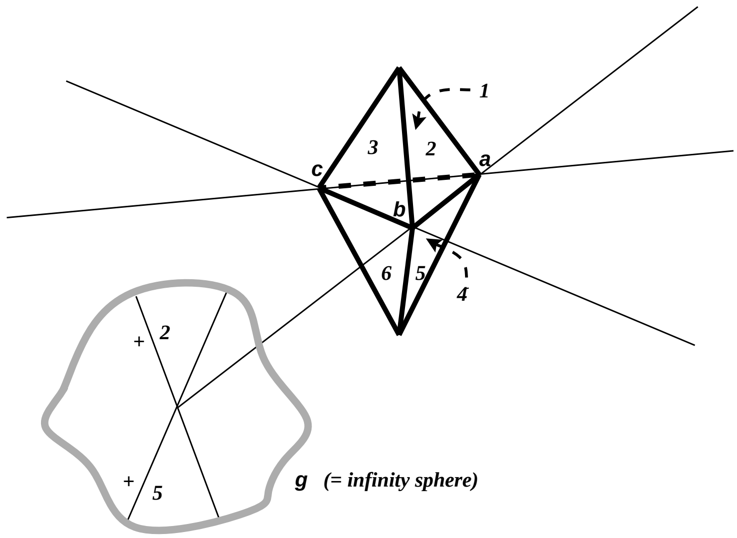


Figure 1.10: Initial Arrangement of 7 hyperplanes with a bipyramid feasible region

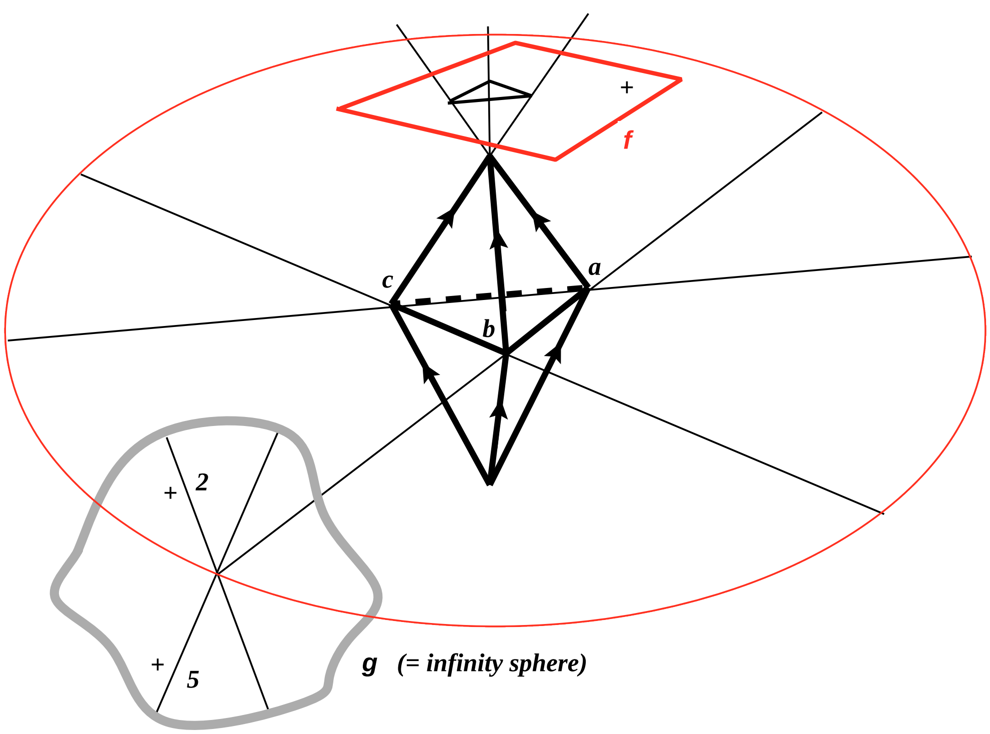


Figure 1.11: Objective element f and the optimum solution X^*

Now, we are at the final stage of the construction, where the three edges of the triangle $\{a, b, c\}$ will get a cyclic orientation by deforming the objective hyperplane f. This deformation operation will be called *perturbation* and will be shown to preserve the OM-ness.

The deformation will be performed at three infinity points, but by symmetry we only have to explain one. Since the objective hyperplane f is parallel to the triangle $\{a, b, c\}$, it meets the affine lines spanned by ab, bc and ca at infinity as shown in Figure 1.11. The key idea is simply to deform f at the two infinity points so that the edge ab gets the orientation from a to b, as shown in Figure 1.12. This deformation must be done symmetrically at two antipodal infinity points. By applying the same perturbation for other two lines, we will get an OMP in which the three feasible vertices form a directed cycle: $a \to b \to c \to a$.

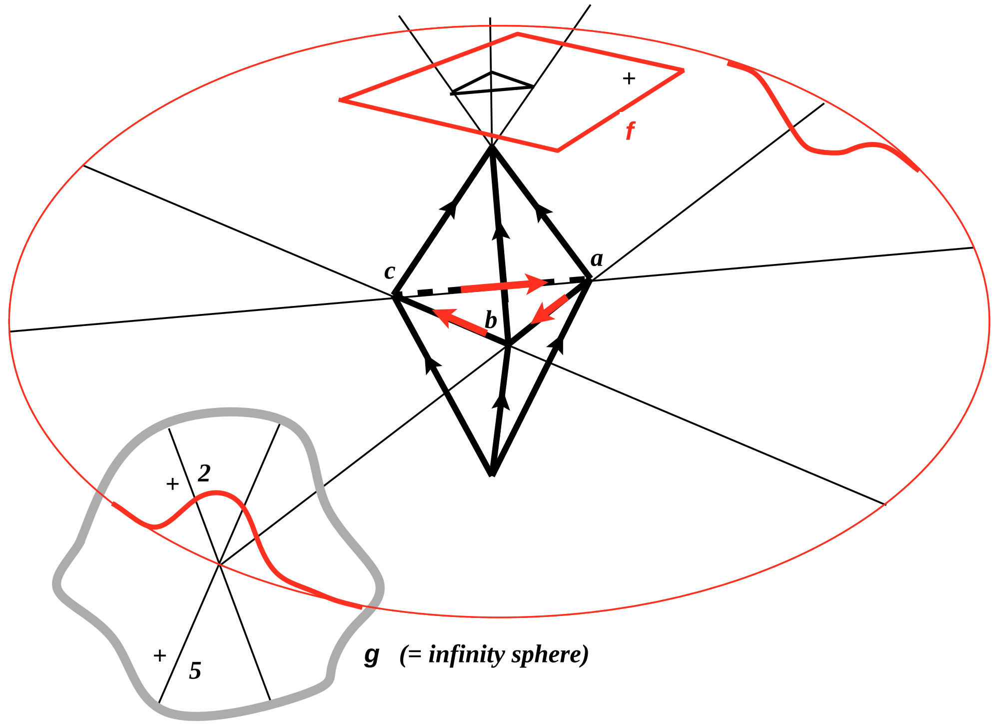


Figure 1.12: Local deformation of f at infinity

2 Geometric Computation

This chapter concerns algorithmic aspects of polyhedral geometry and related structures.

2.1 Minkowski-Weyl's Theorem and Constructive Proofs

Minkowski-Weyl's Theorem states that every polyhedron has two representations, the (finite) inequality representation and the (finite) generator reprentation. This theorem is actually of two theorems, Minkowski's Theorem states that if a set is given by an inequality representation, then it admits a generator representation, and Weyl's Theorem states the reverse.

In this section, we study these theorems in various forms and constructive proofs.

2.1.1 Minkowski-Weyl's Theorem in Various Forms

Let us recall some definitions. For vectors $v_1, v_2, \ldots, v_k \in R^d$, we define $conv\{v_1, v_2, \ldots, v_k\}$ as their *convex hull*:

$$conv\{v_1, v_2, \ldots, v_k\} \quad = \quad \{x : x = \sum_{j=1}^{k} \lambda_j v_j \ , \ \sum_{j=1}^{k} \lambda_j = 1 \text{ and } \lambda_j \geq 0 \ \forall j = 1, 2, \ldots, k\},$$

and define $nonneg\{v_1, v_2, \ldots, v_k)$ as their *nonnegative hull*:

$$nonneg\{v_1, v_2, \ldots, v_k\} \quad = \quad \{x : x = \sum_{j=1}^{k} \lambda_j v_j \text{ and } \lambda_j \geq 0 \ \forall j = 1, 2, \ldots, k\}.$$

For two subsets P and Q of R^d, $P + Q$ denotes the *Minkowski sum* of P and Q:

$$P + Q \quad = \quad \{p + q : p \in P \text{ and } q \in Q\}.$$

The most general form of Minkowski-Weyl's Theorem is the following.

Theorem 2.1 (Minkowski-Weyl's Theorem for Polyhedra). *For a subset P of R^d, the following statements are equivalent:*

(a) *P is a polyhedron, i.e., $P = \{x : Ax \leq b\}$ for some matrix A and vector b ;*

(b) *$P = conv\{v_1, v_2, \ldots, v_n\} + nonneg\{r_1, r_2, \ldots, r_s\}$ for some vectors $v_1, v_2, \ldots, v_n$ and $r_1, r_2, \ldots, r_s$.*

In both statements, the "finiteness" of representation is essential. There are two implications in the theorem. The implication (a) $\Longrightarrow$ (b) is known as Minkowski's Theorem, and the other (b) $\Longrightarrow$ (a) is Weyl's Theorem. These two theorems appear to be significantly different, as remarked by Jack Edmonds. He pointed out that Minkowski's theorem can be proved constructively by a simple strongly polynomial algorithm, while no such proof for Wely's Theorem is known and every known proof relies on some proof of Farkas' Lemma. It should be noted that there is no simple polynomial proof for Farkas' Lemma. We shall discuss this issue later in this section.

While this general form of Minkowski-Weyl' Theorem is most useful for various applications, the following homogeneous form is mathematically more ellegant and easier to deal with.

Theorem 2.2 (Minkowski-Weyl's Theorem for Cones). *For a subset P of R^d, the following statements are equivalent:*

(a) *P is a homogeneous polyhedral cone, i.e., $P = \{x : Ax \geq 0\}$ for some matrix A;*

(b) *$P = nonneg\{r_1, r_2, \ldots, r_s)$ for some vectors $r_1, r_2, \ldots, r_s$.*

The last form is another special case of the first form.

Theorem 2.3 (Minkowski-Weyl's Theorem for Polytopes). *For a subset P of R^d, the following statements are equivalent:*

(a) *P is a bounded and $P = \{x : Ax \leq b\}$ for some matrix A and vector b ;*

(b) *P is a polytope, i.e., $P = conv\{v_1, v_2, \ldots, v_n\}$ for some vectors $v_1, v_2, \ldots, v_n$.*

What are the relationships between these three theorems? Do we need to prove three theorems separately? Of course, not, since there are trivial implications such as Theorem 2.1 $\Longrightarrow$ Theorem 2.3 . Since we are going to prove the theorem for cones, we need to do the following.

Proposition 2.4. *Theorem 2.2 $\Longrightarrow$ Theorem 2.1.*

Proof. Exercise.

□

2.1.2 Minkowski-Weyl's Theorem and Farkas' Lemma

In order to prove Minkowski-Weyl's Theorem, it is important to understand the relation between two sub-statements in the theorem, Minkowski's Theorem and Weyl's Theorem. The main purpose of this section is to prove they are equivalent under Farkas' Lemma.

For this purpose, the most convenient form of Farkas Lemma is the following: (Note: We do not prove Farkas' Lemma here. But I plan to add a small section to prove it constructively.)

Theorem 2.5 (Farkas' Lemma). *For any matrix $A \in R^{m \times d}$ and any vector $b \in R^m$, exactly one of the following statements hold:*

(a) *$\exists x \in R^d$ such that $A\, x \geq \mathbf{0}$ and $b^T\, x < 0$;*

(b) *$\exists z \in R^m$ such that $z \geq \mathbf{0}$ and $b^T = z^T\, A$.*

In order to relate Farkas' Lemma and Minkowski-Weyl's Theorem, we first rewrite the latter theorem in matrix form. Also this matrix form is very convenient for our presentation of algorithms to prove Minkowski's Theorem constructively.

A pair (A, R) of real matrices A and R is said to be a *double description pair* or simply a *DD pair* if the relationship

$$Ax \geq 0 \quad \text{if and only if} \quad x = R\,\lambda \text{ for some } \lambda \geq 0$$

holds. Clearly, for a pair (A, R) to be a DD pair, it is necessary that the column size of A is equal to the row size of R, say d. The term "double description" was introduced by Motzkin et al. [MRTT53], and it is quite natural in the sense that such a pair contains two different descriptions of the same object. Namely, the set $P(A)$ represented by A as

$$P(A) = \{x \in R^d : Ax \geq 0\} \tag{2.1}$$

is simultaneously represented by R as

$$\{x \in R^d : x = R\,\lambda \text{ for some } \lambda \geq 0\}. \tag{2.2}$$

A matrix A is called a *representation matrix* of the polyhedral cone $P(A)$. When a cone P is represented by (2.2), we say R is a *generating matrix* for P or a matrix R *generates* the polyhedral cone. Clearly, each column vector of a generating matrix R lies in the cone P and every vector in P is a nonnegative combination of some columns of R.

Now we are ready to rewrite Minkowski and Weyl Theorems for cones, Theorem 2.2. Minkowski's Theorem states that every polyhedral cone admits a generating matrix.

Theorem 2.6 (Minkowski's Theorem for Cones). *For any $m \times d$ real matrix A, there exists some $d \times n$ real matrix R such that (A, R) is a DD pair, or in other words, the cone $P(A)$ is generated by R.*

Here again, the nontriviality is in that the row size n of R is a finite number. If we allow the size n to be infinite, there is a trivial generating matrix consisting of all vectors in the cone. In this respect, the essence of the theorem is the statement "every cone is finitely generated."

Weyl's Theorem is the converse.

Theorem 2.7 (Weyl's Theorem for Cones). *For any $d \times n$ real matrix R, there exists some $m \times d$ real matrix A such that (A, R) is a DD pair, or in other words, the set generated by R is the cone $P(A)$.*

Now we show the following basic result using Farkas' Lemma, Theorem 2.5.

Proposition 2.8. *For any real matrices A and R, the pair (A, R) is a DD pair if and only if the pair (R^T, A^T) is a DD pair.*

Proof. The proof is left to the reader. □

The importance of this proposition is that it is sufficient to prove one of the two theorems, Theorem 2.6 and Theorem 2.7, since one follows directly from the other by the proposition. We give two different proofs of Minkowski's Theorem, Theorem 2.6 in the following two sections.

2.1.3 First Proof of Minkowski's Theorem

In this section, we give a constructive proof of Minkowski's Theorem using a finite algorithm, known as the double description (DD) method [MRTT53].

Suppose that an $m \times d$ matrix A is given, and let $P(A) = \{x : Ax \geq 0\}$. We call any vector $r \in P(A)$ *ray* of $P(A)$. The DD method is an incremental algorithm to construct a $d \times n$ matrix R such that (A, R) is a DD pair.

Let K be a subset of the row indices $\{1, 2, \ldots, m\}$ of A and let A_K denote the submatrix of A consisting of rows indexed by K. Suppose we already found a generating matrix R for $P(A_K)$, or equivalently (A_K, R) is a DD pair. If $A = A_K$, clearly we are done. Otherwise we select any row index i not in K and try to construct a DD pair (A_{K+i}, R') using the information of the DD pair (A_K, R).

Once this basic procedure is described, we have an algorithm to construct a generating matrix R for $P(A)$. This procedure can be easily understood geometrically and the reader is strongly encouraged to draw some simple example in the three dimensional space.

The newly introduced inequality $A_i\, x \geq 0$ partitions the space R^d into three parts:

$$\begin{aligned} H_i^+ &= \{x \in R^d : A_i\, x > 0\} \\ H_i^0 &= \{x \in R^d : A_i\, x = 0\} \\ H_i^- &= \{x \in R^d : A_i\, x < 0\}. \end{aligned} \tag{2.3}$$

Let J be the set of column indices of R and let r_j denote the jth column of R. The rays r_j $(j \in J)$ are then partitioned into three parts:

$$\begin{aligned} J^+ &= \{j \in J : r_j \in H_i^+\} \\ J^0 &= \{j \in J : r_j \in H_i^0\} \\ J^- &= \{j \in J : r_j \in H_i^-\}. \end{aligned} \tag{2.4}$$

We call the rays indexed by J^+, J^0, J^- the *positive*, *zero*, *negative* rays with respect to i, respectively. To construct a matrix R' from R, we generate new $|J^+| \times |J^-|$ rays lying on the ith hyperplane H_i^0 by taking an appropriate positive combination of each positive ray r_j and each negative ray $r_{j'}$ and by discarding all negative rays.

The following lemma ensures that we have a DD pair (A_{K+i}, R'), and provides the key procedure for the most primitive version of the DD method.

Lemma 2.9 (Main Lemma for Double Description Method). *Let (A_K, R) be a DD pair and let i be a row index of A not in K. Then the pair (A_{K+i}, R') is a DD pair, where R' is the $d \times |J'|$ matrix with column vectors r_j ($j \in J'$) defined by*

$$\begin{aligned} J' &= J^+ \cup J^0 \cup (J^+ \times J^-), \text{ and} \\ r_{jj'} &= (A_i\, r_j) r_{j'} - (A_i\, r_{j'}) r_j \quad \text{for each } (j, j') \in J^+ \times J^-. \end{aligned}$$

Proof. Let $P = P(A_{K+i})$ and let P' be the cone generated by the matrix R'. We must prove that $P = P'$. By the construction, we have $r_{jj'} \in P$ for all $(j, j') \in J^+ \times J^-$ and $P' \subset P$ is clear.

Let $x \in P$. We shall show that $x \in P'$ and hence $P \subset P'$. Since $x \in P$, x is a nonnegative combination of r_j's over $j \in J$, i.e., there exist $\lambda_j \geq 0$ for $j \in J$ such that

$$x = \sum_{j \in J} \lambda_j r_j. \tag{2.5}$$

If there is no positive λ_j with $j \in J^-$ in the expression above then $x \in P'$. Suppose there is some $k \in J^-$ with $\lambda_k > 0$. Since $x \in P$, we have $A_i\ x \geq 0$. This together with (2.5) implies that there is at least one $h \in J^+$ with $\lambda_h > 0$. Now by construction, $hk \in J'$ and

$$r_{hk} = (A_i\ r_h) r_k - (A_i\ r_k) r_h. \tag{2.6}$$

By subtracting an appropriate positive multiple of (2.6) from (2.5), we obtain an expression of x as a positive combination of some vectors r_j ($j \in J'$) with new coefficients λ_j where the number of positive λ_j's with $j \in J^+ \cup J^-$ is strictly smaller than in the first expression. As long as there is $j \in J^-$ with positive λ_j, we can apply the same transformation. Thus we must find in a finite number of steps an expression of x without using r_j such that $j \in J^-$. This proves $x \in P'$, and hence $P \subset P'$. □

Now we can finish the first proof of Minkowski's Theorem.

Proof. (**of Theorem 2.6**) By Lemma 2.9, it is enough to show that if A has only one row, there exists a matrix R such that (A, R) a DD pair. One can use any set of vectors to span the space $\{x : Ax = 0\}$, their negatives and the vector A^T itself to form the columns of R (why?). □

Here we write the DD method in procedural form:

```
procedure DoubleDescriptionMethod(A);
begin
    Obtain any initial DD pair (A_K, R);
    while K ≠ {1, 2, ..., m} do
    begin
        Select any index i from {1, 2, ..., m} \ K;
        Construct a DD pair (A_{K+i}, R') from (A_K, R);
            /* by using Lemma 2.9 */
        R := R';    K := K + i;
    end;
    Output R;
end.
```

The DD method given here is very primitive, and the straightforward implementation is not quite useful, because the size of J increases very fast and goes beyond any tractable limit. One reason for this is that many (perhaps, most) vectors $r_{jj'}$ the algorithm generates (defined in Lemma 2.9), are unnecessary. We shall discuss later how one can avoid generating redundant vectors.

2.1.4 Second Proof of Minkowski's Theorem

The second proof of Minkowski's Theorem is due to Jack Edmonds.

We assume that an $m \times d$ matrix A is given, and let $P(A) = \{x : A\ x \geq \mathbf{0}\}$. The *zero set* $Z(r)$ of a vector $r \in R^d$ is $\{i : A_i\ r = 0\ \}$. Recall that any nonzero vector $r \in P(A)$ is called a ray of $P(A)$. A ray r of $P(A)$ is said to be an *elementary ray* if its zero set is maximal. When $P(A)$ is a pointed cone (i.e., $\mathbf{0}$ is an extreme point), there are only finitely many elementary rays up to positive multiples. Here is the key lemma for proving Minkowski's Theorem.

Lemma 2.10. *Let A be an $m \times d$ rational matrix A, and suppose $P(A) = \{x : A\ x \geq 0\}$ be a pointed cone (i.e., $\mathbf{0}$ is an extreme point). Given any rational ray r of $P(A)$, there exists a strongly polynomial algorithm to find elementary rays $r_1, \ldots, r_s$ of $P(A)$ such that $s \leq d$ and r is a positive combination of r_i's.*

Proof. We use induction on d. When $d = 1$, every ray is elementary and the the result is trivial. Assume by induction that for any A with column size less than d, the lemma is true.

Assume that $P(A) = \{x : A\, x \geq \mathbf{0}\}$ is a pointed cone, and let r be any ray of $P(A)$. We may assume that no row vector of A is zero vector, since any zero row of A can be eliminated without affecting the cone.

If $Z(r)$ is nonempty, then r can be considered as a ray of the cone $P' = P(A) \cap \{x : A_k\, x = 0\}$ for any fixed $k \in Z(r)$. Clearly P' is pointed. Since A_k is nonzero, we may assume, say the last component a_{kd} is nonzero. Then P' is representable by a system of linear inequalities in $(d-1)$ variables x' as

$$P' = \{(x', x_d) : A'\, x' \geq \mathbf{0} \text{ and } x_d = a^T x'\}$$

for some matrix A' and vector a. This case reduces to the case with smaller column size for which there is a strongly polynomial algorithm to find at most $d-1$ elementary rays of P' to represent r. Every elementary ray of P' is elementary in P, and the result follows.

Suppose $Z(r)$ is empty set. Since $P(A)$ is pointed, starting from the point r, one can obtain an elementary ray, say r_1, of $P(A)$ by exactly d pivot operations for the system of equations $y = A\, x$ in $(m+d)$ variables. This procedure is clearly strongly polynomial. Now we draw a line L from r_1 toward r. Since $P(A)$ is pointed, this line must leave the cone $P(A)$ at some point, say r'. (See Figure 2.13 illustrating a cut section of the cone $P(A)$ by some hyperplane.) Clearly $Z(r')$ is not empty, and we can apply the first argument to r' to obtain at most $d-1$ elementary rays to represent r'. These rays together with r_1 represent r as their positive combinations. This completes the proof. □

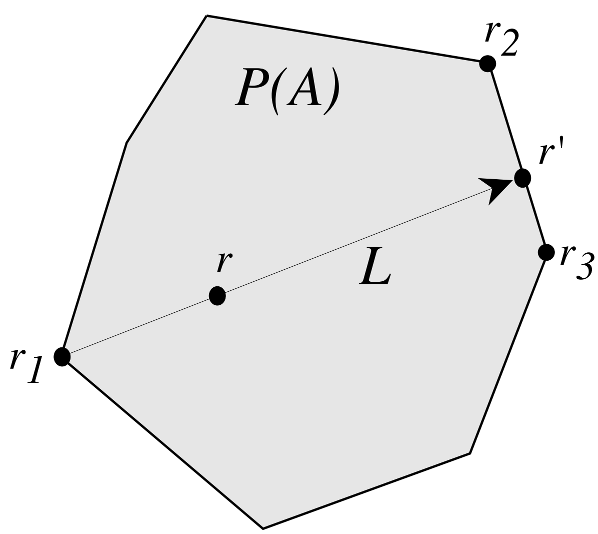


Figure 2.13: A Cut Section of $P(A)$ in Edmonds' Proof

Since there are only finitely many elementary rays in a pointed cone, Minkowski's Theorem for pointed cones follows directly from the lemma above. Observe that the proof is a simple constructive algorithm that is strongly polynomial. As far as we know, there is no similar proof for Weyl's Theorem.

it is not difficult to extend the proof above of Minkowski's Theorem to the non-pointed case, since such a case can be reduced to the pointed case.

Lemma 2.11. *Let A be an $m \times d$ rational matrix A, and let $N(A)$ and $R(A)$ be the null space and the row space of A, respectively.*

(a) *$P(A)$ is pointed if and only if $N(A) = \{\mathbf{0}\}$.*

(b) *$P'(A) := P(A) \cap R(A)$ is a pointed cone.*

(c) *Every ray of $P(A)$ is representable by a sum of two rays, one in $P'(A)$ and the other in $N(A)$.*

Proof. (a) Suppose $N(A) \neq \{\mathbf{0}\}$. Then for any ray $r \in N(A)$, the zero vector is the sum of $\frac{1}{2}r$ and $\frac{1}{2}(-r)$ which are both in $P(A)$. Thus the zero vector is not an extreme point of $P(A)$. Conversely, suppose the zero vector is not extreme. Then $\mathbf{0}$ is a convex combination of two rays, say r_1 and r_2. It follows that both r_1 and r_2 must be in $N(A)$, implying $N(A) \neq \{\mathbf{0}\}$.

(b) Let B be any matrix whose rows form a basis of $N(A)$. Then $P'(A) = \{x : A\,x \geq \mathbf{0} \text{ and } B\,x = \mathbf{0}\}$. Clearly the zero vector is the only vector x orthogonal to all rows of A and B. By (a), this implies $P'(A)$ is pointed.

(c) Let r be a ray of $P(A)$. Let r' be the orthogonal projection of r to the subspace $R(A)$, that is, $r - B^T(BB^T)^{-1}Br$. It is easy to verify that $r = r' + s$ and $r' \in P'(A)$, where $s = B^T(BB^T)^{-1}Br \in N(A)$. This completes the proof. □

Finally we have the second proof of Minkowski's Theorem.

Proof. (**of Theorem 2.6**) Let $P'(A) := P(A) \cap R(A)$, and let $b_1, \ldots, b_k$ be a basis of $N(A)$. By Lemma 2.11, $P'(A)$ is pointed and by Lemma 2.10, every ray of $P'(A)$ is in the nonnegative hull of the finite set of all elementary rays, say $r_1, \ldots, r_s$ with length 1. Observing that every vector in $N(A)$ is a nonnegative combination of $b_1, \ldots, b_k$ and their negatives, Lemma 2.11 (c) implies that

$$P(A) = \mathit{nonneg}\{r_1, \ldots, r_s, b_1, \ldots, b_k, -b_1, \ldots, -b_k\}.$$

This completes the proof. □

2.2 Computational Complexity

In the first chapter, we have presented some fundamental geometric problems, Geometric Computation 1.1 1.6, associated with convex polytopes and arrangements of hyperplanes. These problems are of similar nature. Namely the size of output might be very large with respect to the size of input. In particular, the output size cannot be bounded by a polynomial in the input size, in general. For this sort of problem, the usual notion of *efficient algorithms* is not appropriate since such an algorithm is required to run in time polynomial in the input size. Thus we are motivated to relax the notion of efficient algorithms so that the time complexity can depend on the output size as well. Basically, we shall allow a time complexity that grows polynomially (preferably linearly) in the output size. We shall give formal definitions below.

Let Σ denote the set $\{0, 1\}$ (alphabet) and Σ^* denote the set of ordered sequences (words) of 0's and 1's. A *relation* R is a boolean function from the product space $\Sigma^* \times \Sigma^*$ to Σ. An instance of the *enumeration problem associated with a relation* R is to list all members of the *enumeration* set $Enum(R, x) = \{y : R(x, y) = 1\}$, for a given x.

A *polynomial-time* algorithm for the enumeration problem associated with a relation R is one whose time complexity is polynomial in the input size $|x|$ and the output size $|Enum(R, x)|$ for any input x. A *linear-time* algorithm is a polynomial time algorithm whose time complexity is linear in the output size. Finally, a *compact* algorithm for the enumeration problem associated with a relation R is one whose space complexity is polynomial in the input size $|x|$. A relation R is called (*strongly*) *P-enumerable* if there exists a (compact) linear-time algorithm for the enumeration problem.

Here we take some examples. Let E be a finite set. For a rational $E \times d$ matrix$A$, let

(2.7) $$C(A) := \{Ax : x \in R^d\} \quad \text{(the column space of } A \text{)}$$

(2.8) $$\mathcal{F}(A) := \sigma(C(A)) \quad \text{(the oriented matroid represented by } A \text{).}$$

The *arrangement face relation* R_f (the *arrangement vertex relation* R_v) is defined by $R_f(A, Y) = 1$ ($R_v(A, V) = 1$) if and only if Y is in $\mathcal{F}(A)$ (V is in $\min(\mathcal{F}(A)\backslash\{\mathbf{0}\})$).

2.3 Reverse Search

In the introduction we have given a basic idea of reverse search for enumeration. Here we shall present it formally with more generality.

Let $G = (V, E)$ be a (undirected) graph with vertex set V and edgecite set E. We shall call a triple (G, S, f) *local search* if S is a subset of V, f is a mapping: $V\backslash S \Rightarrow V$ satisfying

(L1) $\{v, f(v)\} \in E$ for each $v \in V\backslash S$.

and *finite* local search if in addition,

(L2) for each $v \in V\backslash S$, there exists a positive integer k such that $f^k(v) \in S$.

Here is a procedural form of a local search (G, S, f):

```
procedure LocalSearch(G,S,f,v0:vertex of G);
    v := v0;
    while v ∉ S do
        v := f(v)
    endwhile;
    output v.
```

The function f is said to be the *local search function*, and G the *underlying graph structure*. Naturally, we consider the set V to be the set of *candidates* for a solution, the set S to be the set of *solutions*. The local search function f is simply an algorithm for finding one solution.

It is not difficult to find examples of local search. To list a few,

- the simplex method for linear programming, where V is the set of feasible bases, E is induced by the pivot operation, f is the simplex pivot, and S is the set of optimal bases,
- the edge-exchange algorithm for finding a minimum spanning tree in a weighted graph [AHU87, Section 10.5], where V is the set of all spanning trees, E is induced by the edge-exchange operation, f is the best-improvement exchange algorithm, and S is the set of all minimum spanning trees,
- the flip procedure for finding a Delaunay triangulation in the plane for a given set of points, where V is the set of all possible triangulations, E is induced by the flip operation, f is the flip algorithm, and S is the set of Delaunay triangulations.

It will be helpful for us to keep at least one of these examples in mind for better understanding of several new notions to be introduced below.

The *trace* of a local search (G, S, f) is a directed subgraph $T = (V, E(f))$ of G where

$$E(f) \quad = \quad \{(v, f(v)) : v \in V \backslash S\}.$$

The trace T is simply the digraph with all vertices of G and those edges of G used by the local search. We also define the *height* $h(T)$ of a trace T as the length of a longest directed path in T. An obvious but important remark is

Property 2.12. *If (G, S, f) is a finite local search then its trace T is a directed spanning forest of G with each component containing exactly one vertex of S as a unique sink.*

Let (G, S, f) be a finite local search with trace T, and denote by $T(s)$ the component of T containing a vertex s for each $s \in S$. We call the following procedure an abstract reverse search:

```
procedure AbstractReverseSearch(G,S,f);
    for  each vertex s ∈ S do
        traverse the component T(s) and output all its vertices
    endfor.
```

Here we are purposely vague in describing how we traverse $T(s)$. The actual traversal depends on how the local search is given: in almost all cases for which reverse search is useful, G is not explicitly given. Also, we shall deal with the case where $|S| > 1$, and the set S is not explicitly given. In many cases, we can consider S to be the output of another reverse search for which the solution set is a singleton.

Thus, it is extremely useful to discuss a special implementation of reverse search when the local search is given in a certain way which is general enough for our applications to be described later but yet restricted enough for us to make interesting statements about the time complexity of reverse search.

We say that a graph G is given by *adjacency-oracle* or simply *A-oracle* when the following conditions are satisfied:

(A1) The vertices are represented by nonzero integers.

(A2) An integer δ is explicitly given which is an upper bound on the maximum degree of G, i.e., for each vertex $v \in V$, the degree $deg(v)$ is at most this number.

(A3) the adjacency list oracle *Adj* satisfying (i), (ii) and (iii) is given:

- **(i)** for each vertex v and each number k with $1 \le k \le \delta$ the oracle returns $Adj(v, k)$, a vertex adjacent to v or extraneous 0 (zero),
- **(ii)** if $Adj(v, k) = Adj(v, k') \neq 0$ for some $v \in V$, k and k', then $k = k'$,
- **(iii)** for each vertex v, $\{Adj(v, k) : Adj(v, k) \neq 0, 1 \le k \le \delta\}$ is exactly the set of vertices adjacent to v.

The conditions (i) - (iii) imply that *Adj* returns each adjacent vertex to v exactly once during the δ inquiries $Adj(v, k)$, $1 \le k \le \delta$, for each vertex v.

The conditions (A2) and (A3) may not seem to be natural, but as we will see, in many cases, we have no knowledge of the maximum degree of the underlying graph but only an upper bound. Consider the simplex method. For each feasible basis, some pivot operations lead to feasible bases, and others lead to non-feasible bases. In general (with possible degeneracy and an unbounded feasible region), we do not know the maximum number of adjacent feasible bases. However, we have a trivial bound, that is, the number of pivot positions (= number of basic variables times number of nonbasic variables). Associated with each feasible basis and each k-th pivot position we have either an adjacent feasible basis or something else (i.e. a nonfeasible basis or impossible pivot), that determines our A-oracle. For the flip algorithm, one can naturally see that the underlying graph is given by A-oracle.

A local search (G, S, f) is said to be *given by an A-oracle* if the underlying graph G is.

When a local search is given by an A-oracle, we can write a particular implementation of abstract local search. The following procedure, ReverseSearch, which will be used in all of the applications in the present paper, is the one where the traversal of each component is done by depth first search and the set S is explicitly given:

```
          procedure ReverseSearch(Adj,δ,S,f);
              for each vertex s ∈ S do
                  v := s; j := 0; (* j: neighbor counter *)
                  repeat
                      while j < δ do
                          j := j + 1;
(r1)                      next := Adj(v, j);
                          if next ≠ 0 then
(r2)                          if f(next) = v then  (* reverse traverse *)
                                  v := next; j := 0
                              endif
                          endif
                      endwhile;
                      if v ≠ s then (* forward traverse *)
(f1)                      u := v;    v := f(v);
(f2)                      j := 0;    repeat j := j + 1 until Adj(v, j) = u (* restore j *)
                      endif
                  until v = s and j = δ
              endfor.
```

Note that for each vertex $v \in V \backslash S$, exactly one "forward traverse" is performed in the procedure ReverseSearch. The time complexity of ReverseSearch can be now evaluated. For a local search (G, S, f) given by an A-oracle, let $t(f)$ and $t(Adj)$ denote the time to evaluate f and Adj, respectively.

Theorem 2.13. *Suppose that a local search (G, S, f) is given by an A-oracle. Then the time complexity of ReverseSearch is $O(\delta\ t(Adj)|V| + t(f)|E|)$.*

Proof. It is easy to see that the time complexity is determined by the total time spent to execute the four lines (r1), (r2), (f1) and (f2). The first line (r1) is executed at most δ times for each vertex, and the total time spent for (r1) is $O(\delta\ t(Adj)|V|)$. The line (r2) is executed as many times as the degree $deg(v)$ for each vertex v, and thus the total time for (r2) is $O(t(f)|E|)$. The third line (f1) is executed for each vertex v in $V \backslash S$, and hence the total time for (f1) is $O(t(f)(|V| - |S|))$. Similarly, the total time for (f2) is $O(\delta\ t(Adj)(|V| - |S|))$. Since $|V| - |S| \leq |E|$, by adding up the four time complexities above, we have the claimed result. □

Corollary 2.14. *Suppose that a local search (G, S, f) is given by an A-oracle. Then the time complexity of ReverseSearch is $O(\delta\ (t(Adj) + t(f))|V|)$. In particular, if δ, $t(f)$ and $t(Adj)$ are independent of the number $|V|$ of vertices in G, then the time complexity is linear in the output size $|V|$.*

Proof. The claim follows immediately from Theorem 2.13 and the fact that $2|E| \leq \delta|V|$. □

The assumption that δ, $t(f)$ and $t(Adj)$ are independent of the number $|V|$ is not satisfied in general (e.g. when G is a complete graph), but is for many cases it can be assumed. In fact, all applications to be presented in the next section satisfy this assumption.

We have already two simple formulas, Theorem 2.13 and Corollary 2.14, for the time complexity of the reverse search. We shall use these to evaluate the time complexity of some of our applications. However, a stronger result is possible in some cases. One of such cases is when the lines (r1) and (r2) have a shortcut, i.e., it is possible to check for any vertex v of G and any integer $1 \le j \le \delta$ whether $f(Adj(v,j)) = v$ without executing Adj and f. Another case is when the procedure (f2) has a shortcut, i.e., it is possible for any vertex v of G to determine the integer j such that $Adj(f(v), j) = v$ without executing Adj explicitly. In order to deal with these cases more clearly we shall give below an alternative version of reverse search. Here we use the convention that $f(0) = 0$.

```
      procedure ReverseSearch2(Adj,δ,S,f);
          for each vertex s ∈ S do
              v := s; j := 0; (* j: neighbor counter *)
              repeat
                  while j < δ do
                      j := j + 1;
(r1')                 if f(Adj(v, j)) = v then (* reverse traverse *)
(r2')                     v := Adj(v, j); j := 0
                      endif
                  endwhile;
                  if v ≠ s then (* forward traverse *)
(f1)                  u := v;   v := f(v);
(f2')                 determine j such that Adj(v, j) = u (* restore j *)
                  endif
              until v = s and j = δ
          endfor.
```

In order to describe the time complexity of this procedure, we define $t^R(Adj, f)$ to be the time necessary to decide for any vertex v of G and any integer $1 \le j \le \delta$ whether $f(Adj(v,j)) = v$ (i.e., $t^R(Adj, f)$ is the time to decide whether moving from v to $Adj(v,j)$ is a reverse of f). Similarly, we define $t^F(Adj, f)$ to be the time necessary for any vertex v of G to determine the integer j such that $Adj(f(v), j) = v$.

Theorem 2.15. *Suppose that a local search (G, S, f) is given by an A-oracle. Then the time complexity of ReverseSearch2 is $O((t(Adj) + \delta\, t^R(Adj, f) + t(f) + t^F(Adj, f))|V|)$.*

Proof. The proof is similar to that of Theorem 2.13. The total time for (r1') is $O(\delta\, t^R(Adj, f)|V|)$, and that for (r2') is $O(t(Adj)(|V| - |S|))$. Similarly, the total time for (f1) is $O(t(f)(|V| - |S|))$, and that for (f2') is $O(t^F(Adj, f)(|V| - |S|))$. Adding up all these yields the result. □

Looking into this theorem, we notice a possibility of further refinement of reverse search. Remark that the part $t_1 = (t(Adj) + \delta\, t^R(Adj, f))$ of the time complexity is the time necessary for the reverse traversal, i.e., moving away from the top vertex, and the remaining part $t_2 = (t(f) + t^F(Adj, f))$ is the time for the forward traversal, i.e. , moving toward the top vertex. One cannot really shorten the first part, but interestingly one can shorten the latter part by storing the forward traverse paths. More precisely, if we store the forward sequence to return to the top vertex while reversing, Neither f nor Adj need to be evaluated to go forward. This remark can be particularly important when the trace of a local search has a short height and t_1 is an order of magnitude smaller than t_2, though presently we do not have any such applications.

Finally we should remark that the space complexity is usually independent of the cardinality of the output. At the moment, we cannot evaluate precisely the space complexity, but it only depends linearly on the space necessary to store a single vertex and the space necessary to realize the functions (oracles) f and Adj.

2.4 Vertex Enumeration in Polyhedra by Reverse Search

The vertex enumeration problem is to list all vertices of the convex polyhedron given by $P = \{x : Ax \leq b, x \geq 0\}$, where A is an $m \times n$ matrix and b is a m-vector.

Consider the linear program of form: maximize cx subject to $Ax \leq b$ and $x \geq 0$. The simplex algorithm can be considered as a finite local search (G_{LP}, S_{LP}, f_{LP}) where $G_{LP} = (V_{LP}, E_{LP})$ is a graph with V_{LP} is the set of all feasible bases; where two bases are adjacent if and only if one can be obtained from the other by a pivot operation; S_{LP} being the set of all optimal bases; and f_{LP} is the simplex algorithm with Bland's smallest subscript rule. Moreover, represent each basis by the set of indices of basic variables, and define Adj_{LP} to be $Adj_{LP}(B, (i,j)) = B - i + j$ if B is a basis and $B - i + j$ is a basis, and 0 otherwise, for each basic and nonbasic indices i and j. Then the local search is given by an A-oracle Adj_{LP} with $\delta_{LP} = m \times n$.

Now, how can we find all vertices of P? We can easily find one feasible basis of the linear inequality system $Ax \leq b$ and $x \geq 0$, say B, by the simplex method or the interior-point method. Then we can set up an LP with objective function cx for which the current basic solution is the unique optimal solution and B is an optimal basis. The associated local search (G_{LP}, S_{LP}, f_{LP}) immediately yields the reverse search ReverseSearch(Adj_{LP},δ_{LP},S_{LP},f_{LP}) to list all feasible bases as long as the set S_{LP} of all optimal bases are explicitly given.

If the set S_{LP} is the singleton $\{B\}$ and we are done. Otherwise, we can enumerate all optimal bases from B by another reverse search with respect to the dual simplex method applied to an auxiliary problem. See [AF92a] for details.

If the system $Ax \leq b, x \geq 0$ is nondegenerate, then one can design a much simpler algorithm. The critical difference is that for each feasible basis B and each nonbasic index j, there exists at most one basic index $i = i(j)$ such that $B - i + j$ is again a feasible basis. Define Adj_{LP} to be $Adj_{LP}(B, j) = B - i + j$ if there exists i such that $B - i + j$ is a feasible basis, and 0 otherwise, for each nonbasic index j. Then the local search is given by an A-oracle Adj_{LP} with smaller $\delta_{LP} = n$.

It is well known that the simplex method with Bland's rule might take an exponential number of pivots to find an optimal solution, see [AC78]. This means that the height of the trace T_{LP} cannot be bounded by a polynomial function of m and n. Theoretically, this means even the best parallel implementation may not be much faster. The expected behavior, however, might turn out to be quite different.

A C-implementation of the vertex enumeration algorithm is available in [Avi94].

2.5 Enumeration of Cells in Arrangements by Reverse Search

Let $E = \{1, 2, \ldots, m\}$, let $\mathcal{A}$ be an arrangement of distinct hyperplanes $\{H_i : i \in E\}$ in R^n, where each hyperplane is given by a linear equality $H_i = \{x : a^i x = b_i\}$. The two sides of H_i are $H_i^+ = \{x : a^i x \geq b_i\}$ and $H_j^- = \{x : a^j x \leq b_j\}$. For each $x \in R^n$, the signed vector $SV(x)$ of x is the vector in $\{-, 0, +\}^E$ defined by

$$SV(x)_i = \begin{cases} - & \text{if } x \in H_i^- \\ 0 & \text{if } x \in H_i \\ + & \text{if } x \in H_i^+ \end{cases} \qquad (i \in E).$$

Let V_{CELL} be the set of signed vectors of points in R^n whose nonzero support is E. We can identify each vector c in V_{CELL} with the open cell (open n-face) of the arrangement defined by $\{x : SV(x) = c\}$. For two cells c and c', let $sep(c, c')$ be the set of separators of c and c', that is, the set of elements i of E such that c_i and c'_i have opposite signs. We say that two cells c and c' are *adjacent* in G_{CELL} if they differ in only one component, or equivalently, $|sep(c, c')| = 1$. The following lemma is important.

Lemma 2.16. *For any two distinct cells c and c' in V_{CELL}, there exists a cell c'' which is adjacent to c and $sep(c, c'') \subset sep(c, c')$.*

Proof. Let c and c' be two distinct cells, and let x (x') be a point in c (in c', respectively) in general position. Moving from x toward x' on the line segment $[x, x']$, we encounter the sequence of cells: $c_o = c, c_1, c_2, \ldots, c_k = c'$, and we can easily verify that c_1 is adjacent to c and $sep(c, c_1) \subset sep(c, c')$. ☐

Let us assume that V contains the cell c^* of all $+$'s. Lemma 2.16 implies that for each cell c different from c^*, there is a cell c'' which is adjacent to c and $sep(c^*, c'') \subset sep(c^*, c)$. Let us define $f_{CELL}(c)$ as such c'' that is lexico-largest (i.e., the unique element in $sep(c, c'')$ is smallest possible). Then, $(G_{CELL}, S_{CELL}, f_{CELL})$ is a finite local search with $S_{CELL} = \{c^*\}$. By Lemma 2.16, one immediately obtains:

Corollary 2.17. *The height of the trace T_{CELL} of the local search $(G_{CELL}, S_{CELL}, f_{CELL})$ is at most m.*

Figure 2.14 describes the trace of the local search on a small example with $n = 2$ and $m = 4$.

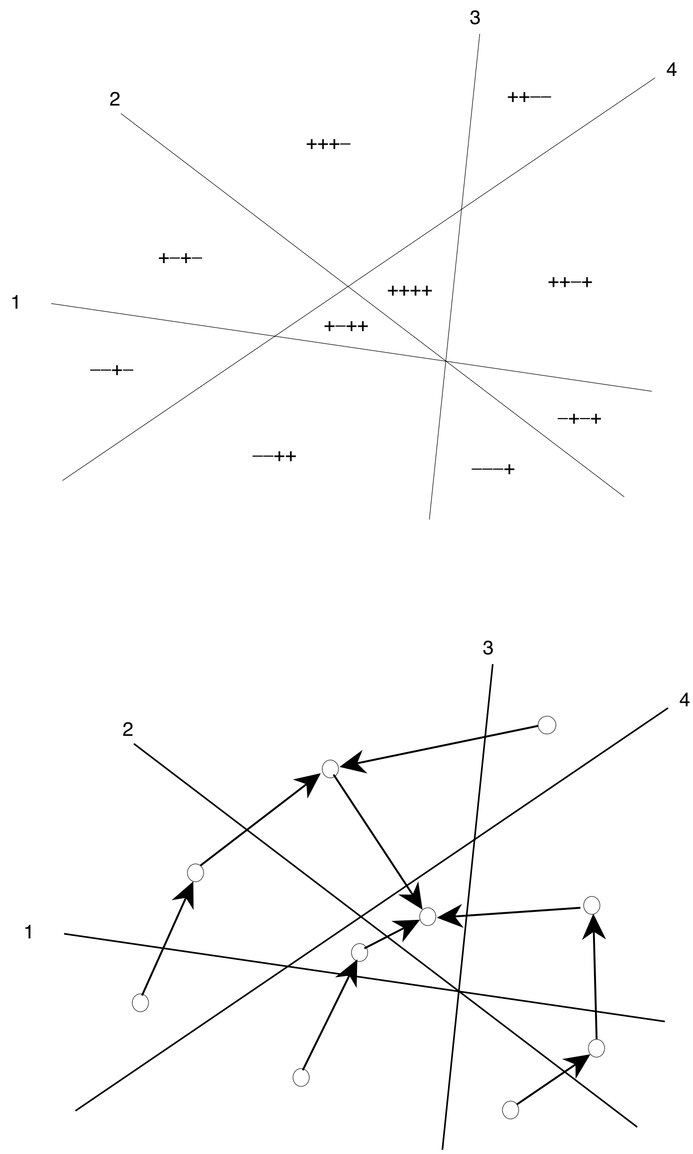


Figure 2.14: An arrangement of hyperplanes and the trace of f_{CELL}

By reversing this local search, we obtain an algorithm to list all cells in an arrangement. There are a few things to be explained for an implementation. First, we assumed that the cell c^* of all +'s is given, but we can pick up any cell c in the arrangement, and consider it as the cell of all +'s since replacing some equality $a^i x = b_i$ by $-a^i x = -b_i$ does not essentially change the arrangement. Note that one can obtain an initial cell by picking up any random point in R^n and perturbing it if it lies on some hyperplanes.

Now, how can we realize ReverseSearch(Adj_{CELL},δ_{CELL},S_{CELL},f_{CELL}) in an efficient way? First we can set $\delta_{CELL} = m$ and $S_{CELL} = \{c^*\}$. For any cell $c \in V_{CELL}$ and $k \in E$, the function $Adj_{CELL}(c, k)$ can be realized via solving an LP of the form

$$
\begin{array}{lll}
 & \text{minimize (maximize)} & y_k \\
(2.9) & \text{subject to} & y = Ax - b, \\
 & & y_i \geq 0 \text{ for all } i \neq k \text{ with } c_i = +, \\
 & & y_i \leq 0 \text{ for all } i \neq k \text{ with } c_i = -,
\end{array}
$$

where minimization (maximization) is chosen when $c_k = +$ ($c_k = -$, respectively). The function returns the adjacent cell c' with $sep(c, c') = \{k\}$ if and only if LP (2.9) has a feasible solution with negative (positive) objective value. The time $t(Adj_{CELL})$ depends on how an LP with n variables and $m-1$ inequalities is solved. We denote this as a function $l(m,n)$ of m and n.

There is a straightforward implementation of f_{CELL}, which solves a sequence of LP's similar to (2.9) with objective functions $y_1, y_2, y_3, \ldots$. This means we may have to solve $O(m)$ LP's in the worst case. Presently we don't know how to implement it in a more efficient manner.

Theorem 2.18. *There is an implementation of ReverseSearch(Adj_{CELL}, δ_{CELL}, S_{CELL}, f_{CELL}) for the cell enumeration problem with time complexity $O(m\, n\, l(m,n)|V_{CELL}|)$ and space complexity $O(m\, n)$.*

Proof. To prove this, first we recall that Theorem 2.13 says, the time complexity of ReverseSearch is $O(\delta\, t(Adj)|V| + t(f)|E|)$. As we remarked earlier, $\delta_{CELL} = m$, $t(Adj_{CELL}) = O(l(m,n))$, and $t(f_{CELL}) = O(m\, l(m,n))$. Since $|E_{CELL}| \le n\, |V_{CELL}|$ holds for any arrangement (see, e.g., [FST91, FSTT91]), the claimed time complexity follows. The space complexity is clearly same as the input size $O(m\, n)$. □

We believe that there is no previously known algorithm to enumerate all cells of an arrangement whose time complexity is polynomial in the size of output.

The cardinality of output is of course exponential in m and n, and the maximum, explicitly given by Buck's formula $\sum_{i=0}^{n} \binom{m}{i}$, is attained for any simple arrangements, see [Buc43, Ede87]. By Corollary 2.17, the cell enumeration can profit a lot from parallel implementation.

2.6 Enumeration of Triangulations by Reverse Search

Let P be a set $\{p_1, \ldots, p_n\}$ of n distinct points in the plane. A pair $\{p, q\}$ of distinct points in P is called an *edge* if the line segment connecting p and q does not contain any other point of P. A triple $\{p, q, r\}$ of points in P is called a *triangle* if they are not collinear and the their convex hull (triangle region) does not contain any other points in P. A point or edge in P is called *external* if it is contained in the boundary of the convex hull of P, and *internal* otherwise.

A *triangulation* of P is a set Δ of triangles in P such that (1) each external edge is contained in exactly one triangle of Δ, (2) each internal edge is contained in either no triangle of Δ or exactly two triangles of Δ. An *edge* of a triangulation Δ is an edge contained in at least one triangle of Δ.

By using Euler's relation, one can easily see that the number of triangles and edges of a triangulation are independent of the choice of triangulation.

Proposition 2.19. *Let Δ be a triangulation of P, and let f_1 and f_2 be the number of edges and triangles of Δ, respectively. Then, they are determined by $f_1 = 3n - n^* - 3$, $f_2 = 2n - n^* - 2$, where n^* denotes the number of external points.*

It is clear from the definition that the number of triangulations is finite. The enumeration of all possible triangulations of P is the problem in this section. In order to apply the reverse search technique, the notion of Delaunay triangulation and the flip algorithm is very useful.

Let Δ be a triangulation with f_2 triangles whose interior angles $\alpha_1, \alpha_2, \ldots, \alpha_{3f_2}$ are indexed in such a way that $\alpha_i \le \alpha_j$ for any $1 \le i < j \le 3f_2$. The vector $\alpha(\Delta) = (\alpha_1, \alpha_2, \ldots, \alpha_{3f_2})$ is called the angle vector of Δ . A triangulation is said to be *Delaunay* if its angle vector is lexicographically maximal over all possible triangulations of the same point set, where the comparison of components is done from left to right.

Let Δ be a triangulation. Let $\{a, b\}$ be any internal edge of Δ, and let $\{a, b, c\}$ and $\{a, b, d\}$ be the two triangles of Δ containing it. We call $\{a, b\}$ *flippable* if the set $Flip(\Delta, \{a, b\}) := \Delta \setminus \{\{a, b, c\}, \{a, b, d\}\} \cup \{\{a, c, d\}, \{b, c, d\}\}$ is again a triangulation of P. One can easily see that an internal edge $\{a, b\}$ is flippable if and only if the points a, b, c, d form a convex quadrangle.

We call $\{a, b\}$ *legal* if the circumscribing disk of one of the triangles abc, abd does not contain the other, and *illegal* otherwise. When an edge $\{a, b\}$ is illegal, it is always flippable and the operation $Flip(\Delta, \{a, b\})$ is called a *Delaunay flip*.

It is known that a triangulation Δ is Delaunay if and only if it does not contain any illegal edges. The flip algorithm is simply a procedure to use the Delaunay flip operation repeatedly in any order until no such operation is possible, see Figure 2.15. The following theorem states that the flip algorithm is finite.

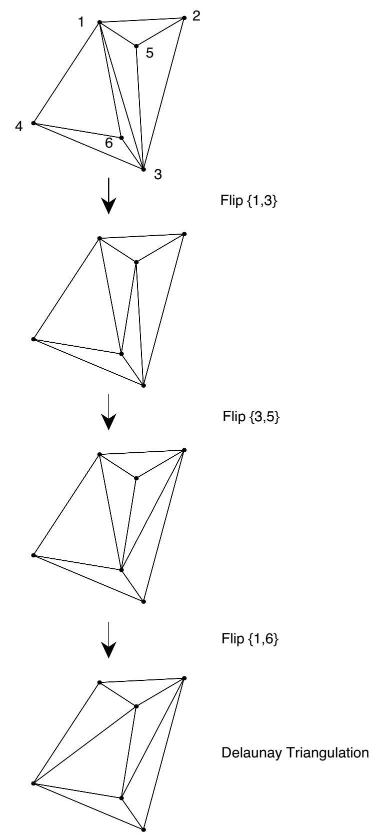


Figure 2.15: Flip Algorithm and Delaunay Triangulation

Theorem 2.20 ([For87]). *The flip algorithm terminates in $O(n^2)$ steps and finds a Delaunay triangulation of P, starting with any initial triangulation.*

To apply reverse search, let V_{TRI} (S_{TRI}) be the set of all (Delaunay, respectively) triangulations of P. The underlying graph G_{TRI} is (V_{TRI}, E_{TRI}) where two vertices are adjacent if and only if one is a flip of the other. We define a local search f_{TRI} as the function from $V_{TRI} \backslash S_{TRI}$ to V_{TRI} such that

$$f_{TRI}(\Delta) := Flip(\Delta, \{a, b\})$$

where $\{a, b\}$ is the lexico-smallest illegal edge of Δ.

Now, how one should design an A-oracle? For any triangulation Δ, let L be the list of interior edges ordered lexicographically and let L_k be the kth member of L. By Proposition 2.19, the cardinality $|L|$ of L is exactly $3n - 2n^* - 3$, which we denote by δ_{TRI}. The adjacency oracle $Adj_{TRI}(\Delta, k)$ is then defined as

$$Adj_{TRI}(\Delta, k) := \begin{cases} Flip(\Delta, L_k) & \text{if } L_k \text{ is flippable,} \\ 0 & \text{otherwise,} \end{cases}$$

for each $k = 1, \ldots, \delta_{TRI}$.

Theorem 2.21. *There is an implementation of ReverseSearch2(Adj_{TRI}, δ_{TRI}, S_{TRI}, f_{TRI}) for the triangulation enumeration problem with time complexity $O(n\, |V_{TRI}|)$ and space complexity $O(n)$.*

Proof. For the implementation, we can use the quad-edge data structure [GS85] for storing a triangulation. Also we store L as a linked list of edges each with a flag indicating either nonflippable, legal or illegal, and store the lexico-smallest illegal edge of L. For the analysis of time complexity, we apply Theorem 2.15. First, note that we can evaluate Adj_{TRI} and f_{TRI} in $O(n)$ time, including time to update L and the triangulation data. Since we store the lexico-smallest illegal edge of L, $t^R(Adj, f)$ and $t^F(Adj, f)$ are both $O(1)$. Since $\delta_{TRI} = O(n)$ and by Theorem 2.15, we have the stated time complexity. The space complexity is clearly $O(n)$. □

We remark that enumerating all elements in S_{TRI} (i.e. the enumeration of all Delaunay triangulations) is unnecessary. It is possible to transform any Delaunay triangulation to another by a sequence of flips, and one can extend the local search f_{TRI} so that it finds the lexico-smallest Delaunay triangulation by flipping some legal edge at a non-lexico-smallest Delaunay triangulation.

Theorem 2.20 shows that a parallel implementation can be quite fast.

It should be mentioned that essentially the same algorithm has been discovered independently by H. Telley.

3 OM Theory I

3.1 Face Axioms

Recall that we defined oriented matroids by using the face axioms, Axiom 1.1 (F1)∼(F4). In this section, we present two variations of the strong elimination axiom (F4) which are useful. One is stronger than (F4) and the other is weaker than (F4), and yet they are all equivalent under the rest of axioms (F1)∼(F3).

For proving theorems in oriented matroids, the stronger axiom is useful, and to prove some given object is an oriented matroid, obviously the weaker axiom is useful.

First we present the following stronger one:

(F4c) $Y, Y' \in \mathcal{F}$ and $I \subseteq D(Y, Y') \Longrightarrow$
there exist $f \in I$ and $Z \in \mathcal{F}$ such that $Z_f = 0$, $Z_I \preceq Y_I$ and
$Z_j = (Y \circ Y')_j$ for all $j \notin D(Y, Y')$. (conformal elimination)

Axiom 3.1: Conformal Elimination Axiom

Proposition 3.1. *For any set $\mathcal{F}$ of sign vectors on a finite set E, the conformal elimination axiom (F4c) is equivalent to the strong elimination axiom (F4).*

Proof. It is clear that (F4c) implies (F4). We shall prove the other implication: (F4) implies (F4c).

For this, we denote by $(\text{F4c})_k$ the statement (F4c) with the restriction that $|I| = k$, for any positive integer k. We shall prove the implication: (F4) implies $(\text{F4c})_k$ for any k, by induction on k. Clearly $(\text{F4c})_1$ coincides with (F4). We assume as an induction hypothesis that the implication (F4) $\Longrightarrow$ $(\text{F4c})_i$ holds when $i < k$, and we show (F4) $\Longrightarrow$ $(\text{F4c})_k$.

Now let (F4) is satisfied, and let $Y, Y' \in \mathcal{F}$ and $I \subseteq D(Y, Y')$ with $|I| = k > 0$. Let f be any element of I, and apply (F4) to Y and Y'. Then there exists $Z \in \mathcal{F}$ such that $Z_f = 0$ and $Z_j = (Y \circ Y')_j$ for all $j \notin D(Y, Y')$. If $Z_I \preceq Y_I$, then $(\text{F4c})_k$ is valid and we are done. Suppose $Z_I \not\preceq Y_I$. Let $Y'' = Z \circ Y$, and let $I' = I \cap D(Y, Y'')$. Clearly $|I'| < k$, and thus by induction hypothesis, we can apply (F4c) to Y, Y'' and I'. This will produce a vector $Z' \in \mathcal{F}$ and $f' \in I'$ such that $Z'_{f'} = 0$, $Z'_I \preceq Y_I$ and $Z'_j = (Y \circ Y')_j$ for all $j \notin D(Y, Y')$. Thus $(\text{F4c})_k$ is valid, and this completes the proof. □

Corollary 3.2. *A set $\mathcal{F}$ of sign vectors on a finite set E is the set of faces of an oriented matroid if and only if it satisfies the following axioms:*

(F1) $\mathbf{0} \in \mathcal{F}$;

(F2) $Y \in \mathcal{F} \Longrightarrow -Y \in \mathcal{F}$; (symmetry)

(F3) $Y, Y' \in \mathcal{F} \Longrightarrow Y \circ Y' \in \mathcal{F}$; (composition)

(F4c) $Y, Y' \in \mathcal{F}$ and $I \subseteq D(Y, Y') \Longrightarrow$
there exist $f \in I$ and $Z \in \mathcal{F}$ such that $Z_f = 0$, $Z_I \preceq Y_I$ and
$Z_j = (Y \circ Y')_j$ for all $j \notin D(Y, Y')$. (conformal elimination)

Axiom 3.2: OM Face Axioms II

Now we present a weaker version of the elimination axiom (F4):

(F4w)	$Y, Y' \in \mathcal{F}$, $f \in D(Y, Y')$ and $g \in \underline{Y} \backslash D(Y, Y') \Longrightarrow$ there exists $Z \in \mathcal{F}$ such that $Z_f = 0$, $Z_g = Y_g$ and $Z_j = (Y \circ Y')_j$ for all $j \notin D(Y, Y')$.	(weak elimination)

Axiom 3.3: Weak Elimination Axiom

One can easily verify that (F4w) is clearly weaker than (F4), but they are equivalent under the condition (F3). Thus we have

Proposition 3.3. *A set $\mathcal{F}$ of sign vectors on a finite set E is the set of faces of an oriented matroid if and only if it satisfies the axioms: (F1), (F2), (F3) and (F4w).*

3.2 Proofs for Topes and Vertices

For this section, we assume $M = (E, \mathcal{F})$ is an oriented matroid on a finite set E with $\mathcal{F}$ being the set of faces.

The first application of the conformal elimination axiom (F4c) in the previous section is a proof for Proposition 1.3, which state that both the set of topes and the set of vertices,

$$\mathcal{T} = Max(\mathcal{F}) \tag{3.1}$$

$$\mathcal{V} = Min(\mathcal{F} \backslash \{\mathbf{0}\}). \tag{3.2}$$

determine $\mathcal{F}$ uniquely, and thus determine the oriented matroid. First we prove the following:

Proposition 3.4. *For any face $Y \in \mathcal{F}$, and any $j \in \underline{Y}$, there exists a vertex $V \in \mathcal{V}$ such that $V \preceq Y$ and $V_j = Y_j$.*

Proof. Let Y be any nonzero face and let j be any index of $\underline{Y}$. If $Y \in \mathcal{V}$, there is nothing to prove. Assume $Y \notin \mathcal{V}$. Thus there is a vertex V such that $V \prec Y$. If $j \in \underline{V}$, we are done. Otherwise, $V_j = 0$. Now let $I = D(Y, -V)$ and apply (F4c) with setting $Y' = -V$. Then we have $Z \in \mathcal{F}$ such that $Z_j = Y_j$, $Z \prec Y$ and $\underline{Z} \subset \underline{Y}$. SInce Z has a smaller support, by applying the same procedure to Y replaced by Z repeatedly, we must obtain in finite steps a vertex we are looking for. This completes the proof. □

This immediately implies

Corollary 3.5 (Conformal Decomposition). *Every face $Y \in \mathcal{F}$ admits a* conformal decomposition *into vertices, that is, $Y = V_1 \circ V_2 \circ \cdots \circ V_k$ for some vertices $V_1, V_2, \ldots, V_k$ such that $V_j \preceq Y$ for all j.*

Now we are ready to prove the important proposition stating the vertices determine the faces:

Proof. (for Proposition 1.3 (b)) We must show $\mathcal{F} =< \mathcal{V}, \circ >$. Let $\mathcal{F}' =< \mathcal{V}, \circ >$. Since an oriented matroid is closed under composition, clearly $\mathcal{F}' \subseteq \mathcal{F}$. The other inclusion is a direct consequence of Corollary 3.5. This completes the proof. □

Next we shall prove the remaining Proposition 1.3 (a). For this purpose, we first prove the following:

Lemma 3.6. *Let X be any sign vector on E.*

(3.3) $$X \circ T \in \mathcal{T} \text{ for all } T \in \mathcal{T} \Longrightarrow X \circ Y \in \mathcal{F} \text{ for all } Y \in \mathcal{F}$$

Proof. We prove this by contradiction. Suppose that the statement (3.3) is false and let (X^1, Y^1) be a counterexample: $X^1 \circ T \in \mathcal{T}$ for all $T \in \mathcal{T}$, $Y^1 \in \mathcal{F}$ and $X^1 \circ Y^1 \notin \mathcal{F}$. Let Y be a minimal face with the property that $X^1 \circ Y^1 \prec Y$. Such a vector exists because $\mathcal{T} \neq \emptyset$ and for any tope T, the vector $X^1 \circ Y^1 \circ T$ is a face because $Y^1 \circ T$ is a tope and by the assumptions. Now let $Y' = X^1 \circ Y^1 \circ (-Y)$. Clearly $D(Y, Y') \neq \emptyset$. By applying the conformal elimination axiom (F4c) to (Y, Y') with $I = D(Y, Y')$, we obtain a vector $Z \in \mathcal{F}$ such that $Z_f = 0$ and $X^1 \circ Y^1 \preceq Z$. Now Z is strictly smaller than Y, and it is strictly larger than $X^1 \circ Y^1$ (because Z is a face). This contradicts the choice of Y, and thus the statement (3.3) must hold. □

Proof. (for Proposition 1.3 (a)) Let $\mathcal{F}' = \{X \in \{+, 0, -\}^E : X \circ T \in \mathcal{T} \quad \forall T \in \mathcal{T}\}$ We must show $\mathcal{F} = \mathcal{F}'$. Since an oriented matroid is closed under composition, clearly $\mathcal{F} \subseteq \mathcal{F}'$. Let $X \in \mathcal{F}'$. By Lemma 3.6, $X \circ Y$ for all $Y \in \mathcal{F}$. In particular, $X = X \circ \mathbf{0}$ must be in $\mathcal{F}$. This completes the proof. □

3.3 Matroids

In this section, we review some basic results of Matroid Theory that are necessary for our presentation of results in oriented matroids.

First we introduce an axiomatization of matroids using flats. A matroid N on a finite set E is a pair $(E, \mathcal{L})$ where $\mathcal{L}$ is a set of subsets of E satisfying the following axioms:

(L1) $E \in \mathcal{L}$;

(L2) $F, F' \in \mathcal{L} \Longrightarrow F \cap F' \in \mathcal{L}$;

(L3) $F, F' \in \mathcal{L}$, $g \in F \backslash F'$ and $f \in E \backslash (F \cup F') \Longrightarrow \exists G \in \mathcal{L}$ such that $f \in G \not\ni g$ and $F \cap F' \subseteq G$.

Axiom 3.4: Matroid Flat Axioms

In order to see the relation between oriented matroids and matroids, we observe:

Proposition 3.7. *For an oriented matroid $M = (E, \mathcal{F})$, the set of zero supports of faces in $\mathcal{F}$:*

(3.4) $$\mathcal{L}(M) = \{E \backslash \underline{Y} \ : \ Y \in \mathcal{F}\}$$

is the set of flats of an matroid.

Proof. Let M be an oriented matroid. We shall verify Axiom 3.4 for $\mathcal{L} = \mathcal{L}(M)$. First of all, (L1) is implied by the oriented axiom (F1) of Axiom 1.1, and (L2) by (F3). To verify (L3), let $F, F' \in \mathcal{L}$, $f \in F \backslash F'$ and $g \in E \backslash (F \cup F')$, and let Y and Y' be faces of M whose zero supports are F and F', respectively. Without loss of generality, we can assume $f \in D(Y, Y')$, since otherwise we may take the negative of Y as Y by (F2). By the choice of g, $Y_g = 0$ and thus $g \notin D(Y, Y')$. By (F4), there exists $Z \in \mathcal{F}$ such that $Z_f = 0$ and $Z_j = (Y \circ Y')_j$ for all $j \notin D(Y, Y')$. This implies that the zero support of Z satisfies all conditions required for G in (L3). This completes the proof. □

We call this matroid $(E, \mathcal{L}(M))$ the underlying matroid of M, denoted by $\underline{M}$. A matroid can be alternately defined by the set of its cocircuits, as defined in 1.4, but for our purposes the flat axioms, Axiom 3.4 is most convenient and we shall derive all fundamental results on matroids using the flat axioms. Recall that the span *span*(S) of a subset S in a matroid N is the smallest flat containing S:

(3.5) $$span(S) = \cap \{F \in \mathcal{L} : S \subseteq F\}.$$

Some obvious properties of the function *span* are

$$S \subseteq S' \Longrightarrow span(S) \subseteq span(S'), \text{ and} \tag{3.6}$$

$$span(span(S)) = span(S). \tag{3.7}$$

Also, recall that a subset S of E is independent in N if the span of S properly contains the span of $S - j$ for all $j \in S$. The following lemma is basic and very important.

Lemma 3.8. *Let N be a matroid on a finite set E. and let S be a subset of E. Then,*

(a) *If I is independent, then every subset of I is independent.*

(b) *If I is independent and $f \in E \backslash span(I)$, then $I + f$ is independent.*

Proof. To prove (a), observe that S is independent if and only if for each $j \in S$, there exists a flat F such that $S - j \subseteq F \not\ni j$. This condition is clearly satisfied for every subset of an independent set S.

To prove (b), suppose that I is independent, $f \in E \backslash span(I)$ but $I + f$ is not independent. This implies that there exists $g \in I + f$ such that

$$span(I + f - g) = span(I + f). \tag{3.8}$$

Since $f \notin span(I)$, $span(I) \neq span(I + f)$. It follows that $g \neq f$ and thus $g \in I$. Let $F \in \mathcal{L}$ such that $I \subseteq F \not\ni f$. Since I is independent, there exists $F' \in \mathcal{L}$ such that $I - g \subseteq F' \not\ni g$. By (3.8), $f \notin F'$. Noting that $f \notin F \cup F'$ and $g \in F \backslash F'$, we can apply (L3) to (F, F') to obtain a vector $G \in \mathcal{L}$ with $f \in G \not\ni g$. This contradicts to (3.8). This completes the proof. □

Let S be a subset of E. A subset B of S is called a *basis* of S if it is a maximal independent subset of S. A basis of the ground set E is simply called a *basis* (of the matroid). The following is an immediate consequence of Lemma 3.8.

Corollary 3.9. *Let N be a matroid on a finite set E, and let S be a subset of E. If B is a basis of S then $span(B) = span(S)$.*

Now we are ready to prove a fundamental property of bases in matroids.

Proposition 3.10. *Let N be a matroid on a finite set E. Then,*

(Basis Exchange Property) *B and B' are bases and $f \in B' \backslash B$*

$$\Longrightarrow \exists g \in B \backslash B' \text{ such that } (B' - f + g) \text{ is a basis.}$$

Proof. Let B and B' be bases, and let $f \in B' \backslash B$. Since $B' - f$ is independent and $span(B' - f) \subset span(B') = span(B)$ (by Corollary 3.9), there exists $g \in B \backslash span(B' - f) = B \backslash B'$, and for each such g, the set $B' - f + g$ is independent. We only have to show that $B' - f + g$ is a basis. Suppose that there are two elements g and g' in $B \backslash B'$ such that $B'' := B' - f + g + g'$ is independent. Since B'' is independent, there exist two flats F and F' such that

$$B'' - g \subseteq F \not\ni g$$
$$B'' - g' \subseteq F' \not\ni g'.$$

This implies that each of F and F' does not span $span(B') = E$, and since B' is a basis, $f \notin F \cup F'$. By the flat axiom (L3) applied to (F, F'), there exists a flat G such that $f \in G \not\ni g$ and $F \cap F' \subseteq G$. This means that G contains B'. By Corollary 3.9, $G = span(G) \supseteq span(B') = E$, but this contradicts $g \notin G$. This completes the proof. □

A very important corollary of the proposition above is

Corollary 3.11. *Every basis of a matroid has the same cardinality.*

It is easily shown that the two fundamental properties above of bases of a matroid N are still valid when the term "basis" is replaced with " basis of S", for any subset S of the ground set E (Exercise: verify this claim!). Using these, we can define the *rank* $r(S)$ *of a subset* S in N as the size of any basis of S, and the *rank* $r(N)$ *of a matroid* N as $r(E)$.

Finally, consider the set $\mathcal{L}$ of flats of a matroid N as the poset ordered by set inclusion. This is a lattice because it contains the largest element E and the smallest element $\mathbf{0}_{\mathcal{L}}$ that is the intersection of all flats.

Proposition 3.12. *The lattice $\mathcal{L}$ of flats of a matroid satisfies the J-D property. Furthermore, the height $h(F)$ of a flat is determined by $h(F) = r(F) + 1$.*

Proof. Since the rank of the smallest flat is zero, it is sufficient to prove for any ordered flats $F \subseteq F'$ the followings:

(a) $F' \subset F$ if and only if $r(F) - r(F') \geq 1$;

(b) If $r(F) - r(F') \geq 2$, then there exists $G \in \mathcal{L}$ such that $F' \subset G \subset F$.

Since the span of any flat is itself, the statement (a) follows directly from Lemma 3.8.

To prove (b), assume $r(F) - r(F') \geq 2$. Let B' be a basis of F'. Since F' is independent, it can be extended (enlarged) to a basis B of F. By the assumption, B has at least two extra elements, say f and g. Let G be the span of $B - f$. It is clearly a flat with the required property $F' \subset G \subset F$. □

Exercise 3.1. *Show that the following variation of the Basis Exchange property is valid for matroids:*

$$\begin{aligned} &B \text{ and } B' \text{ are bases and } f \in B' \backslash B \\ &\qquad \Longrightarrow \exists g \in B \backslash B' \text{ such that } (B + f - g) \text{ is a basis.} \end{aligned} \tag{3.9}$$

Exercise 3.2. *Show that the rank function r of a matroid satisfies the following properties:*

(R1) $r(\emptyset) = 0$;
(R2) $R \subseteq S \subseteq E \Longrightarrow r(R) \leq r(S)$;
(R3) $R, S \subseteq E \Longrightarrow r(R \cap S) + r(R \cup S) \leq r(R) + r(S)$. *(Submodularity)*

3.4 Basic Properties of OM Lattices and Tope Lattices

In this section, we shall prove Proposition 1.2 and Proposition 1.6.

We assume $M = (E, \mathcal{F})$ is an oriented matroid where $\mathcal{F}$ is the set of its faces ordered by conformal relation. Also, we assume $\underline{M} = (E, \mathcal{L})$ is the underlying matroid with $\mathcal{L} == \mathcal{L}(M)$ being the set of its flats, and r is the rank function of $\underline{M}$.

Lemma 3.13. *Let Y and Y' be faces of M with $Y \preceq Y'$.*

(a) *$Y \prec Y'$ if and only if $r(E \backslash \underline{Y}) - r(E \backslash \underline{Y'}) \geq 1$;*

(b) *If $r(E \backslash \underline{Y}) - r(E \backslash \underline{Y'}) \geq 2$, then there exists $Z \in \mathcal{F}$ such that $Y \prec Z \prec Y'$.*

Proof. Statement (a) is obvious. To prove (b), assume $r(E \backslash \underline{Y}) - r(E \backslash \underline{Y'}) \geq 2$. Let F and F' be the zero supports of Y and Y', respectively. By Proposition 3.12, there is a flat G such that $F' \subset G \subset F$. Let Z be a face whose zero support is G. We may assume $Y \prec Z$, since otherwise we can replace Z with the face $Y \circ Z$. Now if $Z \prec Y'$, we are done. Suppose $Y \prec Z \not\prec Y'$. Now we apply the conformal elimination axiom (F4c) to (Y', Z) with $I := D(Y', Z)$. This produces a face Z' such that $Y \prec Z' \prec Y'$. This completes the proof. □

3.5 Farkas, Coloring and Duality

The main goal of this sections is to prove the basic duality results presented in Section 1.9, in particular, Theorem 1.17. Recall the face axioms (F1), (F2), (F3), (F4) of oriented matroids, Axioms 1.1 ,

Proposition 3.14. *Let $\mathcal{F}$ be a set of sign vectors on a finite set E. Then $\mathcal{F}^*$ satisfies the first three OM axioms (F1), (F2) and (F3).*

Proof. The first two properties (F1) and (F2) are trivially satisfied by $\mathcal{F}^*$. To verify (F3), let $X, X' \in \mathcal{F}^*$ and let $Y \in \mathcal{F}$. Setting $X'' = X \circ X'$, we need to show $X'' * Y$. If $\underline{X} \cap \underline{Y} \neq \emptyset$, then clearly $X'' * Y$. Otherwise, $X' * Y$ implies $X'' * Y$. This completes the proof. □

To see that the elimination axiom (F4) is not satisfied by the dual set $\mathcal{F}^*$ in general, consider

$$\mathcal{F} = \{(+,+,+), (+,-,-), (0,+,+), \text{ and their negatives}\}$$
$$\mathcal{F}^* = \{(0,0,0), (+,-,+), (+,+,-), (0,-,+), \text{ and their negatives}\}.$$

It is easy to see that one cannot eliminate 2nd (or 3rd) component from the second and the third vectors.

Now we prove Farkas' Lemma (Theorem 1.21) which I repeat here:

Theorem [Farkas' Lemma for OM's] (Edmonds-Fukuda-Mandel, see [Fuk82]). Let $\mathcal{F}$ be a set of sign vectors on a finite set E satisfying (F2) and (F3). Then for any fixed index $g \in E$, exactly one of the following statements holds:

(a) $\exists Y \in \mathcal{F}$ such that $Y \geq \mathbf{0}$ and $Y_g > 0$;

(b) $\exists X \in \mathcal{F}^*$ such that $X \geq \mathbf{0}$ and $X_g > 0$.

Proof. Clearly both statements cannot hold simultaneously. We shall prove at least one of them must hold. Let G be the set of indices g for which the statement (a) is valid. Since $\mathcal{F}$ is closed under composition, i.e. satisfying (F3), the vector $\tilde{Y}$ defined by $\tilde{Y}^+ = G$ and $\tilde{Y}^- = \emptyset$ is in $\mathcal{F}$. It is sufficient to prove that the vector $\tilde{X}$ defined by $\tilde{X}^+ = E \backslash G$ and $\tilde{X}^- = \emptyset$ is in $\mathcal{F}^*$. Suppose $X \notin \mathcal{F}^*$. This together with (F2) implies that there exists $Y \in \mathcal{F}$ such that $Y_{E \backslash G} \geq \mathbf{0}$ and $Y_j = +$ for some $j \in E \backslash G$. Again by (F3), $Y' = \tilde{Y} \circ Y$ is in $\mathcal{F}$. Clearly $Y' \geq \mathbf{0}$ and $G \subsetneq \underline{Y'}$. This contradicts the definition of G. Therefore, $X \in \mathcal{F}^*$ and this completes the proof. □

Next, we prove a strengthening of Theorem 1.18 below. As you see, the first statement (a) does not need any assumptions but the second requires all the OM axioms except (F1).

Theorem 3.15. *Let $\mathcal{F}$ be a set of signed vectors on a finite set E. For any subsets R, S of E the following statements hold.*

(a) $(\mathcal{F} \backslash R)^* = \mathcal{F}^* / R$*;*

(b) $(\mathcal{F}/S)^* = \mathcal{F}^* \backslash S$*, provided that $\mathcal{F}$ satisfies (F2), (F3) and (F4).*

Proof. **(a)** The inclusion $(\mathcal{F} \backslash R)^* \supseteq \mathcal{F}^*/R$ is obvious. For the other inclusion, let $X \in (\mathcal{F} \backslash R)^*$. Set $\tilde{X}$ be the vector on E with $\tilde{X}_R = X$ and $\tilde{X}_{E \backslash R} = \mathbf{0}$. It follows that $\tilde{X} \in \mathcal{F}^*$ and thus $X \in \mathcal{F}^*/R$.

(b) To prove the inclusion $(\mathcal{F}/S)^* \supseteq \mathcal{F}^* \backslash S$, let $X \in \mathcal{F}^* \backslash S$. This means there exists $\tilde{X} \in \mathcal{F}^*$ with $\tilde{X}_{E \backslash S} = X$. Take any $Y \in \mathcal{F}$ with $Y_S = \mathbf{0}$. Since $\tilde{X} * Y$ and $Y_S = \mathbf{0}$, $X * Y_{E \backslash S}$. Therefore $X \in (\mathcal{F}/S)^*$. (Note that we have not used any OM assumptions yet.)

For the other inclusion, we may assume that $|S| = 1$, since the inclusion with larger $|S|$ follows inductively. Let $S = \{s\}$ and let $X \in (\mathcal{F}/\{s\})^*$. Let us denote by $X(\alpha)$ the sign vector on E with $X(\alpha)_{E-s} = X$ and $X(\alpha)_s = \alpha$, for each $\alpha = +, 0, -$. We need to prove that at least one of $X(+)$, $X(0)$, $X(-)$ is in $\mathcal{F}^*$. By contradiction, we suppose that none of them belongs to $\mathcal{F}^*$. This means that there exist $Y(+)$, $Y(0)$, $Y(-)$ in $\mathcal{F}$ such that $X(\alpha) \not* Y(\alpha)$, for $\alpha = +, 0, -$. Since $\mathcal{F}$ satisfies (F2), we may assume that $D(X(\alpha), Y(\alpha)) = \emptyset$ for all $\alpha = +, 0, -$. It follows from $X(0) \not* Y(0)$ that there exists $f \in E - s$ such that $X(0)_f = Y(0)_f \neq 0$. Set $Y^1 = Y(+) \circ Y(0)$ and $Y^2 = Y(-)$. Since $\mathcal{F}$ satisfies (F3), $Y^1 \in \mathcal{F}$. Now we apply the elimination properly (F4) to (Y^1, Y^2) to eliminate s-component to generate a new vector $Z \in \mathcal{F}$ with $Z_s = 0$, $Z_f = X_f$ and $D(Z, X(0)) = \emptyset$. Clearly $Z_{E-s} \in \mathcal{F}/\{s\}$ but Z_{E-s} cannot be orthogonal to X. This contradicst the choice of X. □

Corollary 3.16. *Let $\mathcal{F}$ be the set of faces (covectors) of an oriented matroid with ground set E. For any disjoint subsets R, S of E*

$$(\mathcal{F}\backslash R/S)^* = \mathcal{F}^*/R\backslash S.$$

Farkas' Lemma for OM's and this corollary imply the 4-coloring property (Corollary 1.22):

Theorem [Coloring Lemma for OM's]. Let $\mathcal{F}$ be the set of faces of an oriented matroid on a finite set E. Then for any partition ("coloring") of E into disjoint subsets R, G, B, W and for any fixed index $r \in R\cup$, exactly one of the following statements holds:

(a) $\exists Y \in \mathcal{F}$ such that $Y_R \geq \mathbf{0}$, $Y_G \leq \mathbf{0}$, $Y_B = \mathbf{0}$ and $Y_r \neq 0$;

(b) $\exists X \in \mathcal{F}^*$ such that $X_R \geq \mathbf{0}$, $X_G \leq \mathbf{0}$, $X_W = \mathbf{0}$ and $X_r \neq 0$.

Now we are ready to prove the key theorem, Thoerem 1.17:

Theorem [OM Duality]. Let $M = (E, \mathcal{F})$ be an oriented matroid on E. Then the following statements hold.

(1) the pair $M^* := (E, \mathcal{F}^*)$ is an oriented matroid;

(2) the dual M^{**} of the dual is the original M, that is, $\mathcal{F}^{**} = \mathcal{F}$.

Proof. **(1)** By Proposition 3.14, we only need to show that $\mathcal{F}^*$ satisfies (F4). Let X, $X' \in \mathcal{F}^*$ and let $f \in D(X, X')$. Set

$$\begin{aligned} R &= (X \circ X')^+ \backslash D(X, X') \\ G &= (X \circ X')^- \backslash D(X, X') \\ B &= D(X, X') \backslash \{f\} \\ W &= (X \circ X')^0 \cup \{f\}. \end{aligned}$$

By Coloring Lemma, for any $r \in R \cup G$, one of the two statements:

(a) $\exists Y \in \mathcal{F}$ such that $Y_R \geq \mathbf{0}$, $Y_G \leq \mathbf{0}$, $Y_B = \mathbf{0}$ and $Y_r \neq 0$;

(b) $\exists X \in \mathcal{F}^*$ such that $X_R \geq \mathbf{0}$, $X_G \leq \mathbf{0}$, $X_W = \mathbf{0}$ and $X_r \neq 0$

must hold. We claim that the second statement (b) holds for every $r \in R \cup G$. This implies that $\mathcal{F}^*$ satisfies (F4) because the composition of all certificate vectors $X(r) \in \mathcal{F}^*$ ($r \in R \cup G$) is one satisfying the conclusion of (F4) for $\mathcal{F}^*$. By contradiction, we suppose that (a) holds for some $r \in R \cup G$, i.e. there exists $Y \in \mathcal{F}$ such that $Y_R \geq \mathbf{0}$, $Y_G \leq \mathbf{0}$, $Y_B = \mathbf{0}$ and $Y_r \neq 0$. It is easy to see that Y cannot be orthogonal to both X and X', since X_f and X'_f are opposite in sign.

(2) Since $\mathcal{F}^{**} \supseteq \mathcal{F}$ is obvious, we shall prove $\mathcal{F}^{**} \subseteq \mathcal{F}$. Let $Y \in \mathcal{F}^{**}$ and suppose $Y \notin \mathcal{F}$. Set

$$R = Y^+, \qquad G = Y^-, \qquad B = Y^0, \qquad W = \emptyset,$$

and let r be any member of $R \cup G$. Since $\mathcal{F}$ is an oriented matroid, it satisfies Coloring Lemma. Using that fact that $Y \notin \mathcal{F}$, the second statement (b) must hold for some $X \in \mathcal{F}^*$. It is easy to see that such a vector X cannot be orthogonal to Y, a contradiction to $Y \in \mathcal{F}^{**}$. This completes the proof.

□

Open Problems

Here I present a list of open problems discussed in this lecture. I believe that all problems are fundamental and closely related, directly or indirectly, to the efficiency of computation. Furthermore, all problems are quite difficult but they are "reasonable" problems, meaning that the complete answer to any of the problems does not seem to imply any miracle discovery, such as P=NP.

In what follows, we employ the following definitions. A *polynomial algorithm* is one whose time complexity is bounded by a polynomial in the size of input. A *polynomial enumeration algorithm* is one whose time complexity is bounded by a polynomial in both the sizes of input and output. A *linear enumeration algorithm* is one whose time complexity is bounded by a function polynomial in the size of input and linear in the output size. Thus every polynomial algorithm is linear (and of course polynomial) enumeration algorithm. Finally, A *compact enumeration algorithm* is one whose space complexity is bounded by a polynomial in the size of input and independent in the output size. It should be noted that a polynomial or linear enumeration algorithm is not necessarily compact.

Axiomatics of Oriented Matroids

1. **Is there any axiomatization for the topes of an oriented matroid that can be verified in time polynomial in the size of input?** (Section 1.4)

   Note: There are several characterizations known, e.g. [BC87, Han91, dS91], but they do not yield a (direct) polynomial algorithm.

2. **Is there any polynomially verifiable characterization of the tope graphs of oriented matroids?** (Section 1.10)

   Note: There is a polynomial algorithm to check whether a given graph is isomorphic to the tope graph of some oriented matroid, see [FST91]. Since the tope graph uniquely determines the oriented matroid up to isomorphism, such a characterization makes the theory of oriented matroids a special branch of graph theory. For rank ≤ 3, a good characterization is known, see [FH93].

Combinatorial Problems on Polytopes and Topes

1. **Design a polynomial enumeration algorithm to construct the face lattice from the graph (i.e. 1-skeleton) of a simple d-polytope.**

   Note: The face lattice is uniquely determined by the graph. This result is due to Blind and Mani-Levistska [BML87]. A finite algorithm is given by Kalai [Kal88].

2. **Find a good characterization of good orientations of the graph of a simple polytope.**

   Note: An orientation of the graph of a polytope is said to be *good* if it is acyclic and the subgraph induced by each face contains exactly one sink and one source. Such an orientation always exists for any polytopes since any linear function not constant on each edge induces a good (LP) orientation. A simple but exponential characterization is given in [Kal88].

3. **Does every tope cell admits a good orientation?**

   Note: A tope cell is the interval $[\mathbf{0}, T]$ for some tope T of some oriented matroid. There is a natural extension of the LP orientations of a polytope, namely the OMP orientations of a tope cell. But such an orientation might contain a directed cycle.

4. **Does the graph of a simple tope determine its face lattice?**

   Note: A tope T of dimension d is said to be *simple* if the graph of the cell $[0, T]$ is d-regular graph. The answer is positive if the answer to the previous question is positive.

5. **Find a good characterization of tope lattices.** ("Weak" Steinitz' Problem.)

   Note: The tope lattices are a natural extension of the polytopal lattices. It is known that the problem of checking whether a given lattice is polytopal is NP-hard, even for the restricted case of 4-polytopality, as shown recently by Richter-Gebert [RG95, RGZ95].

6. **Does every oriented matroid have a simplex tope?** (Las Vergnas' Conjecture;Extension of Shannon's Theorem)

   Does every oriented matroid have a simple vertex?

   Note: A simplex tope is a tope whose face lattice is isomorphic to the face lattice of a simplex. A simple vertex is a vertex which have exactly d zero components (after removal of loops and parallel elements). These two questions were positively answered for Euclidean oriented matroids by Edmonds-Fukuda-Mandel (1982).

Computational Problems on Polytopes

Below, A is a given rational $m \times d$ matrix and we assume that the associated cone $P = \{x : A\,x \geq \mathbf{0}\}$ is pointed (i.e. the origin is an extreme point).

1. **(Any Polynomial Extreme Ray Enumeration Algorithm Exists?) Find a polynomial enumeration algorithm to generate all extreme rays of P, or prove no such algorithm exists.** (Section 2)

   Note: A harder questions can be formulated with "polynomial" replaced with "linear" or "linear and compact". These are major open problems in computational geometry and optimization.

   Under the nondegeneracy assumption that every nonzero feasible solution to $A\,x \geq \mathbf{0}$ satisfies at most $(d-1)$ inequalities with equality, there is a linear compact enumeration algorithm, see [AF92a].

2. **(Any Polynomial Double Description Method Exists?) Is there any way to reorder the rows of a given system $A\,x \geq \mathbf{0}$ in such a way that the double description method adding kth inequality with this ordering is a polynomial enumeration algorithm for the extreme ray enumeration.** (Section 2.1.3)

   Note: This problem is settled negatively by David Bremner [Bre99].

3. **(Extreme Ray Enumeration using Edmonds' Proof?) Is it possible to design an extreme ray enumeration algorithm which calls Edmonds' oracle (for Minkowski's Theorem) a finite (and hopefully polynomial) number of times?** (Section 2.1.4)

   Note: Edmonds' oracle returns at most d extreme rays of P to represent any given point $\overline{x}$ of P as their nonnegative combination. It can be implemented efficiently.

LP, LCP and OM Programming

Below, a general linear programming problem (LP) is assumed to have d nonnegative variables restricted by m $(\leq d)$ linear equality constraints with rational coefficients. Also, a linear complementarity problem (LCP) is assumed to have $2d$ variables.

1. **(Any Strongly Polynomial LP Algorithm Exists?) Find a pivoting algorithm to solve any linear programming problem which terminates in a number of pivots polynomial in d.**

   Note: Of course a harder question can be formulated by requiring the algorithm to stay feasible once a feasible solution is obtained (e.g. the simplex algorithm). These are major open problems in optimization. See [FLN97] for the best-case analysis for a certain class of pivot algorithms.

2. **(Any Strongly Polynomial LCP Algorithm Exists?) Find a pivoting algorithm which solves any linear programming problem in a number of pivots polynomial in d, whenever a given matrix is sufficient.**

   Note: See [FNT95] for a simple finite pivot algorithm, and see [FLN97] for the best-case analysis for a certain class of pivot algorithms.

3. **(Any Strongly Polynomial OMP Algorithm Exists?) Find a pivoting algorithm which solves any OMP problem OMP(M, g, f) in a number of pivots polynomial in d, where d is the size of the ground set of M.**

   Note: This is a harder problem than the first LP problem, but it might be more appropriate question, since it might be difficult to take advantage of linearity anyway.

Exercise

Convex Polyhedra and Faces

1. Prove Lemma 0.2.

Polyhedral Realization of Arrangements

1. Let
$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 1 \end{bmatrix}$$
and let $\mathcal{A}$ be the central arrangement with representation $(A, 0)$.

   (a) Draw the arrangement.

   (b) Compute $\mathcal{F}(\mathcal{A})$.

   (c) Determine P_A and its dual $(P_A)^*$, and verify Theorem 0.14 for the arrangement.

2. Let $A = I^{(d)}$, the identity matrix of size d, and let $\mathcal{A}$ be the central arrangement with representation $(A, 0)$.

   (a) Determine $\mathcal{F}(\mathcal{A})$.

   (b) Draw P_A and its dual $(P_A)^*$. What are these polytopes? Verify Theorem 0.14 for the arrangement.

3. Prove Theorem 0.14.

   Hint: For $y \in R^m$, let $\mathrm{sg}(y)$ denote the $(+1, 0, -1)$-valued sign vector of y. For any nonzero vector x of R^d, let $p(x)$ denote the scaled vector $p(x) := \frac{x}{\mathrm{sg}(Ax)^T Ax}$. Here the denominator is positive by Assumption 0.13. Finally, for any $x \in R^d$, define $a(x)$ ("active set" at x for P_A) by
$$a(x) := \{y \in \{+1, -1\}^m : y^T A\, x = 1\}.$$
   Show the following for nonzero $x \in R^d$ and $y \in \{+1, -1\}^m$:
   (a) $p(x) \in P_A$.
   (b) $y \in a(p(x))$ if and only if $\mathrm{sg}(y)_i = \mathrm{sg}(A\, x)_i$ for all i with $(A\, x)_i \neq 0$.

Faces, Vertices and Topes

1. Shannon' theorem states that every arrangement of m spheres on S^d admits a face isomorphic to a d-simplex, under our regularity assumption, Assumption 0.13. (Actually the theorem is stronger.)

   (a) State the equivalent statement in terms of $(d + 1)$-zonotopes.

   (b) Prove the theorem for $d = 2$ (in the form you prefer) by using Euler's relation $f_0 - f_1 + f_2 = 2$, where f_k is the number of k-faces.

2. The Sylvester-Gallai theorem states that every set of n points in the plane not all of which are colinear admits an ordinary line. Here a line is called *ordinary* if it contains exactly two points from the set.

   (a) Dualize the statement of the theorem. In particular, state a theorem for arrangements of spheres on S^2 and also for 3-zonotopes.

   (b) Prove the theorems in (a) by using Euler's relation.

3. A colorful generalization of the Sylvester-Gallai theorem states:

   For any set of n points in the plane not all of which are colinear, and for any balanced non-Radon partition $P = B \cup R$ (colored blue and red), there is a "colorful" (i.e. blue and red) ordinary line.

Here a partition is *balanced* if the cardinalities of the two sets are as equal as possible (i.e. the difference is at most one).

(a) State a dual version of the statement for arrangements of spheres on S^2.

(b) Can you prove it? This problem is still **open**, although quite a few discrete geometers tried to prove or disprove it.

4. Prove the Conformal Decomposition Theorem for OMs:
Theorem. Let $M = (E, \mathcal{F})$ be an oriented matroid. Then, every face $Y \in \mathcal{F}$ admits a *conformal decomposition* into vertices, that is, $Y = V_1 \circ V_2 \circ \cdots \circ V_k$ for some vertices $V_1, V_2, \ldots, V_k$ such that $V_j \preceq Y$ for all j.

5. Let $G = (V, E)$ be a connected graph. Fix one orientation of G. and let $A = [a_{ij} : i \in E; j \in V]$ denote the $E \times V$ $(1, 0, -1)$ incidence matrix of G. Here a_{ij} is 1 (-1) if a vertex j is the head (the tail) of an edge i, and 0 otherwise. Consider the oriented matroid $\mathcal{F} = \sigma(\mathcal{V})$ where V is the cut space of G:

$$V = Cut(G) := \{y : y = Ax \text{ and } x \in R^V\}.$$

An orientation is called *acyclic* if it does not contain a directed cycle, or equivalently each edge is contained in a directed cut.

(a) Suppose the initial orientation is acyclic. Show that $\mathcal{F}$ contains the vector of all +'s.

(b) What are the topes of the oriented matroid? Can you interprete them in terms of orientations?

(c) What does Shannon's theorem imply?

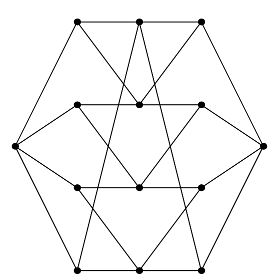
Matroids

1. Prove the basis exchange property from the flat axiom.

2. Consider an oriented matroid with a sphere system representation. Give a characterization of independent sets (and bases) in terms of the associated sets of spheres.

Tope and Cocircuit Graphs

1. The following graph is either the tope graph or the cocircuit of a simple oriented matroid.



(a) Which one is it? Determine the rank of the oriented matroid and the size of the ground set.

(b) Recover the tope vectors of the OM up to isomorphism. Is your method efficient (i.e. polynomial) for any correct input?

2. Design a finite algorithm to generate all acyclic orientation of a given graph. Analyze its time and space complexity.

Duality

1. Let

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 1 \end{bmatrix}$$

   and let V be its column space $\{y : y = Ax, x \in R^2\}$. Compute $\sigma(V)$, $\sigma(V^\perp)$ and $\sigma(V)^*$. Note that this matrix is the one given in Problem 2.1.

2. Prove the following theorem using Farkas' Lemma (for matrices): For any vector subspace V of R^E, $\sigma(V)^* = \sigma(V^\perp)$. Hint: one inclusion is trivial.

3. Let $\mathcal{F}$ be any set of sign vectors on a finite set E. Which of the OM face axioms (F1) $\sim$ (F4) are generally satisfied by $\mathcal{F}^*$? Give a counterexample for each axiom not satisfied.

4. Using the Coloring Lemma for OMs, prove that the dual $\mathcal{F}^*$ satisfies the strong elimination property (F4).

Shelling

1. Let P be a d-dimensional $\mathcal{H}$-polytope:

$$P = \{x \in R^d : a_i\, x \le 1,\ i = 1, 2, \ldots, m\}.$$

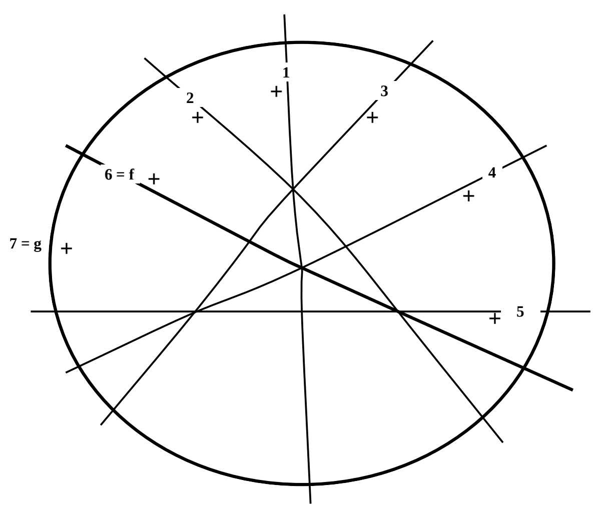
   Note that the origin $\mathbf{0}$ is in the interior of P. For any nonzero vector $c \in R^d$, let $L(c)$ denote the oriented line through the origin $L(c) = \{x(\lambda) \in R^d : x(\lambda) = \lambda\, c, \lambda \in R\}$. Assume that the face

$$F_i = \{x \in P : a_i\, x = 1\}.$$

   determined by ith inequality is facet, for each i. Find conditions on a_i's and c under which F_1, F_2, …, F_m is the line shelling induced by $L(c)$. Interprete the conditions for the dual polytope $P^* = conv(\{a_1, a_2, \ldots, a_m\})$.

OM Programming

1. Consider the OM programming $P = (M, g, f)$ on seven elements $E = \{1, 2, 3, 4, 5, f = 6, g = 7\}$:



   (a) Find a primal optimal solution.

   (b) Find a dual optimal solution. Explain using the arrangement why the dual solution is in fact in $\mathcal{F}^*$.

Hint: For $X \in \{+, 0, -\}^E$, $X \in\in \mathcal{F}^*$ if and only if $Y \in \mathcal{F}$ and $Y_j = X_j$ or 0 for all $j \in \underline{X}$ imply $Y_j = 0$ for all $j \in \underline{X}$.

(c) Write down the dictionary $D(B)$ for the basis $B = \{g, 1, 2\}$.

(d) Find an optimal basis B^* and write down its dictionary. Where in the dictionary are the primal and the dual optimal solutions represented. How many admissible pivots do we need to get to B^* from B?

(d) Apply the criss-cross method to B. How many steps does it take to get to the optimal basis?

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