Normalization by Evaluation for Cocon, Part 1: Evaluation (Work in progress)

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INTRODUCTION
COCON

COCON: A two-level type theory designed for meta-programming.

[Pientka et al., 2019]

- Data-level: Edinburgh logical framework LF.
  Used to define languages in higher-order abstract syntax (HOAS).
- Meta-level: Martin-Löf type-theory (MLTT).
  Used to reason about LF datatypes.
- The two levels are linked with a contextual box/unbox modality.
- First-class LF contexts and LF context variables.

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Heterogeneous meta-programming

- We write programs (proofs) in a meta-language.
  For us, the the meta-language is MLTT.
- We manipulate programs from an object language (OL).
  For us, OLs are defined in LF.

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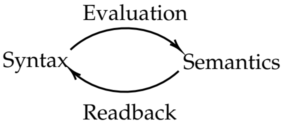
Normalization by Evaluation



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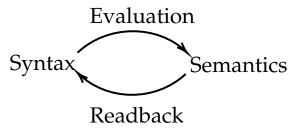

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What we know

- NbE for MLTT [Abel et al., 2007].
  $\Rightarrow$ NbE for LF since LF $\subseteq$ MLTT.
- NbE for modal dependent type theory [Hu et al., 2023].

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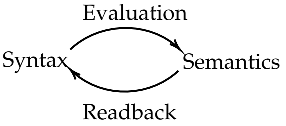

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What we don't know

- How to deal with first-class contexts and context variables.
- How to deal with contextual modality.

We can focus on these aspects by restricting our attention to CLF instead of the full Cocon.

Syntax of CLF

Sorts	s	$::=$	$\mathsf{type} \mid \mathsf{kind}$
Constants	$\mathbf{c}, \mathbf{a}$	$::=$	$\mathbf{tp} \mid \mathbf{base} \mid \mathbf{arr} \mid \mathbf{tm} \mid \mathbf{lam} \mid \mathbf{app}$
Expressions	M, A, K, E, F	$::=$	$s \mid \mathbf{c} \mid x_i \mid \lambda M \mid M\, M' \mid \Pi A.B \mid [\sigma]M \mid [\![\theta]\!]M \mid \lfloor C \rfloor_\sigma$
Contexts	Γ	$::=$	$\cdot \mid X_i \mid \Gamma, A \mid [\![\theta]\!]\Gamma$
Erased contexts	$\hat{\Gamma}$	$::=$	$\cdot, n \mid X_i, n \mid [\![\theta]\!]\hat{\Gamma}$
Substitutions	σ	$::=$	$\cdot \mid \mathsf{id}_{\hat{\Gamma}} \mid \uparrow \mid \sigma, M \mid [\sigma]\sigma' \mid [\![\theta]\!]\sigma$
Meta-types	U	$::=$	$\#\lceil \Gamma \vdash A \rceil \mid \lceil \Gamma \vdash A \rceil \mid \mathsf{ctx} \mid [\![\theta]\!]U$
Meta-terms	C	$::=$	$X_i \mid \lceil \hat{\Gamma} \vdash M \rceil \mid \Gamma \mid [\![\theta]\!]C$
Meta-contexts	Δ	$::=$	$\cdot \mid \Delta, U$
Meta-substitutions	θ	$::=$	$\cdot \mid \Uparrow \mid \theta, C \mid [\![\theta]\!]\theta'$

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$\boxed{(\Gamma)^- = \hat{\Gamma}}$ – Context erasure function:

$$(\cdot)^- = \cdot, 0 \qquad (\Gamma, A)^- = \begin{cases} \cdot, n+1 & \text{if } (\Gamma)^- = \cdot, n \\ X_i, n+1 & \text{if } (\Gamma)^- = X_i, n \end{cases}$$

$$(X_i)^- = X_i, 0 \qquad ([\![\theta]\!]\Gamma)^- = [\![\theta]\!](\Gamma)^-$$

Typing Judgment

$\boxed{\Delta; \Gamma \Vdash^{\mathrm{LF}} M : A}$ and $\boxed{\Delta \Vdash^{\mathcal{M}} C : U}$ – LF-layer and meta-layer typing.

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- Normal and parameter boxes:

$$\frac{\Delta; \Gamma \Vdash^{\mathrm{LF}} M : A \quad (\Gamma)^{-} = \hat{\Gamma}}{\Delta \Vdash^{\mathcal{M}} \lceil \hat{\Gamma} \vdash M \rceil : \lceil \Gamma \vdash A \rceil} \qquad \frac{\Delta \Vdash^{\mathcal{M}} C : \lceil \Gamma' \vdash A \rceil \quad \Delta; \Gamma \Vdash^{\mathrm{LF}} \sigma : \Gamma'}{\Delta; \Gamma \Vdash^{\mathrm{LF}} \lfloor C \rfloor_{\sigma} : [\sigma]A}$$

$$\frac{\Delta; \Gamma \Vdash^{\mathrm{LF}} x_i : A \quad (\Gamma)^{-} = \hat{\Gamma}}{\Delta \Vdash^{\mathcal{M}} \lceil \hat{\Gamma} \vdash x_i \rceil : \#\lceil \Gamma \vdash A \rceil} \qquad \frac{\Delta \Vdash^{\mathcal{M}} C : \#\lceil \Gamma' \vdash A \rceil \quad \Delta; \Gamma \Vdash^{\mathrm{LF}} \sigma : \Gamma'}{\Delta; \Gamma \Vdash^{\mathrm{LF}} \lfloor C \rfloor_{\sigma} : [\sigma]A}$$

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- Context variables an contexts as meta-terms:

$$\frac{\Delta \Vdash^{\mathcal{M}} X_i : \mathsf{ctx}}{\Delta \Vdash^{\text{LF}} X_i \ \mathsf{ctx}} \qquad \frac{\Delta \Vdash^{\text{LF}} \Gamma \ \mathsf{ctx}}{\Delta \Vdash^{\mathcal{M}} \Gamma : \mathsf{ctx}}$$

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- Identity substitutions and conversion for substitutions:

$$\frac{\Delta \Vdash^{\mathrm{LF}} \Gamma\ \mathsf{ctx} \quad (\Gamma)^- = \hat{\Gamma}}{\Delta; \Gamma \Vdash^{\mathrm{LF}} \mathsf{id}_{\hat{\Gamma}} : \Gamma} \qquad \frac{\Delta; \Gamma \Vdash^{\mathrm{LF}} \sigma : \Gamma'' \quad \Delta; \Gamma \Vdash^{\mathrm{LF}} \Gamma' \equiv \Gamma''\ \mathsf{ctx}}{\Delta; \Gamma \Vdash^{\mathrm{LF}} \sigma : \Gamma'}$$

Equational Theory

Contexts

- Equivalence of contexts is non-trivial with explicit meta-substitutions:

$$\frac{\Delta \Vdash^{\mathcal{M}} \theta : \Delta' \quad \Delta \Vdash^{\mathcal{M}} [\![\theta]\!]X_i \equiv \Gamma : \mathsf{ctx}}{\Delta \Vdash^{\mathrm{LF}} [\![\theta]\!]X_i \equiv \Gamma\ \mathsf{ctx}}$$

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- Also need equivalence for erased contexts:

$$\frac{\Delta \Vdash^{\mathcal{M}} \theta : \Delta' \quad \Delta \Vdash^{\mathcal{M}} [\![\theta]\!]X_i \equiv \Gamma : \mathsf{ctx} \quad (\Gamma)^{-} = (\cdot, m)}{\Delta \Vdash^{\mathrm{LF}} [\![\theta]\!](X_i, n) \equiv (\cdot, n + m)\ \widehat{\mathsf{ctx}}}$$

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Solution: redefine erasure of $[\![\theta]\!]\Gamma$:

$$([\![\theta]\!]\Gamma)^- = \begin{cases} (\Gamma)^- & \text{if } (\Gamma)^- = \cdot, n \\ \cdot, n+m & \text{if } (\Gamma)^- = X_i, n \text{ and } \theta(i) = \Gamma' \text{ and } (\Gamma')^- = \cdot, m \\ X_j, n+m & \text{if } (\Gamma)^- = X_i, n \text{ and } \theta(i) = \Gamma' \text{ and } (\Gamma')^- = X_j, m \end{cases}$$

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Identity substitutions

- Identities are definable except for context variables. Intuitively:

$$\mathsf{id}_{\hat{\Gamma}} = x_{|\Gamma|}, x_{|\Gamma|-1}, ..., x_1$$

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- Expanding identity substitutions formally:

$$\frac{\Vdash^{\mathcal{M}} \Delta \ \mathsf{mctx}}{\Delta; \cdot \Vdash^{\mathrm{LF}} \mathsf{id}_{(\cdot,0)} \equiv \cdot : \cdot} \qquad \frac{\Delta \Vdash^{\mathrm{LF}} \Gamma \ \mathsf{ctx} \quad (\Gamma)^{-} = \cdot, n+1}{\Delta; \Gamma \Vdash^{\mathrm{LF}} \mathsf{id}_{(\cdot,n+1)} \equiv ([\uparrow]\mathsf{id}_{(\cdot,n)}), x_1 : \Gamma}$$

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- Propagation of meta-substitutions:

$$\frac{\Delta \Vdash^{\mathcal{M}} \theta : \Delta' \quad \Delta' \Vdash^{\mathrm{LF}} \Gamma\ \mathsf{ctx} \quad (\Gamma)^{-} = \hat{\Gamma}}{\Delta; [\![\theta]\!]\Gamma \Vdash^{\mathrm{LF}} [\![\theta]\!]\mathsf{id}_{\hat{\Gamma}} \equiv \mathsf{id}_{[\![\theta]\!]\hat{\Gamma}} : [\![\theta]\!]\Gamma} \qquad \frac{\Delta \Vdash^{\mathrm{LF}} \hat{\Gamma} \equiv \hat{\Gamma}'\ \mathsf{ctx} \quad (\Gamma)^{-} = \hat{\Gamma}}{\Delta; \Gamma \Vdash^{\mathrm{LF}} \mathsf{id}_{\hat{\Gamma}} \equiv \mathsf{id}_{\hat{\Gamma}'} : \Gamma}$$

Properties of syntactic judgments

Lemma (Correctness of erasure)

1. *If* $\Delta \Vdash^{\mathrm{LF}} \Gamma$ ctx*, then* $(\Gamma)^{-}$ *terminates without failing.*
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Theorem (Subderivation)

1. *If* $\mathcal{D} ::\ \Delta \Vdash^{\mathrm{LF}} \Gamma\ \mathsf{ctx}$, *then there is* $\mathcal{D}' :: \Vdash^{\mathcal{M}} \Delta\ \mathsf{mctx}$ *such that* $|\mathcal{D}'| \leq |\mathcal{D}|$
2. *If* $\mathcal{D} ::\ \Delta \Vdash^{\mathrm{LF}} \Gamma \equiv \Gamma'\ \mathsf{ctx}$, *then there are* $\mathcal{D}_1 :: \Vdash^{\mathcal{M}} \Delta\ \mathsf{mctx}$, $\mathcal{D}_2 ::\ \Delta \Vdash^{\mathrm{LF}} \Gamma\ \mathsf{ctx}$, *and* $\mathcal{D}_3 ::\ \Delta \Vdash^{\mathrm{LF}} \Gamma'\ \mathsf{ctx}$ *such that* $|\mathcal{D}_1|, |\mathcal{D}_2|, |\mathcal{D}_3| \leq |\mathcal{D}|$.
3. *If* $\mathcal{D} ::\ \Delta; \Gamma \Vdash^{\mathrm{LF}} \sigma : \Gamma'$, *then there are* $\mathcal{D}_1 ::\ \Delta \Vdash^{\mathrm{LF}} \Gamma\ \mathsf{ctx}$ *and* $\mathcal{D}_2 ::\ \Delta \Vdash^{\mathrm{LF}} \Gamma'\ \mathsf{ctx}$ *such that* $|\mathcal{D}_1|, |\mathcal{D}_2| \leq |\mathcal{D}|$.
4. *If* $\mathcal{D} ::\ \Delta; \Gamma \Vdash^{\mathrm{LF}} \sigma \equiv \sigma' : \Gamma'$, *then there are* $\mathcal{D}_1 ::\ \Delta \Vdash^{\mathrm{LF}} \Gamma : \mathsf{ctx}$, $\mathcal{D}_2 ::\ \Delta \Vdash^{\mathrm{LF}} \Gamma'\ \mathsf{ctx}$, $\mathcal{D}_3 ::\ \Delta; \Gamma \Vdash^{\mathrm{LF}} \sigma : \Gamma'$, *and* $\mathcal{D}_4 ::\ \Delta \Vdash^{\mathrm{LF}} \sigma' : \Gamma'$ *such that* $|\mathcal{D}_1|, |\mathcal{D}_2|, |\mathcal{D}_3|, |\mathcal{D}_4| \leq |\mathcal{D}|$.

And similarly for other judgments.

Normalization by Evaluation

Overview

Part 1: Evaluation procedure

1. Define semantics as an untyped domain.
   All objects in the domain are in canonical(-ish) form.
   - Neutral and normal forms ensure no β-reduction.
   - Reflection and reification ensure no η-expansion.

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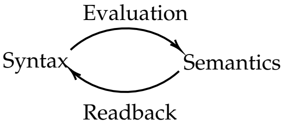

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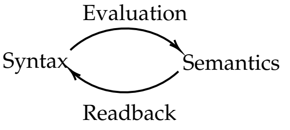

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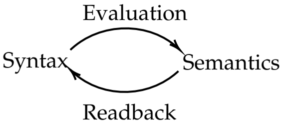

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5. Prove that syntactically equal objects evaluate to PER-related domain objects.
6. Prove that PER-related domain objects readback to the same thing.
7. Conclude syntactically equal objects have the same normal form (Completeness).

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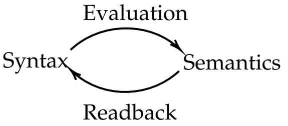

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8. Prove that well-formed objects are syntactically equal to their normal form (Soundness).

Sorts	s	$::=$	$\mathsf{type} \mid \mathsf{kind}$	
Constants	$\mathbf{c}, \mathbf{a}$	$::=$	$\mathbf{tp} \mid \mathbf{base} \mid \mathbf{arr} \mid \mathbf{tm} \mid \mathbf{lam} \mid \mathbf{app}$	
de Bruijn levels	ℓ	$::=$	$n \mid n + o$	where $n \geq 1$
Neutral expressions	e, f	$::=$	$v_\ell \mid e\ d \mid \mathsf{unbox}\ h\ \mathsf{with}\ \rho$	
Normal expressions	m, a, k	$::=$	$\uparrow^a e \mid s \mid (\Lambda M)\tau; \rho \mid (\Pi a.A)\tau; \rho \mid \mathbf{c} \mid \mathbf{arr}\ a \mid \mathbf{arr}\ a\ b$	
		$\mid$	$\mathbf{tm}\ a \mid \mathbf{lam}\ a \mid \mathbf{lam}\ a\ b \mid \mathbf{lam}\ a\ b\ m$	
		$\mid$	$\mathbf{app}\ a \mid \mathbf{app}\ a\ b \mid \mathbf{app}\ a\ b\ m \mid \mathbf{app}\ a\ b\ m\ n$	
Canonical expressions	d	$::=$	$\downarrow^a m$	
Contexts	γ	$::=$	$\cdot \mid V_i \mid \gamma, a$	
Erased contexts	$\hat{\gamma}$	$=$	$\cdot, n \mid V_i, n$	
Environment	ρ	$::=$	$\cdot \mid \mathsf{id}_{V_i} \mid \rho, m$	
Meta-types	u	$::=$	$\#[\![(\gamma \vdash A)\tau]\!] \mid [\![(\gamma \vdash A)\tau]\!] \mid \mathsf{ctx}$	
Meta-neutrals	h	$::=$	V_i	
Meta-normals	c	$::=$	$\uparrow^u h \mid \mathsf{box}(\hat{\gamma} \vdash M)\tau \mid \gamma$	
Meta-canonicals	d	$::=$	$\downarrow^u c$	
Meta-environments	τ	$::=$	$\cdot \mid \tau, c$	

Normalization by Evaluation

Evaluation

Box and unbox

- $\boxed{[\![C]\!]_{\mathcal{M}}(\tau) \searrow c}$ – Meta-term C evaluates to c in meta-environment τ.
- $\boxed{[\![U]\!]^{t}_{\mathcal{M}}(\tau) \searrow u}$ – Meta-type U evaluates to u in meta-environment τ.

$$\frac{[\![\hat{\Gamma}]\!]^{\hat{c}}_{\mathrm{LF}}(\tau) \searrow \hat{\gamma}}{[\![\lceil \hat{\Gamma} \vdash M \rceil]\!]_{\mathcal{M}}(\tau) \searrow \mathsf{box}(\hat{\gamma} \vdash M)\tau} \qquad \frac{[\![\Gamma]\!]^{c}_{\mathrm{LF}}(\tau) \searrow \gamma}{[\![\lceil \Gamma \vdash A \rceil]\!]^{t}_{\mathcal{M}}(\tau) \searrow [\![(\gamma \vdash A)\tau]\!]}$$

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- $\boxed{[\![E]\!]_{\mathrm{LF}}(\tau;\rho) \searrow m}$ – LF expression M evaluates to m in environment $\tau;\rho$
- $\boxed{\mathsf{unbox} \cdot c \ \mathsf{with} \ \rho \searrow m}$ – Unboxing c with environment ρ yields m.

$$\frac{[\![C]\!]_{\mathcal{M}}(\tau) \searrow c \quad [\![\sigma]\!]^{s}_{\mathrm{LF}}(\tau;\rho) \searrow \rho' \quad \mathsf{unbox} \cdot c \ \mathsf{with} \ \rho' \searrow m}{[\![\lfloor C \rfloor_{\sigma}]\!]_{\mathrm{LF}}(\tau;\rho) \searrow m}$$

$$\frac{[\![M]\!]_{\mathrm{LF}}(\tau;\rho) \searrow m}{\mathsf{unbox} \cdot \mathsf{box}(\hat{\gamma} \vdash M)\tau \ \mathsf{with} \ \rho \searrow m} \qquad \frac{[\![A]\!]_{\mathrm{LF}}(\tau;\rho) \searrow a}{\mathsf{unbox} \cdot (\uparrow^{[\![(\gamma \vdash A)\tau]\!]} h) \ \mathsf{with} \ \rho \searrow \uparrow^{a} (\mathsf{unbox} \ h \ \mathsf{with} \ \rho)}$$

Contexts

- $\boxed{[\![\Gamma]\!]^c_{\mathrm{LF}}(\tau) \searrow \gamma}$ – LF context Γ evaluates to γ in meta-environment τ.

$$\frac{}{[\![\cdot]\!]^c_{\mathrm{LF}}(\tau) \searrow \cdot} \qquad \frac{}{[\![X_i]\!]^c_{\mathrm{LF}}(\tau) \searrow \tau(i)} \qquad \frac{[\![\theta]\!]^s_{\mathcal{M}}(\tau) \searrow \tau' \quad [\![\Gamma]\!]^c_{\mathrm{LF}}(\tau') \searrow \gamma}{[\![[\![\theta]\!]\Gamma]\!]^c_{\mathrm{LF}}(\tau) \searrow \gamma}$$

Normalization by Evaluation

Evaluation

Contexts

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$$\frac{[\![\Gamma]\!]^c_{\mathrm{LF}}(\tau) \searrow \gamma \quad [\![A]\!]_{\mathrm{LF}}(\tau; ?) \searrow a}{[\![\Gamma, A]\!]^c_{\mathrm{LF}}(\tau) \searrow \gamma, a}$$

Contexts

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$$\frac{}{[\![\cdot]\!]^c_{\mathrm{LF}}(\tau) \searrow \cdot} \qquad \frac{}{[\![X_i]\!]^c_{\mathrm{LF}}(\tau) \searrow \tau(i)} \qquad \frac{[\![\theta]\!]^s_{\mathcal{M}}(\tau) \searrow \tau' \quad [\![\Gamma]\!]^c_{\mathrm{LF}}(\tau') \searrow \gamma}{[\![[\![\theta]\!]\Gamma]\!]^c_{\mathrm{LF}}(\tau) \searrow \gamma}$$

$$\frac{[\![\Gamma]\!]^c_{\mathrm{LF}}(\tau) \searrow \gamma \quad [\![A]\!]_{\mathrm{LF}}(\tau; \rho^*_\gamma) \searrow a}{[\![\Gamma, A]\!]^c_{\mathrm{LF}}(\tau) \searrow \gamma, a}$$

- Initital LF environment for γ, denoted ρ^*_γ:

$$\begin{array}{lcl} \rho^*_{\cdot} & = & \cdot \\ \rho^*_{V_i} & = & \mathsf{id}_{V_i} \\ \rho^*_{\gamma,a} & = & \rho^*_\gamma, \uparrow^a v_{|\gamma|+1} \end{array} \qquad \begin{array}{lcl} |\cdot| & = & 0 \\ |V_i| & = & o \\ |\gamma, a| & = & |\gamma| + 1 \end{array}$$

Contexts

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$$\frac{[\![\Gamma]\!]^c_{\mathrm{LF}}(\tau) \searrow \gamma \quad [\![A]\!]_{\mathrm{LF}}(\tau; \rho^*_\gamma) \searrow a}{[\![\Gamma, A]\!]^c_{\mathrm{LF}}(\tau) \searrow \gamma, a}$$

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For syntactic contexts: $\rho^*_{\Delta;\Gamma} := \rho^*_\gamma$, where $[\![\Gamma]\!]^c_{\mathrm{LF}}(\tau^*_\Delta) \searrow \gamma$.

Erased contexts

- $\boxed{[\![\hat{\Gamma}]\!]^{\hat{c}}_{\mathrm{LF}}(\tau) \searrow \hat{\gamma}}$ – Erased context $\hat{\Gamma}$ evaluates to $\hat{\gamma}$ in meta-environment τ.

$$\frac{}{[\![\cdot, n]\!]^{\hat{c}}_{\mathrm{LF}}(\tau) \searrow \cdot, n} \qquad \frac{(\tau(i))^{-} = \cdot, m}{[\![\mathrm{X}_i, n]\!]^{\hat{c}}_{\mathrm{LF}}(\tau) \searrow \cdot, m + n} \qquad \frac{(\tau(i))^{-} = V_j, m}{[\![\mathrm{X}_i, n]\!]^{\hat{c}}_{\mathrm{LF}}(\tau) \searrow V_j, m + n}$$

Normalization by Evaluation

Evaluation

Erased contexts

- $\boxed{[\![\hat{\Gamma}]\!]^{\hat{c}}_{\mathrm{LF}}(\tau) \searrow \hat{\gamma}}$ – Erased context $\hat{\Gamma}$ evaluates to $\hat{\gamma}$ in meta-environment τ.

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LF substitutions

- $\boxed{[\![\sigma]\!]^{s}_{\mathrm{LF}}(\tau;\rho) \searrow \rho'}$ – LF substitution σ evaluates to ρ' in environments $\tau;\rho$.

$$\frac{}{[\![\cdot]\!]^{s}_{\mathrm{LF}}(\tau;\rho) \searrow \cdot} \qquad \frac{}{[\![\mathsf{id}_{\hat{\Gamma}}]\!]^{s}_{\mathrm{LF}}(\tau;\rho) \searrow \rho}$$

Note. Domain identity environments id_{V_i} only occur in initial environments.

Normalization by Evaluation

Readback

Box and unbox

- $\boxed{\mathsf{R}_k^{\mathsf{D}^{\mathsf{nf}}_{\mathcal{M}}}(c) \searrow C}$ – Domain canonical meta-term c readbacks to C.

$$\frac{\mathsf{R}_k^{\mathsf{Ctx}_{\mathrm{LF}}}(\gamma) \searrow \Gamma \quad (\Gamma)^- = \hat{\Gamma} \quad [\![A]\!]_{\mathrm{LF}}(\tau; \rho^*_\gamma) \searrow a \quad \textsf{unbox} \cdot c \textsf{ with } \rho^*_\gamma \searrow m \quad \mathsf{R}_{k,|\gamma|}^{\mathsf{D}^{\mathsf{nf}}_{\mathrm{LF}}}(\downarrow^a m) \searrow M}{\mathsf{R}_k^{\mathsf{D}^{\mathsf{nf}}_{\mathcal{M}}}(\downarrow^{[\![(\gamma \vdash A)\tau]\!]} c) \searrow \lceil \hat{\Gamma} \vdash M \rceil}$$

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- $\boxed{\mathsf{R}_{k,l}^{\mathsf{D}^{\mathsf{ne}}_{\mathrm{LF}}}(e) \searrow E}$ – Domain neutral expression e readbacks to E.

$$\frac{\mathsf{R}_k^{\mathsf{D}^{\mathsf{ne}}_{\mathcal{M}}}(h) \searrow E \quad \mathsf{R}_{k,l}^{\mathsf{Env}_{\mathrm{LF}}}(\rho) \searrow \sigma}{\mathsf{R}_{k,l}^{\mathsf{D}^{\mathsf{ne}}_{\mathrm{LF}}}(\mathsf{unbox}\ h\ \mathsf{with}\ \rho) \searrow \lfloor E \rfloor_\sigma}$$

Contexts and erased contexts

- $\boxed{\mathsf{R}^{\mathsf{Ctx}_{\mathrm{LF}}}_{k}(\gamma) \searrow \Gamma}$ – Normal LF domain context γ readbacks to Γ.

$$\frac{}{\mathsf{R}^{\mathsf{Ctx}_{\mathrm{LF}}}_{k}(\cdot) \searrow \cdot} \qquad \frac{}{\mathsf{R}^{\mathsf{Ctx}_{\mathrm{LF}}}_{k}(V_i) \searrow X_{k-i+1}} \qquad \frac{\mathsf{R}^{\mathsf{Ctx}_{\mathrm{LF}}}_{k}(\gamma) \searrow \Gamma \quad \mathsf{R}^{\mathsf{D}^{\mathsf{nf}}_{\mathrm{LF}}}_{k,|\gamma|+1}(a) \searrow A}{\mathsf{R}^{\mathsf{Ctx}_{\mathrm{LF}}}_{k}(\gamma, a) \searrow \Gamma, A}$$

Normalization by Evaluation
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- $\boxed{\mathsf{R}_k^{\widehat{\mathsf{Ctx}}_{\mathrm{LF}}}(\hat{\gamma}) \searrow \hat{\Gamma}}$ – Erased context $\hat{\gamma}$ readbacks to $\hat{\Gamma}$.

$$\frac{}{\mathsf{R}_k^{\widehat{\mathsf{Ctx}}_{\mathrm{LF}}}(\cdot, n) \searrow \cdot, n} \qquad \frac{}{\mathsf{R}_k^{\widehat{\mathsf{Ctx}}_{\mathrm{LF}}}(V_i, n) \searrow X_{k-i+1}, n}$$

Contents and erased contexts

- $\boxed{\mathsf{R}^{\mathsf{Ctx}_{\mathrm{LF}}}_{k}(\gamma) \searrow \Gamma}$ – Normal LF domain context γ readbacks to Γ.

$$\frac{}{\mathsf{R}^{\mathsf{Ctx}_{\mathrm{LF}}}_{k}(\cdot) \searrow \cdot} \qquad \frac{}{\mathsf{R}^{\mathsf{Ctx}_{\mathrm{LF}}}_{k}(V_i) \searrow X_{k-i+1}} \qquad \frac{\mathsf{R}^{\mathsf{Ctx}_{\mathrm{LF}}}_{k}(\gamma) \searrow \Gamma \quad \mathsf{R}^{\mathsf{D}^{\mathsf{nf}}_{\mathrm{LF}}}_{k,|\gamma|+1}(a) \searrow A}{\mathsf{R}^{\mathsf{Ctx}_{\mathrm{LF}}}_{k}(\gamma, a) \searrow \Gamma, A}$$

- $\boxed{\mathsf{R}^{\widehat{\mathsf{Ctx}}_{\mathrm{LF}}}_{k}(\hat{\gamma}) \searrow \hat{\Gamma}}$ – Erased context $\hat{\gamma}$ readbacks to $\hat{\Gamma}$.

$$\frac{}{\mathsf{R}^{\widehat{\mathsf{Ctx}}_{\mathrm{LF}}}_{k}(\cdot, n) \searrow \cdot, n} \qquad \frac{}{\mathsf{R}^{\widehat{\mathsf{Ctx}}_{\mathrm{LF}}}_{k}(V_i, n) \searrow X_{k-i+1}, n}$$

Environments

- $\boxed{\mathsf{R}^{\mathsf{Env}_{\mathrm{LF}}}_{k,l}(\rho) \searrow \sigma}$ – Environment ρ readbacks to substitution σ.

$$\frac{}{\mathsf{R}^{\mathsf{Env}_{\mathrm{LF}}}_{k,l}(\mathsf{id}_{V_i}) \searrow \mathsf{id}_{(X_{k-i+1},0)}}$$

Normalization by Evaluation

Properties of evaluation and readback

Theorem (Determinacy)

All the evaluation and readback relations are deterministic in their last parameter. Precisely:

1. *If* $[\![M]\!]_{\mathrm{LF}}(\tau;\rho) \searrow m$ *and* $[\![M]\!]_{\mathrm{LF}}(\tau;\rho) \searrow m'$*, then* $m = m'$.
2. *If* $[\![\Gamma]\!]^{c}_{\mathrm{LF}}(\tau) \searrow \gamma$ *and* $[\![\Gamma]\!]^{c}_{\mathrm{LF}}(\tau) \searrow \gamma'$*, then* $\gamma = \gamma'$.

   $\vdots$

8. *If* $\mathsf{R}^{\mathsf{D}^{\mathsf{nf}}_{\mathrm{LF}}}_{k,l}(m) \searrow M$ *and* $\mathsf{R}^{\mathsf{D}^{\mathsf{nf}}_{\mathrm{LF}}}_{k,l}(m) \searrow M'$*, then* $M = M'$.
9. *If* $\mathsf{R}^{\mathsf{Ctx}_{\mathrm{LF}}}_{k}(\gamma) \searrow \Gamma$ *and* $\mathsf{R}^{\mathsf{Ctx}_{\mathrm{LF}}}_{k}(\gamma) \searrow \Gamma'$*, then* $\Gamma = \Gamma'$.

   $\vdots$

In other words, evaluation and readback are partial functions.

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   ⋮

8. *If* $\mathsf{R}^{\mathsf{D}^{\mathsf{nf}}_{\mathrm{LF}}}_{k,l}(m) \searrow M$ *and* $\mathsf{R}^{\mathsf{D}^{\mathsf{nf}}_{\mathrm{LF}}}_{k,l}(m) \searrow M'$*, then* $M = M'$*.*
9. *If* $\mathsf{R}^{\mathsf{Ctx}_{\mathrm{LF}}}_{k}(\gamma) \searrow \Gamma$ *and* $\mathsf{R}^{\mathsf{Ctx}_{\mathrm{LF}}}_{k}(\gamma) \searrow \Gamma'$*, then* $\Gamma = \Gamma'$*.*

   ⋮

In other words, evaluation and readback are partial functions.
⇒ We can now define normalization (partial) functions:

$$\mathsf{nbe}^{A}_{\Delta;\Gamma}(M) := \mathsf{R}^{\mathsf{D}^{\mathsf{nf}}_{\mathrm{LF}}}_{k,l}\left(\downarrow^{[\![A]\!]_{\mathrm{LF}}(\tau^*_\Delta;\rho^*_{\Delta;\Gamma})} [\![M]\!]_{\mathrm{LF}}(\tau^*_\Delta;\rho^*_{\Delta;\Gamma})\right)$$

Conclusion

Recap

- Extended CLF with first-class contexts and context variables.
- Extended Abel and Pientka [2010]'s explicit substitution calculus to our version of CLF.

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Next time

- Define PER model.
- Overview of completeness proof.

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- Prove soundness of normalization.

CONCLUSION

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- Defined semantic domain and normalization procedure for CLF.

Next time

- Define PER model.
- Overview of completeness proof.

Future work

- Prove soundness of normalization.
- Scale up from CLF to COCON by extending the meta-level to MLTT.
  - Universe hierarchy
  - Dependent function
  - Recursion over LF objects
- Add first-class substitutions and substitution variables.

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