THE TOPOLOGY OF METRIC SPACES

DANA BERMAN

In the study of analysis, one often begins with the study of continuous functions over the real numbers before generalizing to continuous function on metric spaces. By doing so, we gain generality while simultaneously simplifying our underlying assumptions. In this same spirit, we would like to simplify our model for continuous functions even further. This is where Topology will come into play. Before elaborating on this subject any further, we remind the reader of a few definitions.

Definition 1. *A non-empty set X with a function $d : X \to [0, \infty)$ is called a metric space if for all $x, y, z \in X$ it holds that*

1. $d(x, y) = 0$ *if and only if* $x = y$,
2. $d(x, y) = d(y, x)$, (Reflexivity)
3. $d(x, z) \leq d(x, y) + d(y, z)$. (The triangle inequality)

The function d is called the metric *on X.*

Definition 2. *Suppose that X and Y are metric spaces with metrics d_X and d_Y respectively. A function $f : X \to Y$ is said to be continuous at a point $x_0 \in X$ if for every $\varepsilon > 0$, there exists $\delta > 0$ such that*

$$d_Y\left(f(x_0), f(x)\right) < \varepsilon \quad \textit{whenever} \quad d_X\left(x_0, x\right) < \delta$$

Furthermore, for metric spaces X, Y, we say that a function $f : X \to Y$ is continuous if f is continuous at every $x \in X$. We now turn to the definition of an open set. A subset O of X is said to be open if for every $x \in O$, there exists $\varepsilon > 0$ such that

$$B(x, \varepsilon) = \{y \in X : d_X(x, y) < \varepsilon\} \subseteq O.$$

It is vacuous truth that the empty set is open. Furthermore, a set is said to be closed if it's complement is open. We now turn our attention to a fundamental observation.

Theorem 3. *Suppose X and Y are metric spaces with metrics d_X and d_Y respectively. A function $f : X \to Y$ is continuous at x if and only if the pre-image for every open set $O \subseteq Y$ containing $f(x)$, there exists an open set $\mathfrak{G} \subseteq X$ containing x such that $\mathfrak{G} \subseteq f^{-1}(O)$.*

Proof. Suppose first that f is continuous and let O be an open set containing $f(x)$. By definition of open sets, we may find $\varepsilon > 0$ such that

$$B\left(f(x), \varepsilon\right) \subseteq O$$

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Then, by continuity, there exists $\delta > 0$ such that

$$d_Y\left(f(x), f(y)\right) < \varepsilon \quad \text{whenever} \quad d_X\left(x, y\right) < \delta$$

Thus, we define $\mathfrak{G} = B(x, \delta)$ which is an open set containing x for which it holds that

$$f\left(\mathfrak{G}\right) \subseteq O \iff \mathfrak{G} \subseteq f^{-1}(O).$$

Conversely, fix $\varepsilon > 0$ and note that $B(f(x), \varepsilon)$ is an open set in Y. By assumption, there exists an open set $\mathfrak{G} \subseteq X$ containing x such that

$$f\left(\mathfrak{G}\right) \subseteq B(f(x), \varepsilon).$$

In particular, by definition of an open set we may find $\delta > 0$ such that

$$B(x, \delta) \subseteq \mathfrak{G}$$

But then,

$$f\left(B(x, \delta)\right) \subseteq B(f(x), \varepsilon)$$

which is precisely the statement that

$$d_Y\left(f(x), f(y)\right) < \varepsilon \quad \text{whenever} \quad d_X\left(x, y\right) < \delta.$$

This shows that f is continuous which concludes the proof. □

While the above theorem allows us to define continuity at a point, a more interesting result is given below.

Theorem 4. *Suppose X and Y are metric spaces with metrics d_X and d_Y respectively. A function $f : X \to Y$ is continuous if and only if the pre-image of every open set is open. That is, for every open set $O \subseteq Y$, the set $f^{-1}(O)$ is also open.*

Proof. Suppose first that $f : X \to Y$ is continuous and let O be an arbitrary open set in Y. If O is disjoint from the image of f then $f^{-1}(O) = \varnothing$ which is indeed open, whence we are done. Otherwise, $f^{-1}(O)$ is non-empty. In order to show that $f^{-1}(O)$ is an open set, pick an arbitrary element $x \in f^{-1}(O)$. Now, if $y = f(x)$ then $y \in O$ and so we may find $\varepsilon > 0$ such that

$$B(y, \varepsilon) \subseteq O$$

By continuity of f (at x), there exists $\delta > 0$ be such that

$$f\left(B(x, \delta)\right) \subseteq B(y, \varepsilon) \subseteq O$$

Thus,

$$B(x, \delta) \subseteq f^{-1}(O)$$

Since x was an arbitrary element of $f^{-1}(O)$, we conclude that the set is open. The converse direction follows from the previous theorem. □

From the above theorems, we see that in order to define continuity we only need information about the open sets. This will lead us to the definition of a topological space.

A topological space (which will be defined shortly) is a tuple $(X, \mathcal{T})$ where X is a non-empty set and $\mathcal{T}$ is a collection of subsets of X called the *open* *sets*. There are two core set operations: unions and intersections. In a metric space, we make the following observations about the open set;

(1) An arbitrary union of open sets is open,
(2) A finite intersection of open sets is again open,
(3) The infinite intersection of open sets is not necessarily open.

The proof of the above facts is left as an exercise. Furthermore, if X is a metric space then X and the empty set are themselves open.

The above observations lead us to a rigorous definition of a topological space.

Definition 5. *A non-empty set X with a collection of subsets $\mathcal{T}$ is said to be a topological space if*

(1) $\varnothing, X \in \mathcal{T}$,
(2) *$\mathcal{T}$ is closed under arbitrary unions,*
(3) *$\mathcal{T}$ is closed under finite intersections.*

In this case, we say that $\mathcal{T}$ is a topology on X.

If $(X, \mathcal{T})$ is a topological space, then we call the sets in $\mathcal{T}$ the *open sets* of X. If $S \subseteq X$ is such that S^{c} is open the S is said to be *closed*.

A metric space X with the collection open sets in the sense of a metric space is called the *metric space topology* on X. We note a few properties concerning the relationship between the open balls and open sets in a metric space X. First note that the metric space topology $(X, \mathcal{T})$ is given by

$$\mathcal{T} = \{O \subseteq X : \forall x \in O,\ \exists \text{ open ball } B \text{ s.t. } x \in B \subseteq O\}$$

Furthermore, every open set is the union of open balls. Indeed, if O is open then for each $x \in O$ there exists $\varepsilon_x > 0$ such that $B(x, \varepsilon_x) \subseteq O$. But then,

$$O = \bigcup_{x \in O} B(x, \varepsilon_x)$$

Conversely, it is clear that any arbitrary union of open balls is again open. Note also that for every $x \in X$, there exists an open ball B such that $x \in B$. If B_1, B_2 are balls and $x \in B_1 \cap B_2$ then there exists an open ball $B \subseteq B_1 \cap B_2$ with $x \in B$.

The above observations lead us to the following definitions;

Definition 6. *Given a non-empty set X, a collection $\mathcal{B}$ of subsets of X is called a basis if*

(1) *For all $x \in X$, there exists $B \in \mathcal{B}$ such that $x \in B$,*
(2) *If $x \in B_1 \cap B_2$ for some $B_1, B_2 \in \mathcal{B}$ then there exists $B \in \mathcal{B}$ satisfying $x \in B \in B_1 \cap B_2$.*

Furthermore, if $(X, \mathcal{T})$ is a topological space then $\mathcal{B}$ is said to induce the topology $\mathcal{T}$ if every open set (i.e. set in $\mathcal{T}$) can be written as a union of elements in $\mathcal{B}$.

A few basic results are now in order.

Proposition 7. *If X is a non-empty set and $\mathcal{B}$ is a basis on X then the collection of subsets*

$$\mathcal{T} = \{O : O \text{ is the union of elements of } \mathcal{B}\}$$

is a topology on X. We call this the topology generated *by $\mathcal{B}$.*

Proof. We first notice that $\varnothing, X \in \mathcal{T}$. By convention, the empty union is precisely the empty set. Furthermore, since $\mathcal{B}$ is a basis, for every $x \in X$ there exists $B_x \in \mathcal{B}$ such that $x \in B_x$. Then

$$X = \bigcup_{x \in X} \{x\} = \bigcup_{x \in X} B_x$$

whence $X \in \mathcal{T}$.

Now, it is clear that $\mathcal{T}$ is closed under arbitrary unions. It remains to show closure under finite intersections. To this end, we pick two arbitrary set $O_1, O_2 \in \mathcal{T}$. By assumption, we may write

$$O_1 = \bigcup_{i \in I} A_i, \quad \text{and} \quad O_2 = \bigcup_{j \in J} B_j,$$

where $\{A_i\}_i, \{B_j\}_j \subseteq \mathcal{B}$. We may therefore write

$$O_1 \cap O_2 = \bigcup_{i,j} A_i \cap B_j$$

Now, since $\mathcal{B}$ is a basis, for each $x \in A_i \cap B_j$ we may find $C_{ij}(x) \in \mathcal{B}$ such that

$$x \in C_{ij}(x) \subseteq A_i \cap B_j$$

It is then clear that

$$O_1 \cap O_2 = \bigcup_{i,j} \bigcup_{x \in A_i \cap B_j} C_{ij}(x)$$

We conclude that $O_1 \cap O_2 \in \mathcal{T}$ as desired. □

Corollary 8. *If X is a non-empty set and $\mathcal{B}$ a basis on X then the topology generated by $\mathcal{B}$ is the minimal topology containing $\mathcal{B}$.*

In a metric space, the open balls were our basis. Recall that given the open balls, we had two distinct ways of defining the open sets. First as any arbitrary union of open balls. Equivalently, we say that a set O is open if for every $x \in O$ there exists an open ball B such that $x \in B \subseteq O$. The below proposition shows that this equivalence still holds in the more general case where X is any non-empty set and $\mathcal{B}$ a basis for X.

Proposition 9. *Let X be a non-empty set and $\mathcal{B}$ a basis. We claim that the topology generated by $\mathcal{B}$ is precisely*

$$\mathcal{T} = \{O \subseteq X : \forall x \in O, \exists B \in \mathcal{B} \text{ such that } x \in B \subseteq O\}$$

Proof. We will show that every set in $\mathcal{T}$ is a union of elements of $\mathcal{B}$ and conversely that every union of elements of $\mathcal{B}$ is in particular in $\mathcal{T}$.

Suppose first that O is an arbitrary element of $\mathcal{T}$. Then for each $x \in O$, let $B_x \in \mathcal{B}$ be such that $x \in B_x \subseteq O$. Then it is clear that

$$O = \bigcup_{x \in O} B_x$$

Conversely, suppose that

$$O = \{i \in I\} B_i$$

for some $\{B_i\}_i \subseteq \mathcal{B}$. Then for every $x \in O$, there exists $i \in I$ such that $x \in B_i$. So we have $x \in B_i \subseteq O$ which shows that $O \in \mathcal{T}$. $\square$