Notes on Conformal Mappings and The Riemann Mapping Theorem

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1 Conformal Mappings

1–1.1 Möbius Transformations

Of the mappings that we will study, we begin with the rational composition of linear functions. Of course, any linear function is *entire* or holomorphic in all of $\mathbb{C}$. These will turn out to be an important class of transformations as they preserve much geometry and much of the topological properties of subsets of $\mathbb{C}$. More precisely, these are *fractional linear transformations* or *Möbius transformations* which we define as

Definition 1 (Fractional Linear Transformations). *Let $\mathbb{S} = \mathbb{C} \cup \{\infty\}$. A mapping $T : \mathbb{S} \to \mathbb{S}$ given by*

$$\mathcal{T}(z) = \frac{az+b}{cz+d}, \quad a, b, c, d \in \mathbb{C} \text{ with } ad - bc \neq 0 \tag{1}$$

is called a linear fractional transformation *or* Möbius transformation. *It has an inverse function*

$$\mathcal{T}^{-1}(w) = \frac{dw - b}{a - cw}$$

is also a fractional linear transformation. We naturally define the collection of all such transformations by $\mathrm{PGL}(2, \mathbb{C})$*; this set forms a group under composition and is called the Möbius group. The fact that this is indeed a group when equipped with this operation $\star$ is left as an exercise and may be verified by brute computation.*

These may often be represented in terms of matrices, although in general we shall refrain from doing so. There are many such cases of these transformations. The simplest is when $c = d = 0$. In this case, we are simply left with the map $\mathcal{T}(z) = az + b$ which is certainly entire. Now, if $a = 1$ we are left with $\mathcal{T}(z) = z + c$ which corresponds to a translation in the plane, clearly preserving the structure of whatever subset $\Omega \subseteq \mathbb{S}$. Additionally if $a \neq 0$ and $c = 0$ we are left with $\mathcal{T}(z) = az$ which corresponds to a *rescaling* and/or a rotation. For instance, consider the unit disc $\mathbb{D} \subset \mathbb{C}$, the map $z \mapsto 2z$ will take $\mathbb{D}$ onto the larger disk $\{|z| < 2\}$.

However, if we consider the right-half plane $\mathbb{H}_+ := \{z : \Re(z) > 0\}$ under the transformation $z \mapsto e^{i\frac{\pi}{2}}z$ we are no longer rescaling but *rotating* the half-plane $\mathbb{H}_+$ by $\frac{\pi}{2}$ and hence we send $\mathbb{H}_+$ onto $\mathbb{H}^+ := \{z : \Im(z) > 0\}$. In our case this is not too hard to verify.

Example 1.1. *The map $z \mapsto e^{i\frac{\pi}{2}}$ takes $\mathbb{H}_+$ onto $\mathbb{H}^+$ conformally.*

Solution. Clearly the mapping is holomorphic and hence meromorphic in $\mathbb{C}$ and especially $\mathbb{H}_+ \subset \mathbb{C}$. Now it is clear from the linearity that this mapping is injective in $\mathbb{C}$. To show that it is onto, pick a point $\zeta \in \mathbb{H}_+$, or set $\zeta := a + bi$ for $a > 0$. Then, multiplying through by $e^{i\frac{\pi}{2}}$ corresponds to a rotation by $\frac{\pi}{2}$ or, simply a multiplication by i. Hence, $z \mapsto i(a + bi) = -b + ai$, so we conclude that this map is invertible.

The map $z \mapsto \dfrac{1}{z}$ is referred to as an *inversion*.

We now turn our attention to so-called *cross ratios*. Given $z_2, z_3, z_4 \in \mathbb{S}$, all distinct, we may find a Möbius transformation S taking these aforementioned points into $1, 0, \infty$ respectively. Indeed,

1. If none of these $z_i = \infty$ we may simply take

$$S(z) = \frac{z - z_3}{z - z_4} \cdot \frac{z_2 - z_4}{z_2 - z_3} \tag{2}$$

which is clearly satisfactory and in $PGL(2, \mathbb{C})$.

2. If one of $z_2, z_3, z_4 = \infty$ then respectively:

$$S(z) \in \left\{ \frac{z - z_2}{z - z_4}, \frac{z_2 - z_4}{z - z_4}, \frac{z - z_3}{z_2 - z_3} \right\} \tag{3}$$

Proposition 1. *The transformation S given above is uniquely determined.*

Proof. If $T \sim S$ were any other such transformation, then ST^{-1} would leave the points $1, 0, \infty$ fixed. This can only happen if ST^{-1} is the identity. Indeed, since $\mathrm{PGL}(2, \mathbb{C})$ is a group under composition we know ST^{-1} takes on the form

$$\frac{az + b}{cz + d}$$

for suitable complex numbers a, b, c, d. Since $0 \mapsto 0$ it follows that $b = 0$. Using that $1 \mapsto 1$ we conclude that $a = c + d$, but since $\infty \mapsto \infty$ one also has $c = 0$ so that $a = d$ and $ST^{-1} = z$ or $S = T$.

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Definition 2 (The Crossing Ratio). *The Crossing Ratio denoted (z_1, z_2, z_3, z_4) is the image of $z_1 \in \mathbb{S}$ under the linear transformation taking $z_2, z_3, z_4 \hookrightarrow 1, 0, \infty$ where we of course assume the z_2, z_3, z_4 are all distinct values in the extended complex plane of Riemann sphere.*

THEOREM 2. *Let $z_1, z_2, z_3, z_4 \in \mathbb{S}$ be distinct and let $T : \mathbb{S} \to \mathbb{S}$ be any linear transformation. Then $(Tz_1, Tz_2, Tz_3, Tz_4) = (z_1, z_2, z_3, z_4)$.*

Proof. Let $Sz := (z, z_2, z_3, z_4)$. Then observe that ST^{-1} will take Tz_2, Tz_3, Tz_4 into $1, 0, \infty$ respectively. Hence, since (a, b, c, d) is by definition the image of a under the transformation that takes b, c, d to $1, 0, \infty$ we have

$$(Tz_1, Tz_2, Tz_3, Tz_4) = ST^{-1}(Tz_1) = S(z_1) = (z_1, z_2, z_3, z_4)$$

■

THEOREM 3. *On the Riemann sphere a straight line is a circle. Hence, a Möbius transformation sends circles to circles.*

Proof Sketch. Since a Möbius transformation may be made up of finitely many rotations, scales, translations or inversions we need only argue for each of these. These may be verified by direct calculations. The result is clear for rotations and scalings. Translations are obvious as well.

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DEFINITION 3 (Left Regions). *Let $\Omega \subsetneq \mathbb{C}$ be a non-empty simply connected domain and let $(z_i)_{i=1}^N \subset \partial\Omega$ be a collection of points on a path. We say that Ω is a left-region with respect to $(z_i)_{i=1}^N$ provided Ω lies to the left of the path travelled by $z_1 \to z_2 \to \ldots \to z_N$.*

THEOREM 4 (Left Hand Rule). *Let $\Omega \subsetneq \mathbb{C}$ be a simply non-empty connected domain and let $\Gamma : \mathbb{S} \to \mathbb{S}$ be a Möbius transformation. If Ω is a left region with respect to a collection of ordered points $(z_i)_{i=1}^N \subset \partial\Omega$ and we denote by γ the image of $\partial\Omega$ under Γ and $(w_i)_{i=1}^N := (\Gamma(z_i))_{i=1}^N$ then $\Gamma(\Omega)$ is the left-region with respect to $(w_i)_{i=1}^N \subset \gamma$.*

1–1.2 More General Conformal Mappings

Although these have countless applications in physics and engineering, we will mostly focus on their geometric properties and their topological description of holomorphic functions. It is not, however, easy to determine conformal mappings between subsets of the complex plane. We nonetheless explore some of these in this section.

Given a conformal map $w = f(z)$ defined on a connected subset $\Omega \subset \mathbb{C}$ we may glean some geometric information about the mapping by considering level curves of the map w. This may of course we done by fixing a variable, say $x = \Re(z)$ and writing $f(z) = u(x, y) + iv(x, y)$ for functions $u = \Re(f), v = \Im(f)$. In any case, we may explore some conformal mappings that are **not** Möbius transformations.

The simplest of these aforementioned conformal maps is the power mapping defined by $z \mapsto z^\alpha$ for any $\alpha = \Re(\alpha) > 0$. To see what happens to a domain under this map, we begin we the obvious note that $z \mapsto z^\alpha$ fixes the origin. If $z \neq 0$ then we may express it as

$$z = \rho e^{i\theta}, \quad 0 \leq \theta < 2\pi$$

In which case, it becomes much clearer that

$$z \mapsto z^\alpha = \left(\rho e^{i\theta}\right)^\alpha = \rho^\alpha e^{i(\alpha\theta)}$$

We need to be careful here, if α is fractional then the mapping isn't holomorphic or injective (in fact in the plane it won't even be single valued). The above expression is still useful, it shows that this transformation will preserve circles and preserve the shape of straight lines in $\mathbb{S}$.

To return to the problem at hand, it is usually recommended to proceed in two steps. Suppose we are given two *nice* regions $\Omega, \Sigma \subset \mathbb{C}$ and which to find a conformal mapping $\Psi : \Omega \to \Sigma$. In general, the best method is two find two *auxiliary* conformal mappings

$$\Gamma : \Omega \to \mathbb{D}, \quad \Lambda : \mathbb{D} \to \Sigma \tag{4}$$

so that $\Psi := \Lambda \circ \Gamma$ is a desired conformal mapping.

1–1.3 Automorphisms of $\mathbb{D}$

We begin with an essential definition:

Definition 4. *A conformal mapping from a region $\Omega \subseteq \mathbb{C}$ onto itself is called an automorphism of Ω. The space of all automorphisms of Ω is denoted by* $\mathrm{Aut}(\Omega)$

Yet again, $\mathrm{Aut}(\Omega)$ forms a group under composition $\circ$, where the identity is simply the mapping $z \mapsto z$. All of this is a consequence of what we have shown in general for conformal mappings. Moreover, if we let $f, g \in \mathrm{Aut}(\Omega)$ then clearly

$$(g \circ f)^{-1} = f^{-1} \circ g^{-1} \in \mathrm{Aut}(\Omega)$$

As previously stated, rotations are automorphisms of $\mathbb{D}$. Indeed, letting $\theta \in \mathbb{R}$ we may take $\theta \in [0, 2\pi)$ and see that clearly $z \mapsto e^{i\theta} z$ takes $\mathbb{D}$ onto $\mathbb{D}$. In fact, this is true for any disc of any positive radius in $\mathbb{C}$. This map has the inverse function $w \mapsto e^{-i\theta} w$. Before we study the more interesting automorphisms of the form

$$\varphi_\alpha(z) := \frac{\alpha - z}{1 - \overline{\alpha} z}, \quad \alpha \in \mathbb{C} \text{ with } |\alpha| < 1 \tag{5}$$

we wil give the following lemma due to Schwarz:

Theorem 5 (The Schwarz Lemma). *Let $f : \mathbb{D} \to \mathbb{D}$ be holomorphic along fixing the origin. Then,*

(a) $|f(z)| \leq |z|$ in all of $\mathbb{D}$.

(b) If for some $z_0 \neq 0$ we have equality then f is a rotation.

(c) $|f'(0)| \leq 1$ and f is a rotation of equality holds true.

Proof. We first claim that $\frac{f(z)}{z}$ has a removable singularity at the origin. Indeed, note that since $f(0) = 0$ we may write for z with $|z| < \rho \ll 1$

$$\frac{f(z)}{z} = \frac{f(z) - f(0)}{z - 0} \xrightarrow{z \to 0} f'(0)$$

and hence $\frac{f(z)}{z}$ is bounded in a neighbourhood of the origin and consequently by Riemann's theorem the origin is a removable singularity. Then, we see that we may make $\frac{f(z)}{z}$ holomorphic in all of $\mathbb{D}$. Now on any circle $|z| = r < 1$ we observe that

$$\left| \frac{f(z)}{z} \right| = \frac{|f(z)|}{|z|} \leq \frac{1}{|z|} = \frac{1}{r}$$

where we bound $|f|$ by 1 since f maps $\mathbb{D}$ into $\mathbb{D}$. Now an application of the strong maximum principle for holomorphisms shows that this is true for all z inside and on the circle $|z| = r$. Now as r was arbitrary letting $r \to 1$ yields that

$$|f(z)| \leq |z|, \quad \forall z \in \mathbb{D} \tag{a}$$

If for some interior point $z_0 \in \mathbb{D}$ we have $|f(z_0)| = |z_0|$ then we also have that the holomorphic quotient $\frac{f(z)}{z}$ achieves it's maximum inside some disk $B(0, \rho) \subset \mathbb{D}$ for all sufficiently large $\rho < 1$. Then, again by the strong maximum principle for holomorphic functions we deduce that

$$\frac{|f(z)|}{|z|} = 1, \quad \forall z \in \mathbb{D}$$

Thence we see that $f(z) = \gamma z$ for some $\gamma \in \mathbb{C}$ with $|\gamma| = 1$. As $\gamma \neq 0$ we may express $\gamma = e^{i\theta}$ for $\theta \in \mathbb{R}$ with non-negative modulus strictly less than 2π. Or, equivalently writing

$$f(z) = \gamma z = e^{i\theta} z, \quad \theta \in [0, 2\pi), \quad z \in \mathbb{D} \tag{b}$$

we have that f is a rotation as was required.

Finally, if we let $g(z) = \frac{f(z)}{z}$ we already have that g is holomorphic inside $\mathbb{D}$. It is clear from (a) that the desired inequality in (c) it true. On the other hand, we also note

$$g(0) = \lim_{z \to 0} \frac{f(z)}{z} = \frac{f(z) - f(0)}{z - 0} = f'(0) = 1$$

Hence g achieves it's maximum in the interior of $\mathbb{D}$ and hence by the same argument as before we must have that $g \equiv 1$ whence by the previous argument we also have that f is a rotation.

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Now we return to the briefly described function $\varphi : \mathbb{D} \to \mathbb{D}$. It turns out that mappings of this form will naturally arise in many analytic contexts. We will prove that these are indeed automorphisms of $\mathbb{D}$. Clearly, since we require $|\alpha| < 1$ it follows that $\overline{\alpha} z \neq 1$ in all of $\mathbb{D}$ for otherwise one of $|\alpha|$ or $|z|$ would be no less than 1 which is absurd. Consequently φ_α is holomorphic in $\mathbb{D}$.

We must also show that φ_α indeed maps into $\mathbb{D}$, as this is certainly not obvious. First note that if $|z| = 1$ we have $z = e^{i\theta}$ and hence

$$\begin{aligned} \varphi_\alpha(z) = \frac{\alpha - e^{i\theta}}{1 - \overline{\alpha} e^{i\theta}} = \frac{\alpha - e^{i\theta}}{e^{i\theta}\left(e^{-i\theta} \quad - \overline{\alpha}\right)} &= \frac{\alpha e^{-i\theta} - 1}{e^{-i\theta} - \overline{\alpha}} \\ &= -e^{-i\theta} \cdot \frac{\beta}{\overline{\beta}} \end{aligned}$$

for $\beta := \alpha - e^{\theta}$. Therefore $|\varphi_\alpha(z)| = 1$ on $\partial \mathbb{D}$. We then see that by the strong maximum principle we have $|\varphi_\alpha| \leq 1$ in $\mathbb{D}$ as we wanted. A not so nice calculation (but certainly possible for even a highschool student) verifies that

$$\varphi_\alpha \circ \varphi_\alpha = z$$

and hence that φ_α is it's own inverse. These facts put together yield that this mapping is bijective and holomorphic in $\mathbb{D}$. Therefore $\varphi_\alpha \in \text{Aut}(\mathbb{D})$. This should be nice, but not absolutely amazing...yet. This is only one possible automorphism of $\mathbb{D}$? We prove the following fact to answer this question

Theorem 6. *Let $f \in \text{Aut}(\mathbb{D})$. There exists $\alpha \in \mathbb{D}$ and $\theta \in \mathbb{R}$ so that*

$$f(z) = e^{i\theta} \frac{\alpha - z}{1 - \overline{a} z} \tag{A}$$

Proof. By hypothesis that $f \in \text{Aut}(\mathbb{D})$ there exists a unique point $\alpha \in \mathbb{D}$ so that $f(\alpha) = 0 \in \mathbb{D}$. It is clear that the composite $g := f \circ \varphi_\alpha \in \text{Aut}(\mathbb{D})$ as well. Note that $g(0) = 0$ and so we use the Schwarz lemma to deduce that $|g(z)| \leq |z|$ in all $\mathbb{D}$. Now, $g^{-1} : \mathbb{D} \to \mathbb{D}$ is also an automorphism of $\mathbb{D}$ with $g^{-1}(0) = 0$. Again, by Schwarz we see $\left|g^{-1}(w)\right| \leq |w|$ so that $|z| \leq |g(z)|$ and hence $g \equiv z$ in modulus.

Again by the Schwarz lemma, we may find $\theta \in \mathbb{R}$ so that for $\gamma := e^{i\theta}$ one has $g(z) = \gamma z$ in $\mathbb{D}$. We may now set $z \mapsto \varphi_\alpha(z)$ since $\varphi_\alpha \in \text{Aut}(\mathbb{D})$ to discover that

$$\gamma\varphi_\alpha(z) = g \circ \varphi_\alpha = f(\varphi_\alpha(\varphi_\alpha(z))) = f(z)$$

■

Corollary 7. *Taking $\alpha = 0$ in the above theorem, it follows that the only automorphisms of $\mathbb{D}$ fixing the origin are rotations.*

2 Examples and Problems

In this section we give some actual examples of conformal mappings of subsets of $\mathbb{C}$ that arise naturally in practical situations.

Example 2.1. *Conformally map the disk $|z - i| < 1$ onto the auxiliary disk $|w| < 2$.*

Solution. Our first step here will be to find a conformal mapping

$$\Gamma_1 : \{|z - i| < 1\} \to \mathbb{D}$$

The answer here is clear, simply take $z \mapsto z - i$ and we are left with $\mathbb{D}$. Indeed, to see that this is injective write $\Gamma_1(z) = \Gamma_2(z)$ so that $x + iy = u + iy$. For surjectivity, it is suffices to show that each element in $w \in \mathbb{D}$ is mapped to by some $z \in \{|z - i| < 1\}$. If $w \in \mathbb{D}$ then $w = u + iv$, taking $z = u + i(v + 1) \in \{|z - i| < 1\}$ we see that $z \mapsto w$.

Now let $\Gamma_2 : \mathbb{D} \to \{|w| < 2\}$ be given by $z \mapsto 2z$. This is clearly well defined and one-one. Hence, the composition $\Gamma := \Gamma_2 \circ \Gamma_1$ is a valid mapping and is explicitly given by

$$\Gamma(z) = (\Gamma_2 \circ \Gamma_1)(z) = 2(z - i) = 2z - 2i \tag{6}$$

Example 2.2. *Map (conformally) the inside of the circle $\{|z - i| = 1\}$ to the outside of the circle $|w| = 2$.*

Solution. We will send the origin of the first disk to ∞, so we want a map Γ which takes $i \mapsto \infty$. So, Γ will be of the form

$$\Gamma(z) = \frac{az + b}{cd + z} = \frac{az + b}{z - i}$$

A conformal map preserved angles and so we want to make sure points on the circle $|z - i| = 1$ get mapped to $|w| = 2$. With these conditions imposed, we shall find explicit values for $a, b \in \mathbb{C}$. Consider the points $0, 1 + i, 2i$ which lie on the domain circle. Now, by direct substitution we recover

$$\Gamma(0) = bi, \quad \Gamma(1 + i) = a + ai + b, \quad \Gamma(2i) = \frac{2ai + b}{ai - i}$$

In modulus we want these to satisfy $|\Gamma| = 2$. A simple choice is to take $\alpha = 0$. We would then require

$$\Gamma(z) = \frac{b}{z - i}$$

so for points on $\{|z-i|=1\}$ to be mapped to $\{|w|=2\}$ we would take $b=2$ and therefore

$$\Gamma(z) = \frac{2}{z-i}$$

Let us now recall that we denote the *right-half plane* $\{\Re(z) > 0\}$ by $\mathbb{H}_+$ and the *upper-half plane* $\{\Im(z) > 0\}$ by $\mathbb{H}^+$. With this in mind:

EXAMPLE 2.3. *Find a conformal mapping* $\Gamma : \mathbb{D} \to \mathbb{H}_+$.

Solution. Since conformal maps send circles to circles, and on the Riemann Sphere $\mathbb{S}$ a straight line is a circle we will find a conformal map Γ which takes $\mathbb{D}$ onto $\{\Re(z) = 0\}$. Indeed, to do so we will map a point of the circle to the north pole of $\mathbb{S}$, i.e ∞. Consider now the map

$$\Gamma(z) := \frac{az+b}{z-1}$$

Clearly on $\mathbb{S}$ this will take $1 \mapsto \infty$ and if we want a point, say, -1 to be mapped to the origin we may refine Γ as follows

$$\Gamma(z) = \frac{z+1}{z-1}$$

Hence this takes $\partial\mathbb{D}$ onto the imaginary axis since

$$i \mapsto \frac{i+1}{i-1} = \frac{i+1}{i-1} \cdot -1 - i - 1 - i = -i \in \{\Re(z) = 0\}$$

Now taking an interior point, say the origin, of $\mathbb{D}$ we see

$$\Gamma(0) = \frac{1}{-1} = -1$$

Which is an issue, however this may be corrected by a rotation of π to derive

$$\widetilde{\Gamma}(z) = \frac{1+z}{1-z} \tag{7}$$

EXAMPLE 2.4. *Set* $\Gamma(z) := \dfrac{z}{2z-8}$. *How does this map the interior of the circle* $C := \{|z-2| = 2\}$*?*

Solution. Note that $\Gamma(z)$ is meromorphic in $\mathbb{C}$ since $2z - 8 = 0 \iff z = 4$ which is certainly a point on C and hence we see that Γ is meromorphic at a point on the circle and consequently Γ has a pole on C which together with the fact that $0 \in C$ and $\Gamma(0) = 0$ we conclude that Γ takes C onto a straight line through the origin and the north pole of the sphere $\mathbb{S}$: the imaginary axis. Hence, by angle preservation it will take the interior of C to either $\mathbb{H}_+$ or $\mathbb{H}_-$ where we accordingly define

$$\mathbb{H}_- := \{\Re(z) < 0\}$$

Indeed, the point $z = 2$ is taken to

$$2 \mapsto \frac{2}{4-8} = -\frac{1}{2}$$

So we conclude that Γ maps onto $\Re(z) < 0$.

Example 2.5. *Conformally map $\mathbb{D}^c$ onto $\mathbb{H}_+$.*

Solution. We begin by recalling the *left hand rule*; note that $\mathbb{D}^c$ can be made into a left-region by defining $(z_i)_{i=1}^3$ via $z_1 := -1, z_2 := i$ and $z_3 := 1$. Obviously the arc defined by this triplet in this order along the circle gives $\mathbb{D}^c$ positive orientation.

Now, we may also make $\mathbb{H}_+$ into a left region by simply taking $w_1 := 0, w_2 := -i$ and $w_3 := \infty$. It is clear that in this direction, the half-space $\mathbb{H}_+$ is given positive (left) orientation.

Since we want $-1 \mapsto 0$ and $1 \mapsto \infty$ it is natural to define

$$\Gamma(z) := \frac{z+1}{z-1}$$

Then observe

$$\Gamma(z_2) = \frac{i+1}{i-1} = -i$$

which indeed lies on the imaginary axis and is between 0 and ∞. So our conditions on the sequence $(z_i)_{i=1}^3$ and the transformed sequence $(\Gamma(z_i))_{i=1}^3$ are satisfied. Thence, the map above will then take $\mathbb{D}^c$ onto $\mathbb{H}_+$.

Example 2.6. *Find a conformal mapping $\Gamma : \mathbb{D}^c \to \mathbb{H}^+$*

Solution. Just as before, take $z_1 := -1, z_2 := i$ and finally $z_3 := 1$. In this order, these give $\mathbb{D}$ positive orientation. We wish to send $-1 \mapsto 0, i \mapsto 1$ and $1 \mapsto \infty$. Naturally we begin by considering

$$\gamma(z) := \frac{z+1}{z-1}$$

Plugging in $z = i$ we are left with

$$\gamma(i) = i$$

Thus this transformation leaves i fixed. Now we will rotate by $-\frac{\pi}{2}$ in $\mathbb{C}$ and instead define:

$$\Gamma(z) := \frac{1}{i}\frac{z+1}{z-1} \tag{8}$$

Which takes $\partial\mathbb{D} \hookrightarrow \mathbb{R}$. Yet again, we conclude that Γ takes $\mathbb{D}^c$ onto $\mathbb{H}^+$. Via a similar procedure one can construct a map $\Lambda : \mathbb{D} \to \mathbb{H}^+$.

Problem 2.7. *Conformally map the unit disc $\mathbb{D}$ onto the upper half-plane $\mathbb{H}^+$. Hence, find two biholomorphisms.*

Solution. Considering our *left hand rule* we naturally seek a map φ which takes

$$1 \mapsto -1, \quad i \mapsto 0, \quad -1 \mapsto 1$$

Naturally, we assume $\varphi = \dfrac{z-i}{cz+d}$ and we must find such constants c, d. Plugging in our restrictions on $z = \pm 1$ we obtain a linear system of equations and conclude that solutions are $c = i, d = -1$ by simple highschool algebra. Thus, we have a transformation

$$\varphi(z) = \frac{z-i}{zi-1}, \quad \Phi(z) = -\frac{1+iw}{w+i}$$

where Φ is the inverse transformation and takes $\mathbb{H}^+$ onto $\mathbb{D}$.

Problem 2.8. *Take the region $\left\{|z| < 1 \cap \left|z - \frac{1}{2}\right| > \frac{1}{2}\right\}$ onto $\mathbb{D}$ conformally.*

Solution. This one is more difficult than the others. First, both of these regions are circles. Consider the first mapping

$$\Gamma_1(z) := \frac{1}{z-1}$$

Clearly, this sends $1 \mapsto \infty$. Hence the inner most circle will be a line. More precisely, we that $\Gamma_1(0) = -1$ and hence the image of the inner-most circle under the transformation Γ_1 is a line passing through -1. To see which line, consider $\Gamma_1(1/2 + i/2)$ which is given by

$$\Gamma_1\left(\frac{1+i}{2}\right) = \frac{1}{\frac{1+i}{2} - 1} = -1 - i$$

Thence, we conclude that Γ_1 takes the inner most circle onto the line $\Re(z) = -1$. Similarly, we repeat this process to discover that Γ_1 takes the outer-most circle onto the line $\Re(z) = -\frac{1}{2}$. Putting this facts together,

$$\left\{|z| < 1 \cap \left|z - \frac{1}{2}\right|\right\} \xrightarrow{\Gamma_1} \left\{-1 < \Re(z) < -\frac{1}{2}\right\} \tag{9}$$

We will now perform the following rotation: $\Gamma_2 := e^{-i\frac{\pi}{2}} = -i$ which will take

$$\left\{-1 < \Re(z) < -\frac{1}{2}\right\} \xrightarrow{\Gamma_2} \left\{\frac{1}{2} < \Im(z) < 1\right\} \tag{10}$$

This last transformation is seen by writing an arbitrary $z = x + iy$ for $-1 < x < -\frac{1}{2}$ where multiplying through by $-i$ will yield the above. Introducing a third mapping Γ_3 which takes $z \mapsto z - \frac{i}{2}$ we transform

$$\left\{\frac{1}{2} < \Im(z) < 1\right\} \xrightarrow{\Gamma_3} \left\{0 < \Im(z) < \frac{1}{2}\right\} \tag{11}$$

A rescaling by $\Gamma_4(z) := 2\pi z$ will transform this above region to

$$\left\{0 < \Im(z) < \frac{1}{2}\right\} \xrightarrow{\Gamma_4} \{0 < \Im(z) < \pi\} \tag{12}$$

Now, we will exponentiate this by taking a mapping $\Gamma_5 := e^z$. For any z with $\Im(z) < \pi$ note that writing $z = x + iy$ one has $e^z = e^x e^{iy}$. Thus,

$$\{0 < \Im(z) < \pi\} \xrightarrow{\Gamma_5} \mathbb{H}^+ \tag{13}$$

Finally, let us recall the transformation Φ from the previous problem, which took $\mathbb{H}^+$ onto $\mathbb{D}$ conformally. If we compose all of these, we are left with a conformal mapping Γ which does the job.

3 The Riemann Mapping Theorem

This is a rather intricate proof, and we will instead simply state it and sketch the proof.

Theorem 8 (Riemann). *Let $\Omega \subsetneq \mathbb{C}$ be a non-empty simply connected domain. Then for $z_0 \in \Omega$ there exists a unique conformal mapping $\Gamma : \Omega \to \mathbb{D}$ so that*

$$F(z_0) = 0, \quad F'(z_0) > 0 \tag{$\mathscr{R}$}$$

In particular, by taking the inverses, any two such domains Ω and Ω' must be conformally equivalent.

Proof Sketch. One usually begins by proving that Ω is conformally equivalent to an open subset of $\mathbb{D}$ that also contains the origin. This is done by taking a complex number $\alpha \in \mathbb{C} \setminus \Omega$ and defining a holomorphic (in Ω) logarithm $\log z - \alpha$ and constructing such a conformal mapping by continuity.

One then may consider the family of injective holomorphisms from Ω **into** the unit disk $\mathbb{D}$ fixing the origin $\mathcal{F}$. This is non-empty because it contains the identity. This family is uniformly bounded. One can then find a function $f \in \mathcal{F}$ maximizing $|f'(0)|$. This is done by considering the sup over all such functions at this point and taking a sequence of functions in $\mathcal{F}$ converging pointwise to the sup at the origin in their first derivative. An application of Montel's theorem guarantees a convergent subsequence converging uniformly on compact sets to a function f holomorphic in Ω. One usually shows that this function is a desired function.

Finally one will show that f is indeed conformal; or in our case surjective. Arguing by contradiction one can show that otherwise there exists a function $g \in \mathcal{F}$ with $g'(0)$ larger in modulus than $f'(0)$. This can be done by assuming there is $\alpha \in \mathbb{D}$ so that $f(z) \neq \alpha$ in all Ω and considering the function $\varphi_\alpha \in \mathrm{Aut}(\mathbb{D})$:

$$\varphi_\alpha(z) = \frac{\alpha - z}{1 - \overline{\alpha} z}$$

One can define a square root function on $U := (\varphi_\alpha \circ f)(\Omega)$ by using the logarithm (see other notes). One then defines $F := \varphi_{g(\alpha)} \circ g \circ \varphi_\alpha \circ f$ and shows that $F \in \mathcal{F}$ and may use the Schwarz lemma to conclude that $|F'(0)| > |f'(0)|$ which is a contradiction.

■

4 References

(I) Complex Analysis. Stein and Shakarchi. Princeton Lectures in Analysis, 2003.

(II) Complex Analysis: An Introduction to the Theory of Analytic Functions of One Complex Variable. Lars Ahlfors. 1953.