A Proof of the Uniform Boundedness Principle

Edward Chernysh

In the various references that I have consulted, the so-called *uniform boundedness principle* is presented in an unclear manner. The best proof for this highly non-obvious result I found was in Folland's *Real Analysis: Modern Techniques and Their Applications*, although in my opinion this proof still lacks in detail. In this document, we shall cover this standard proof is greater detail.

Let us now recall some useful notation. If $\mathcal{X}$ and $\mathcal{Y}$ are normed vector spaces over the real or complex numbers, we denote by $\mathscr{L}(\mathcal{X}, \mathcal{Y})$ the space of all continuous linear operators $T : \mathcal{X} \to \mathcal{Y}$. If $(X, \mathfrak{T})$ is a topological space, a subset $V \subseteq X$ is called *meager*[1] in X if it is the countable union of nowhere dense sets. We now state the theorem in question.

Theorem. *Let $\mathcal{X}$ and $\mathcal{Y}$ be normed vector spaces over $\mathbb{R}$ or $\mathbb{C}$. Let $\mathcal{A} \subseteq \mathscr{L}(\mathcal{X}, \mathcal{Y})$ be a non-empty sub-collection and suppose that*

$$\sup_{T \in \mathcal{A}} \|Tx\| < \infty$$

for all x in some non-meager subset of $\mathcal{X}$. Then,

$$\sup_{T \in \mathcal{A}} \|T\| < \infty.$$

Proof of Theorem. Given $n \in \mathbb{N}$ let us define

$$E_n := \left\{ x \in \mathcal{X} : \sup_{T \in \mathcal{A}} \|Tx\| \le n \right\} = \bigcap_{T \in \mathcal{A}} \left\{ x \in \mathcal{X} : \|Tx\| \le n \right\}.$$

We now claim that E_n is topologically closed in $\mathcal{X}$ for every $n \in \mathbb{N}$. Indeed, fix $n \in \mathbb{N}$ and $T \in \mathcal{A}$; let $(u_k)_{k=1}^{\infty}$ be a sequence in $\{x \in \mathcal{X} : \|Tx\| \le n\}$ converging to some $u \in \mathcal{X}$. Now, notice that the map $x \mapsto \|x\|$ is a continuous function $\mathcal{X} \to \mathbb{R}$ so that, by continuity of T,

$$\|Tu\| = \left\| \lim_{k \to \infty} Tu_k \right\| = \lim_{k \to \infty} \|Tu_k\| \le n.$$

[1]Such a subspace is often called a set of the first category.

This gives $u \in \{x \in \mathcal{X} : \|Tx\| \leq n\}$ which shows that $\{x \in \mathcal{X} : \|Tx\| \leq n\}$ is closed in $\mathcal{X}$ for every $T \in \mathcal{A}$. Furthermore,

$$E_n = \bigcap_{T \in \mathcal{A}} \{x \in \mathcal{X} : \|Tx\| \leq n\}$$

is closed as the intersection of closed sets. We now come to the real meat of the proof. We know that there exists some non-meager set $\mathcal{E}$ in $\mathcal{X}$ and $N \in \mathbb{N}$ such that

$$\sup_{T \in \mathcal{A}} \|Tx\| \leq N, \quad \forall x \in \mathcal{E}.$$

This implies precisely that $\mathcal{E} \subseteq E_N$, as defined above. Now, $\overline{\mathcal{E}} \subseteq E_N$ since E_N is closed. Using that $\mathcal{E}$ is not nowhere dense, $\overline{\mathcal{E}}$ contains a closed ball $\overline{B(x_0, r)}$, for some $r > 0$. In particular, $\overline{B(x_0, r)} \subseteq E_N$. Then we also have the inclusion

$$E_{2N} \supseteq \overline{B(0, r)}.$$

Certainly, let $T \in \mathcal{A}$ be given and $\|x\| \leq r$; notice that $x + x_0 \in \overline{B(x_0, r)} \subseteq E_N$ whence

$$\|Tx\| \leq \|T(x - x_0)\| + \|Tx_0\| \leq 2N.$$

This means that $\overline{B(0, r)} \subseteq E_{2N}$. Let now $|x| = 1$ and notice that $rx \in \overline{B(0, r)}$. It follows that

$$\|T(rx)\| \leq 2N.$$

But, by linearity of T: $\|T(rx)\| = r\,\|Tx\|$ whence $\|Tx\| \leq 2N/r$. This implies that $\|T\| \leq 2N/r$. But, $T \in \mathcal{A}$ was arbitrary. It follows that

$$\sup_{T \in \mathcal{A}} \|T\| < \infty$$

as was asserted. ■

The Baire Category Theorem states that a complete metric space is non-meager in itself. Consequently, we obtain the following easy corollary of the uniform boundedness principle.

Corollary. *Let $\mathcal{X}$ be a Banach space and $\mathcal{Y}$ a normed space, over $\mathbb{R}$ or $\mathbb{C}$. Suppose that $\mathcal{A} \subseteq \mathscr{L}(\mathcal{X}, \mathcal{Y})$ is non-empty collection and*

$$\sup_{T \in \mathcal{A}} \|Tx\| < \infty, \quad \forall x \in \mathcal{X}.$$

Then

$$\sup_{T \in \mathcal{A}} \|T\| < \infty.$$