Lecture 10: Sequential Control

Recap so far

- Bandits: all about control! Ie finding the optimal policy
  - But in a one-step decision problem
- Policy evaluation: finding the value function corresponding to a *given policy*
  - Several algorithms: DP, MC, TD and friends
- But what about control in the sequential case? In an MDP?

Recall: Value functions

- The value of a state, given a policy:

$$v_\pi(s) = \mathbb{E}\{G_t \mid S_t = s, A_{t:\infty} \sim \pi\} \qquad v_\pi : \mathcal{S} \to \Re$$

- The value of a state-action pair, given a policy:

$$q_\pi(s,a) = \mathbb{E}\{G_t \mid S_t = s, A_t = a, A_{t+1:\infty} \sim \pi\} \qquad q_\pi : \mathcal{S} \times \mathcal{A} \to \Re$$

- The optimal value of a state:

$$v_*(s) = \max_\pi v_\pi(s) \qquad v_* : \mathcal{S} \to \Re$$

- The optimal value of a state-action pair:

$$q_*(s,a) = \max_\pi q_\pi(s,a) \qquad q_* : \mathcal{S} \times \mathcal{A} \to \Re$$

- Optimal policy: π_* is an optimal policy if and only if

$$\pi_*(a|s) > 0 \text{ only where } q_*(s,a) = \max_b q_*(s,b) \qquad \forall s \in \mathcal{S}$$

  - in other words, π_* is optimal iff it is *greedy* wrt q_*

Bellman Optimality Eqn

$$v_\pi(s) = \sum_a \pi(a|s) \overbrace{\sum_{s',r} p(s',r|s,a)\Big[r + \gamma v_\pi(s')\Big]}^{q_\pi(s,a)}$$

Bellman Optimality Eqn

$$v_\pi(s) = \sum_a \pi(a|s) \overbrace{\sum_{s',r} p(s',r|s,a)\Big[r + \gamma v_\pi(s')\Big]}^{q_\pi(s,a)}$$

$$v_*(s) = \max_{a \in \mathcal{A}(s)} q_{\pi_*}(s,a)$$

Bellman Optimality Eqn

$$v_\pi(s) = \sum_a \pi(a|s) \overbrace{\sum_{s',r} p(s', r|s, a)\Big[r + \gamma v_\pi(s')\Big]}^{q_\pi(s,a)}$$

$$\begin{aligned} v_*(s) &= \max_{a \in \mathcal{A}(s)} q_{\pi_*}(s, a) \\ &= \max_a \mathbb{E}_{\pi_*}[G_t \mid S_t = s, A_t = a] \end{aligned}$$

Bellman Optimality Eqn

$$v_\pi(s) = \sum_a \pi(a|s) \overbrace{\sum_{s',r} p(s', r|s, a)\Big[r + \gamma v_\pi(s')\Big]}^{q_\pi(s,a)}$$

$$\begin{aligned} v_*(s) &= \max_{a \in \mathcal{A}(s)} q_{\pi_*}(s, a) \\ &= \max_a \mathbb{E}_{\pi_*}[G_t \mid S_t = s, A_t = a] \\ &= \max_a \mathbb{E}_{\pi_*}[R_{t+1} + \gamma G_{t+1} \mid S_t = s, A_t = a] \end{aligned}$$

Bellman Optimality Eqn

$$v_\pi(s) = \sum_a \pi(a|s) \overbrace{\sum_{s',r} p(s',r|s,a)\Big[r + \gamma v_\pi(s')\Big]}^{q_\pi(s,a)}$$

$$\begin{aligned} v_*(s) &= \max_{a \in \mathcal{A}(s)} q_{\pi_*}(s,a) \\ &= \max_a \mathbb{E}_{\pi_*}[G_t \mid S_t = s, A_t = a] \\ &= \max_a \mathbb{E}_{\pi_*}[R_{t+1} + \gamma G_{t+1} \mid S_t = s, A_t = a] \\ &= \max_a \mathbb{E}[R_{t+1} + \gamma v_*(S_{t+1}) \mid S_t = s, A_t = a] \end{aligned}$$

Bellman Optimality Eqn

$$v_\pi(s) = \sum_a \pi(a|s) \overbrace{\sum_{s',r} p(s',r|s,a)\Big[r + \gamma v_\pi(s')\Big]}^{q_\pi(s,a)}$$

$$\begin{aligned}
v_*(s) &= \max_{a \in \mathcal{A}(s)} q_{\pi_*}(s,a) \\
&= \max_a \mathbb{E}_{\pi_*}[G_t \mid S_t = s, A_t = a] \\
&= \max_a \mathbb{E}_{\pi_*}[R_{t+1} + \gamma G_{t+1} \mid S_t = s, A_t = a] \\
&= \max_a \mathbb{E}[R_{t+1} + \gamma v_*(S_{t+1}) \mid S_t = s, A_t = a] \\
&= \max_a \sum_{s',r} p(s',r|s,a)\big[r + \gamma v_*(s')\big].
\end{aligned}$$

Bellman Optimality Eqn

$$v_\pi(s) = \sum_a \pi(a|s) \sum_{s',r} p(s', r|s, a)\Big[r + \gamma v_\pi(s')\Big]$$

$$v_*(s) = \max_a \sum_{s',r} p(s', r|s, a)\big[r + \gamma v_*(s')\big]$$

Also as many equations as unknowns (non-linear, this time though).

Value Iteration

Recall the **full policy-evaluation backup**:

$$v_{k+1}(s) = \sum_a \pi(a|s) \sum_{s',r} p(s', r|s, a)\Big[r + \gamma v_k(s')\Big] \qquad \forall s \in \mathcal{S}$$

Here is the **full value-iteration backup**:

$$v_{k+1}(s) = \max_a \sum_{s',r} p(s', r|s, a)\Big[r + \gamma v_k(s')\Big] \qquad \forall s \in \mathcal{S}$$

Value Iteration – One array version

Initialize array V arbitrarily (e.g., $V(s) = 0$ for all $s \in \mathcal{S}^+$)

Repeat
    $\Delta \leftarrow 0$
    For each $s \in \mathcal{S}$:
        $v \leftarrow V(s)$
        $V(s) \leftarrow \max_a \sum_{s',r} p(s', r|s, a)\big[r + \gamma V(s')\big]$
        $\Delta \leftarrow \max(\Delta, |v - V(s)|)$
until $\Delta < \theta$ (a small positive number)

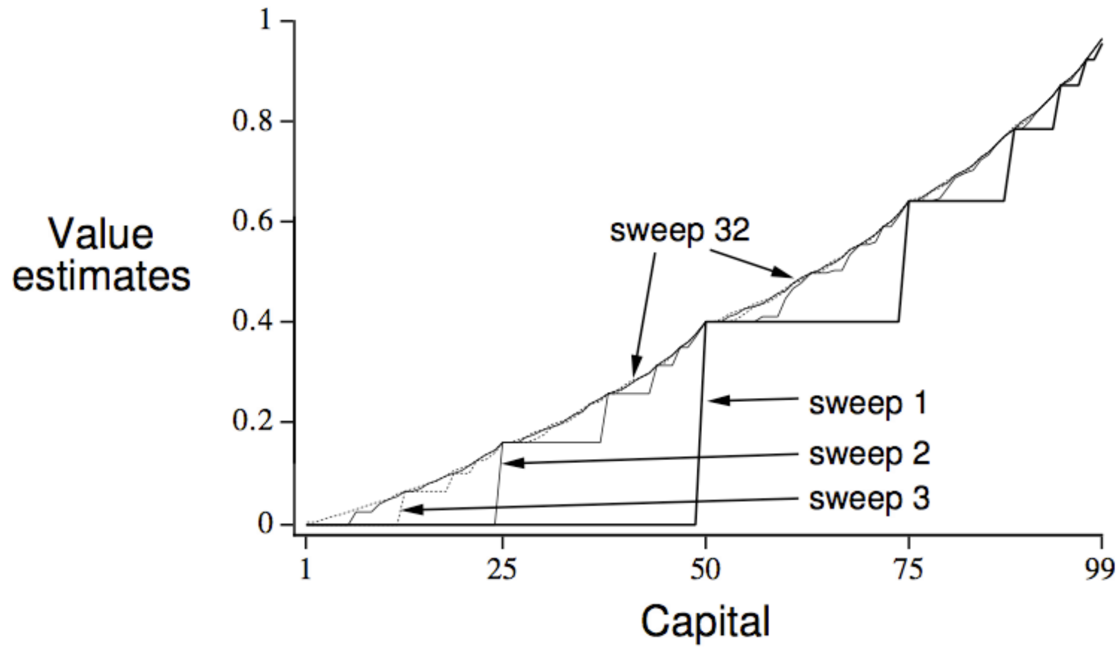
Output a deterministic policy, π, such that
    $\pi(s) = \arg\max_a \sum_{s',r} p(s', r|s, a)\big[r + \gamma V(s')\big]$

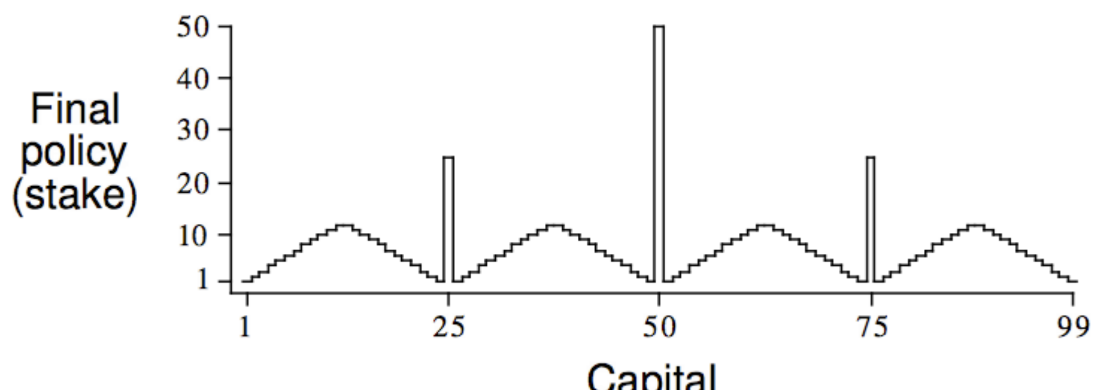
Gambler's Problem

- Gambler can repeatedly bet $ on a coin flip
- Heads he wins his stake, tails he loses it
- Initial capital $\in$ {$1, $2, … $99}
- Gambler wins if his capital becomes $100 loses if it becomes $0
- Coin is unfair
  - Heads (gambler wins) with probability $p = .4$

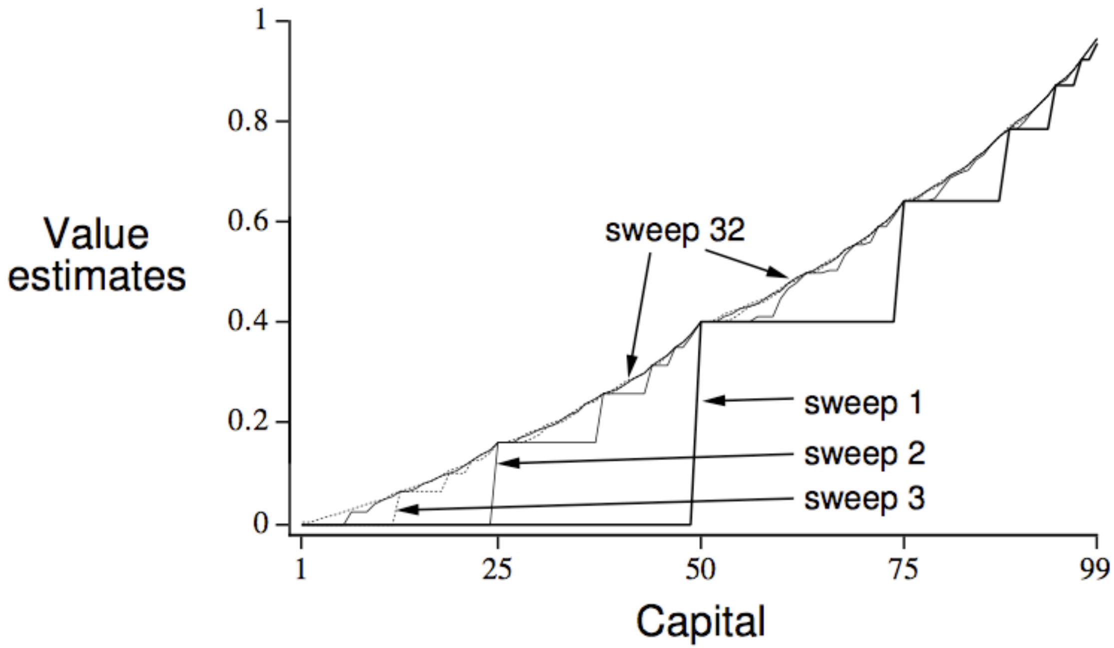
- States, Actions, Rewards? Discounting?

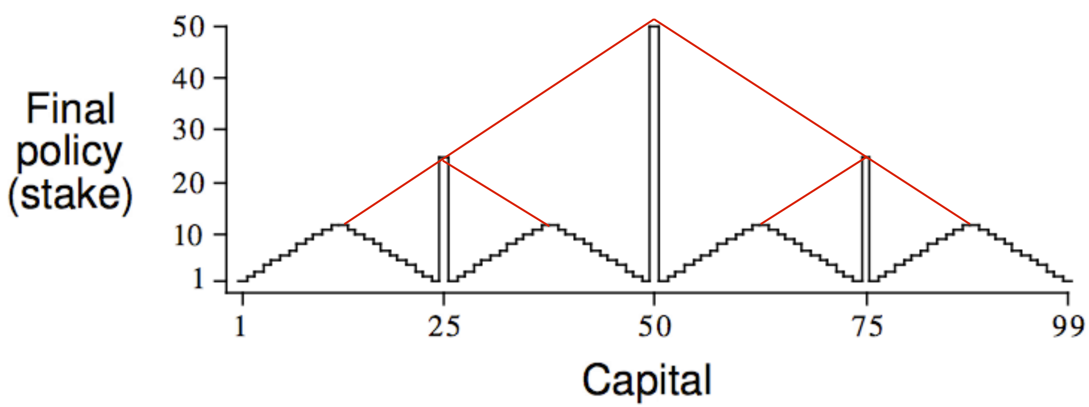
Gambler's Problem Solution





Gambler's Problem Solution





Policy Iteration

$$\pi_0 \xrightarrow{\mathrm{E}} v_{\pi_0} \xrightarrow{\mathrm{I}} \pi_1 \xrightarrow{\mathrm{E}} v_{\pi_1} \xrightarrow{\mathrm{I}} \pi_2 \xrightarrow{\mathrm{E}} \cdots \xrightarrow{\mathrm{I}} \pi_* \xrightarrow{\mathrm{E}} v_*$$

policy evaluation

policy improvement
"greedification"

Policy Improvement

Suppose we have computed v_π for a deterministic policy π.

For a given state s,
would it be better to do an action $a \neq \pi(s)$?

It is better to switch to action a for state s if

$$q_\pi(s,a) > v_\pi(s)$$

Policy Improvement Cont.

Do this for all states to get a new policy $\pi' \geq \pi$ that is **greedy** with respect to v_π :

$$
\begin{aligned}
\pi'(s) &= \arg\max_a q_\pi(s,a) \\
&= \arg\max_a \mathbb{E}[R_{t+1} + \gamma v_\pi(S_{t+1}) \mid S_t = s, A_t = a] \\
&= \arg\max_a \sum_{s',r} p(s',r|s,a)\Big[r + \gamma v_\pi(s')\Big],
\end{aligned}
$$

What if the policy is unchanged by this?

Then the policy must be optimal!

Policy Iteration

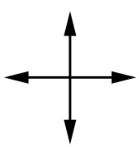
$$\pi_0 \xrightarrow{\text{E}} v_{\pi_0} \xrightarrow{\text{I}} \pi_1 \xrightarrow{\text{E}} v_{\pi_1} \xrightarrow{\text{I}} \pi_2 \xrightarrow{\text{E}} \cdots \xrightarrow{\text{I}} \pi_* \xrightarrow{\text{E}} v_*$$

policy evaluation

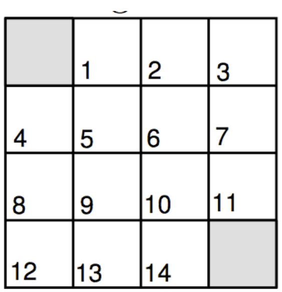
policy improvement
"greedification"

Greedy Policies for the Small Gridworld

$\pi =$ equiprobable random action choices



actions



$R = -1$
on all transitions

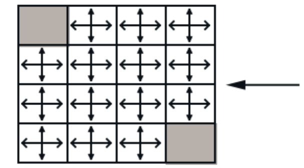
$\gamma = 1$

- An undiscounted episodic task
- Nonterminal states: 1, 2, . . ., 14;
- One terminal state (shown twice as shaded squares)
- Actions that would take agent off the grid leave state unchanged
- Reward is –1 until the terminal state is reached

V_k for the Random Policy | Greedy Policy w.r.t. V_k

$k = 0$

0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0



random policy

$k = 1$

0.0	-1.0	-1.0	-1.0
-1.0	-1.0	-1.0	-1.0
-1.0	-1.0	-1.0	-1.0
-1.0	-1.0	-1.0	0.0

$k = 2$

0.0	-1.7	-2.0	-2.0
-1.7	-2.0	-2.0	-2.0
-2.0	-2.0	-2.0	-1.7
-2.0	-2.0	-1.7	0.0

$k = 3$

0.0	-2.4	-2.9	-3.0
-2.4	-2.9	-3.0	-2.9
-2.9	-3.0	-2.9	-2.4
-3.0	-2.9	-2.4	0.0

$k = 10$

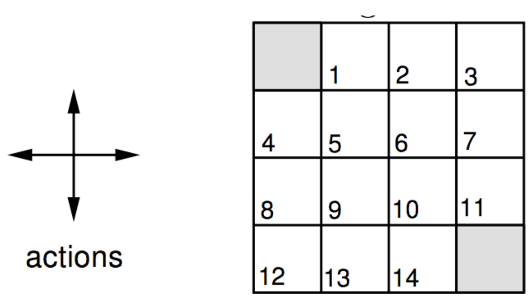
0.0	-6.1	-8.4	-9.0
-6.1	-7.7	-8.4	-8.4
-8.4	-8.4	-7.7	-6.1
-9.0	-8.4	-6.1	0.0

$k = \infty$

0.0	-14.	-20.	-22.
-14.	-18.	-20.	-20.
-20.	-20.	-18.	-14.
-22.	-20.	-14.	0.0

Greedy Policies for the Small Gridworld

π = equiprobable random action choices



$R = -1$
on all transitions

$\gamma = 1$

- An undiscounted episodic task
- Nonterminal states: 1, 2, . . ., 14;
- One terminal state (shown twice as shaded squares)
- Actions that would take agent off the grid leave state unchanged
- Reward is –1 until the terminal state is reached

V_k for the Random Policy

$k = 0$

0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0

$k = 1$

0.0	-1.0	-1.0	-1.0
-1.0	-1.0	-1.0	-1.0
-1.0	-1.0	-1.0	-1.0
-1.0	-1.0	-1.0	0.0

$k = 2$

0.0	-1.7	-2.0	-2.0
-1.7	-2.0	-2.0	-2.0
-2.0	-2.0	-2.0	-1.7
-2.0	-2.0	-1.7	0.0

$k = 3$

0.0	-2.4	-2.9	-3.0
-2.4	-2.9	-3.0	-2.9
-2.9	-3.0	-2.9	-2.4
-3.0	-2.9	-2.4	0.0

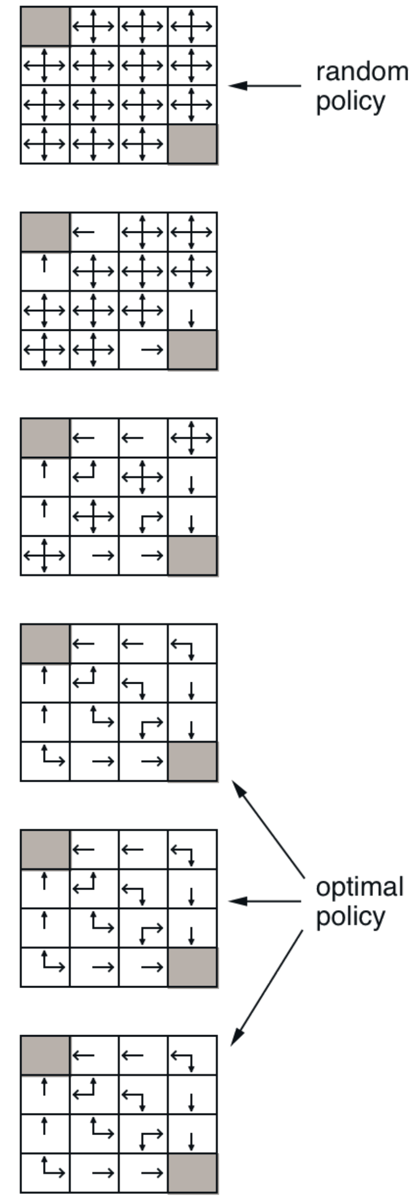
$k = 10$

0.0	-6.1	-8.4	-9.0
-6.1	-7.7	-8.4	-8.4
-8.4	-8.4	-7.7	-6.1
-9.0	-8.4	-6.1	0.0

$k = \infty$

0.0	-14.	-20.	-22.
-14.	-18.	-20.	-20.
-20.	-20.	-18.	-14.
-22.	-20.	-14.	0.0

Greedy Policy w.r.t. V_k



Policy Iteration – One array version (+ policy)

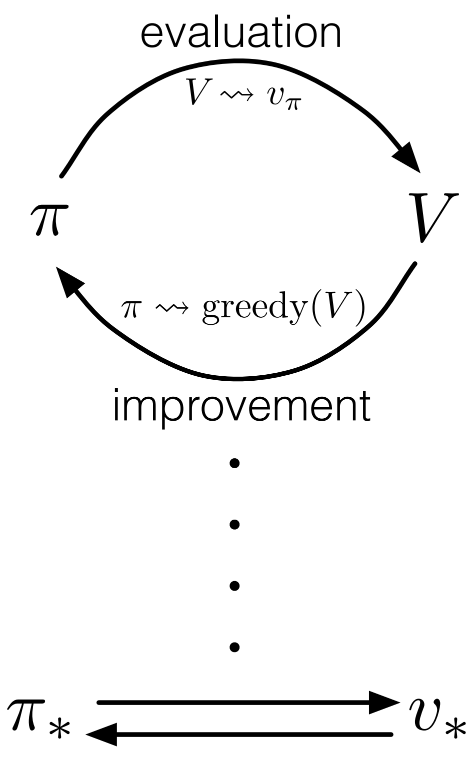
1. Initialization
   $V(s) \in \mathbb{R}$ and $\pi(s) \in \mathcal{A}(s)$ arbitrarily for all $s \in \mathcal{S}$

2. Policy Evaluation
   Repeat
   $\Delta \leftarrow 0$
   For each $s \in \mathcal{S}$:
   $v \leftarrow V(s)$
   $V(s) \leftarrow \sum_{s',r} p(s', r|s, \pi(s))\big[r + \gamma V(s')\big]$
   $\Delta \leftarrow \max(\Delta, |v - V(s)|)$
   until $\Delta < \theta$ (a small positive number)

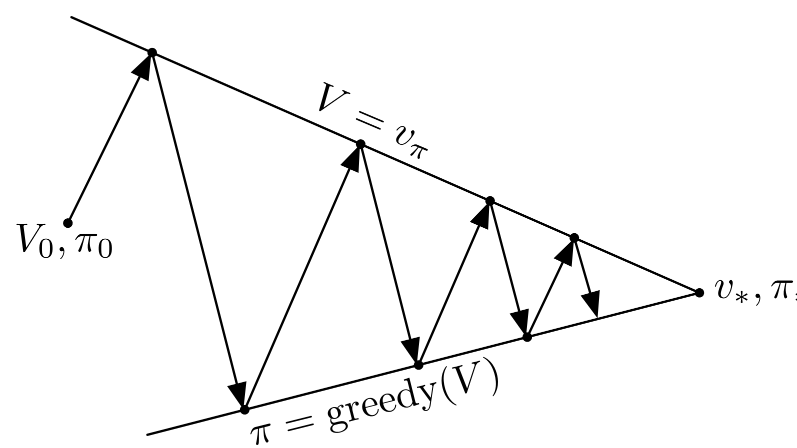
3. Policy Improvement
   policy-stable $\leftarrow$ *true*
   For each $s \in \mathcal{S}$:
   $a \leftarrow \pi(s)$
   $\pi(s) \leftarrow \arg\max_a \sum_{s',r} p(s', r|s, a)\big[r + \gamma V(s')\big]$
   If $a \neq \pi(s)$, then *policy-stable* $\leftarrow$ *false*
   If *policy-stable*, then stop and return V and π; else go to 2

Generalized Policy Iteration

Generalized Policy Iteration (GPI):
any interaction of policy evaluation and policy improvement, independent of their granularity.



A geometric metaphor for convergence of GPI:



Recall: On-policy Monte Carlo Control

- *On-policy:* learn about policy currently executing
- How do we get rid of exploring starts?
  - The policy must be eternally *soft*:
    - $\pi(a|s) > 0$ for all s and a
  - e.g. ε-soft policy:
    - probability of an action = $\frac{\epsilon}{|\mathcal{A}(s)|}$ or $1 - \epsilon + \frac{\epsilon}{|\mathcal{A}(s)|}$

      non-max max (greedy)

- Similar to GPI: move policy *towards* greedy policy (e.g., ε-greedy)
- Converges to best ε-soft policy

On-policy MC Control

Initialize, for all $s \in \mathcal{S}$, $a \in \mathcal{A}(s)$:

$Q(s, a) \leftarrow$ arbitrary

$Returns(s, a) \leftarrow$ empty list

$\pi(a|s) \leftarrow$ an arbitrary ε-soft policy

Repeat forever:

(a) Generate an episode using π

(b) For each pair s, a appearing in the episode:

$G \leftarrow$ return following the first occurrence of s, a

Append G to $Returns(s, a)$

$Q(s, a) \leftarrow \text{average}(Returns(s, a))$

(c) For each s in the episode:

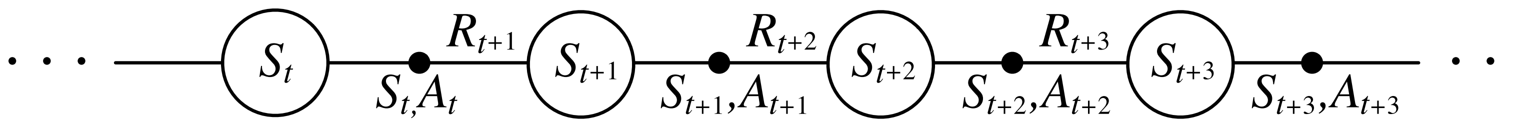
$A^* \leftarrow \arg\max_a Q(s, a)$

For all $a \in \mathcal{A}(s)$:

$$\pi(a|s) \leftarrow \begin{cases} 1 - \varepsilon + \varepsilon/|\mathcal{A}(s)| & \text{if } a = A^* \\ \varepsilon/|\mathcal{A}(s)| & \text{if } a \neq A^* \end{cases}$$

TD-Style Learning for Action-Values

Estimate q_π for the current policy π



After every transition from a nonterminal state, S_t, do this:

$$Q(S_t, A_t) \leftarrow Q(S_t, A_t) + \alpha\left[R_{t+1} + \gamma Q(S_{t+1}, A_{t+1}) - Q(S_t, A_t)\right]$$

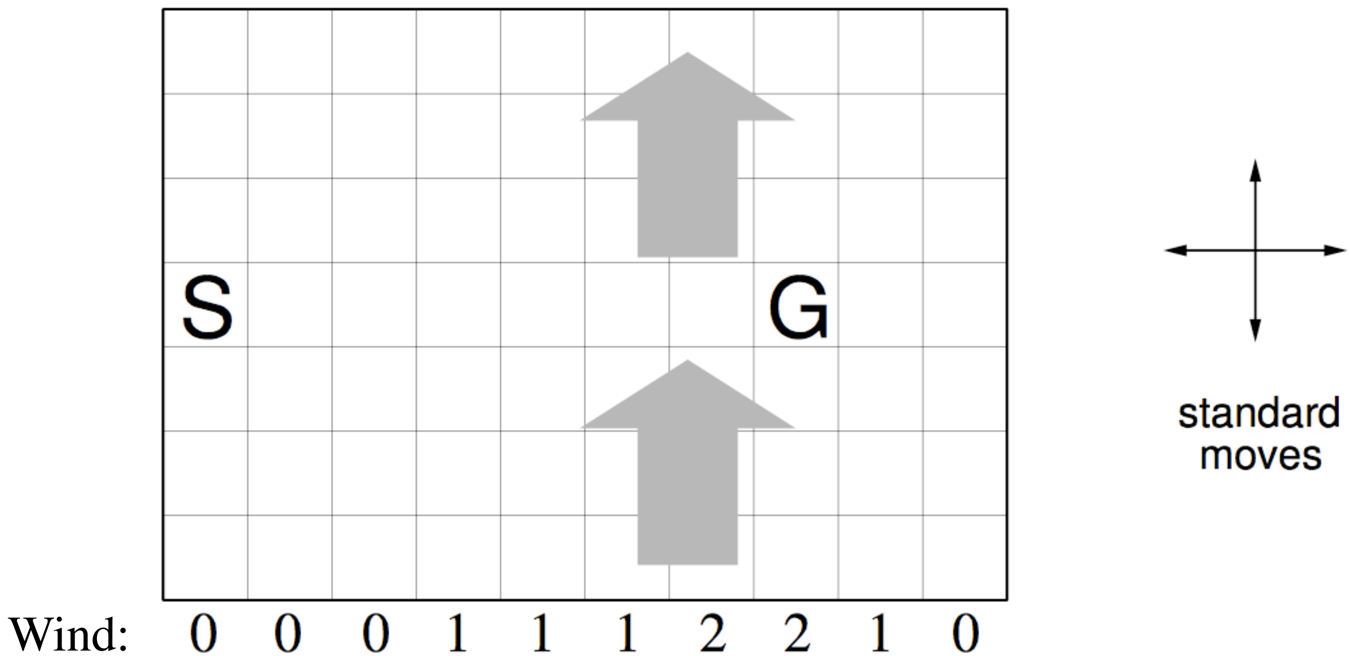
If S_{t+1} is terminal, then define $Q(S_{t+1}, A_{t+1}) = 0$

Sarsa: On-Policy TD Control

Turn this into a control method by always updating the policy to be greedy with respect to the current estimate:

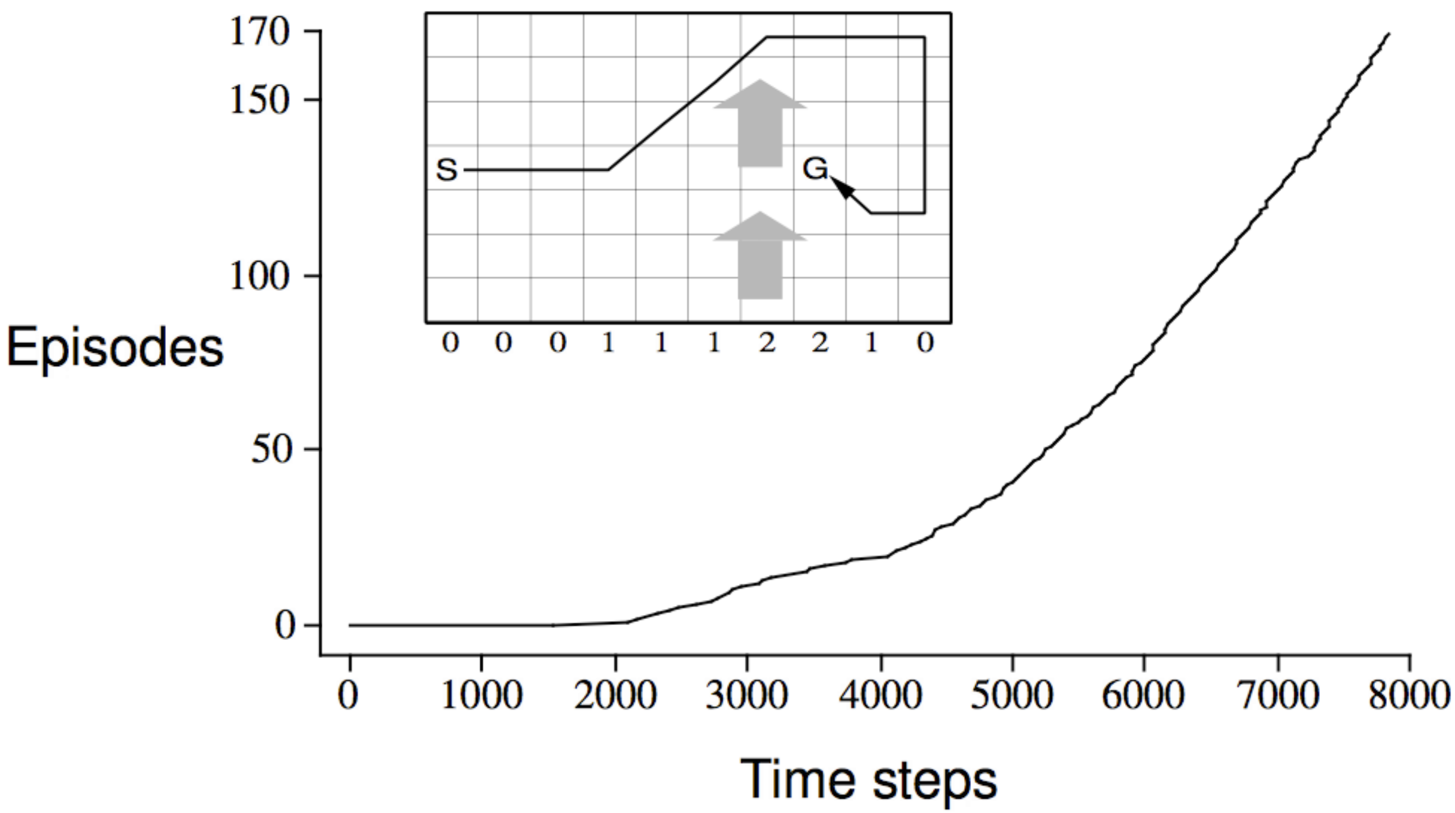
Initialize $Q(s, a), \forall s \in \mathcal{S}, a \in \mathcal{A}(s)$, arbitrarily, and $Q(\textit{terminal-state}, \cdot) = 0$
Repeat (for each episode):
    Initialize S
    Choose A from S using policy derived from Q (e.g., ε-greedy)
    Repeat (for each step of episode):
        Take action A, observe R, S'
        Choose A' from S' using policy derived from Q (e.g., ε-greedy)
        $Q(S, A) \leftarrow Q(S, A) + \alpha[R + \gamma Q(S', A') - Q(S, A)]$
        $S \leftarrow S'$; $A \leftarrow A'$;
    until S is terminal

Windy Gridworld



undiscounted, episodic, reward = –1 until goal

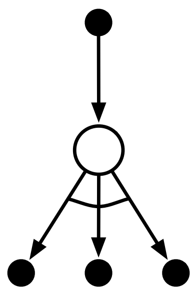
Results of Sarsa on the Windy Gridworld



Q-Learning: Off-Policy TD Control

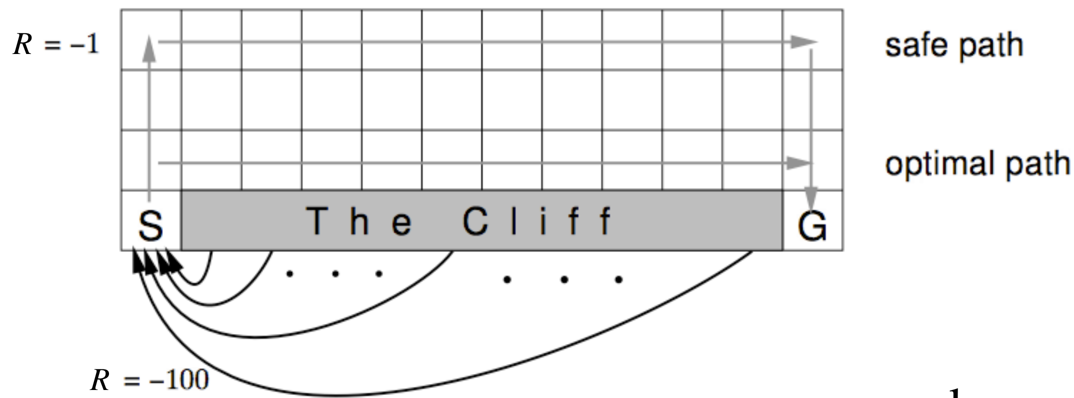
One-step Q-learning:

$$Q(S_t, A_t) \leftarrow Q(S_t, A_t) + \alpha \Big[R_{t+1} + \gamma \max_a Q(S_{t+1}, a) - Q(S_t, A_t) \Big]$$

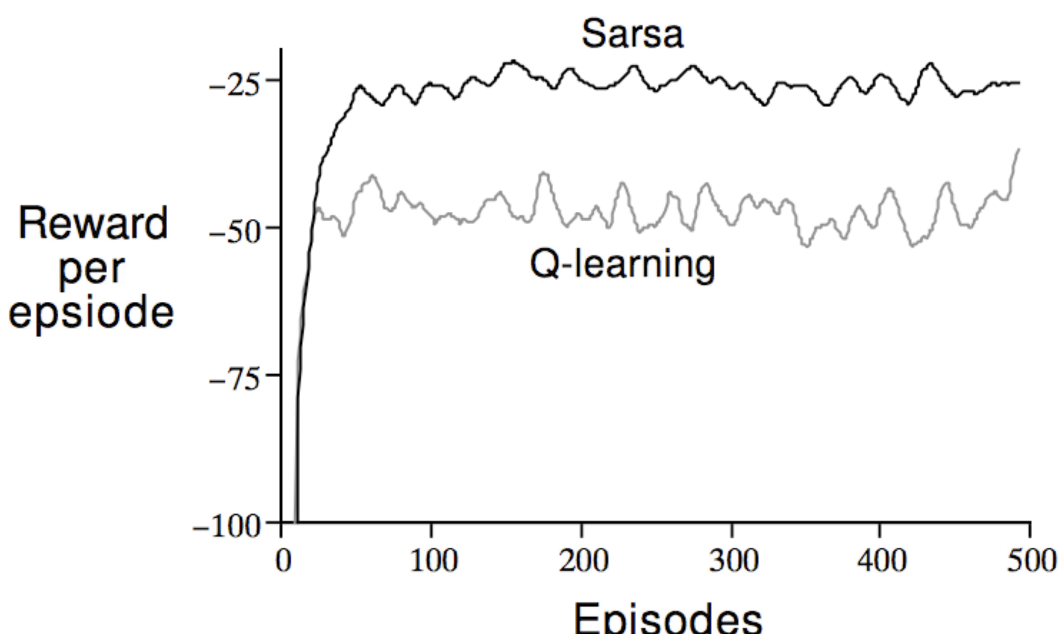


Initialize $Q(s,a), \forall s \in \mathcal{S}, a \in \mathcal{A}(s)$, arbitrarily, and $Q(\textit{terminal-state}, \cdot) = 0$
Repeat (for each episode):
    Initialize S
    Repeat (for each step of episode):
        Choose A from S using policy derived from Q (e.g., ε-greedy)
        Take action A, observe R, S'
        $Q(S,A) \leftarrow Q(S,A) + \alpha[R + \gamma \max_a Q(S',a) - Q(S,A)]$
        $S \leftarrow S'$;
    until S is terminal

Cliffwalking



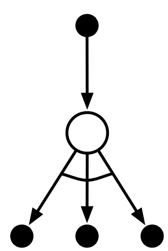
ε−greedy, ε = 0.1



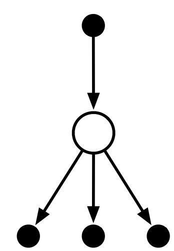
Expected Sarsa

- Instead of the *sample* value-of-next-state, use the expectation!

$$
\begin{aligned}
Q(S_t, A_t) &\leftarrow Q(S_t, A_t) + \alpha\Big[R_{t+1} + \gamma\mathbb{E}[Q(S_{t+1}, A_{t+1}) \mid S_{t+1}] - Q(S_t, A_t)\Big] \\
&\leftarrow Q(S_t, A_t) + \alpha\Big[R_{t+1} + \gamma\sum_a \pi(a|S_{t+1})Q(S_{t+1}, a) - Q(S_t, A_t)\Big]
\end{aligned}
$$



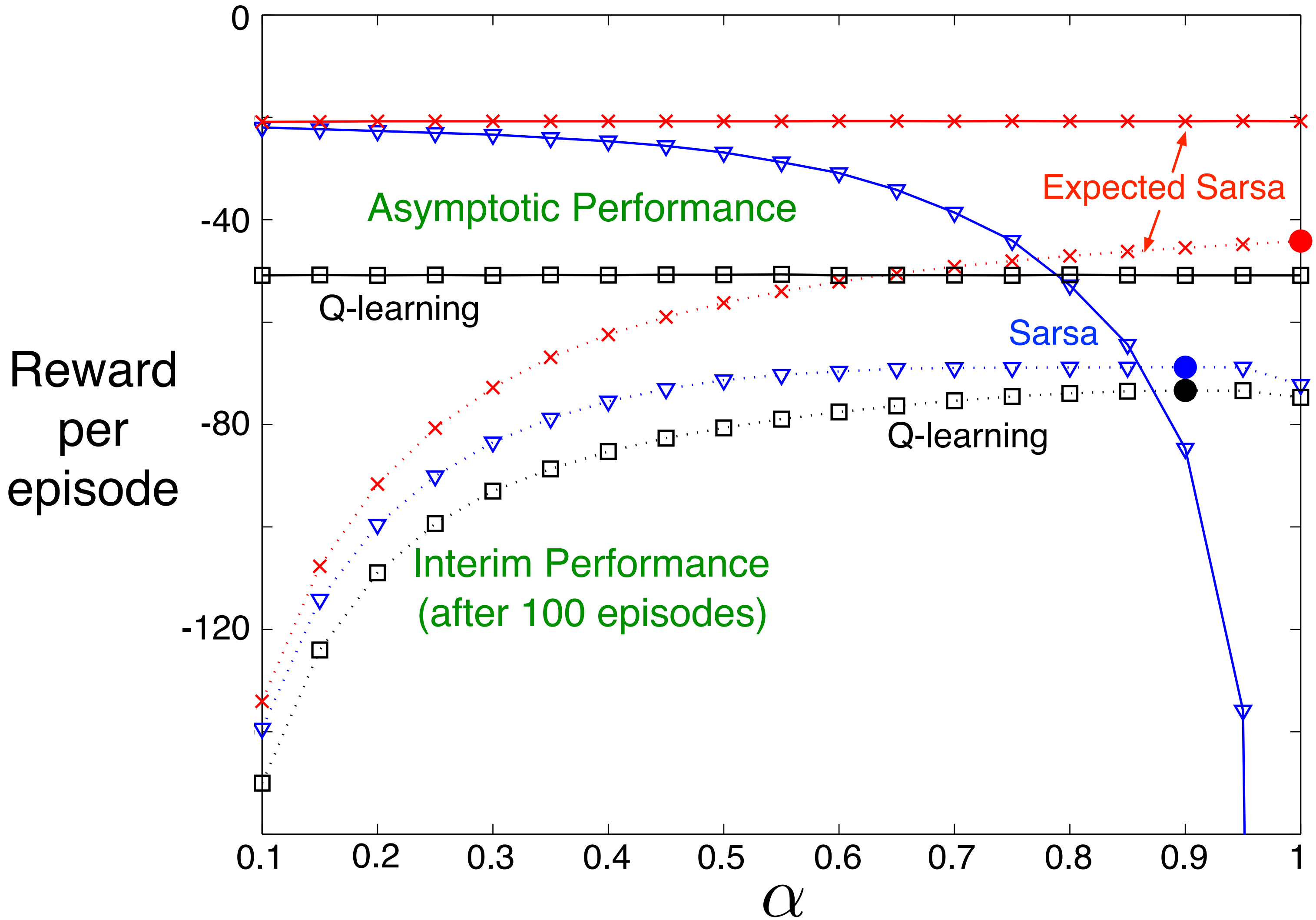
Q-learning



Expected Sarsa

- Expected Sarsa's performs better than Sarsa (but costs more)

Performance on the Cliff-walking Task



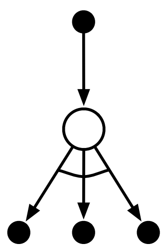
Off-policy Expected Sarsa

- Expected Sarsa generalizes to arbitrary behaviour policies μ
  - in which case it includes Q-learning as the special case in which π is the greedy policy

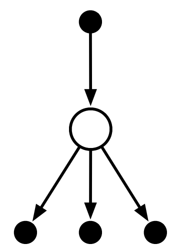
$$Q(S_t, A_t) \leftarrow Q(S_t, A_t) + \alpha \Big[R_{t+1} + \gamma \mathbb{E}[Q(S_{t+1}, A_{t+1}) \mid S_{t+1}] - Q(S_t, A_t) \Big]$$

$$\leftarrow Q(S_t, A_t) + \alpha \Big[R_{t+1} + \gamma \sum_a \pi(a|S_{t+1}) Q(S_{t+1}, a) - Q(S_t, A_t) \Big]$$

Nothing changes here



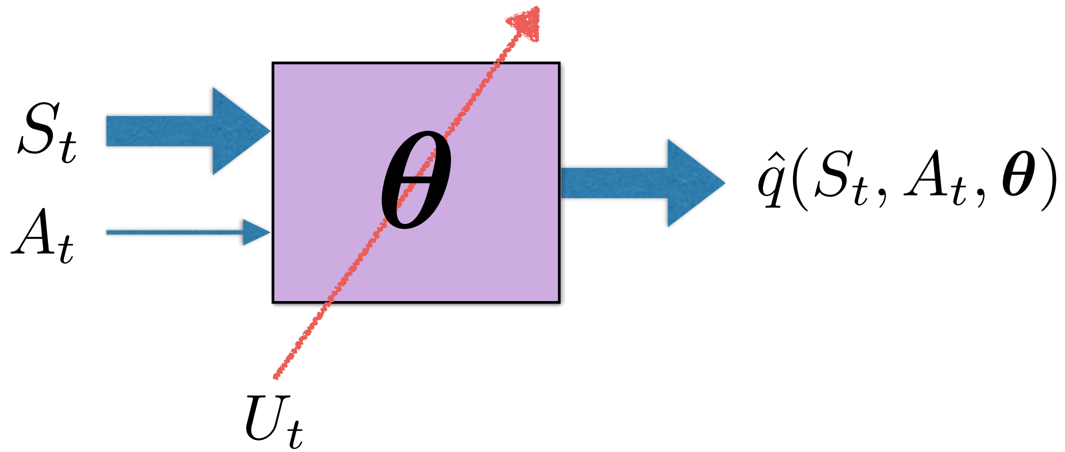
Q-learning



Expected Sarsa

- This idea seems to be new

Value function approximation (VFA) for control



(Semi-)gradient methods carry over to control in the usual on-policy GPI way

- Always learn the action-value function of the current policy
- Always act near-greedily wrt the current action-value estimates
- The learning rule is:

$$\boldsymbol{\theta}_{t+1} \doteq \boldsymbol{\theta}_t + \alpha \Big[U_t - \hat{q}(S_t, A_t, \boldsymbol{\theta}_t) \Big] \nabla \hat{q}(S_t, A_t, \boldsymbol{\theta}_t)$$

update target, e.g. $U_t = G_t$ (MC) $U_t = R_{t+1} + \gamma \hat{q}(S_{t+1}, A_{t+1}, \boldsymbol{\theta}_t)$ (Sarsa)

$U_t = R_{t+1} + \gamma \sum_a \pi(a|S_{t+1}) \hat{q}(S_{t+1}, a, \boldsymbol{\theta}_t)$ (Expected Sarsa)

$U_t = \sum_{s',r} p(s', r|S_t, A_t) \Big[r + \gamma \sum_{a'} \pi(a'|s') \hat{q}(s', a', \boldsymbol{\theta}_t) \Big]$ (DP)

(Semi-)gradient methods carry over to control

$$\boldsymbol{\theta}_{t+1} \doteq \boldsymbol{\theta}_t + \alpha\Big[U_t - \hat{q}(S_t, A_t, \boldsymbol{\theta}_t)\Big]\nabla\hat{q}(S_t, A_t, \boldsymbol{\theta}_t)$$

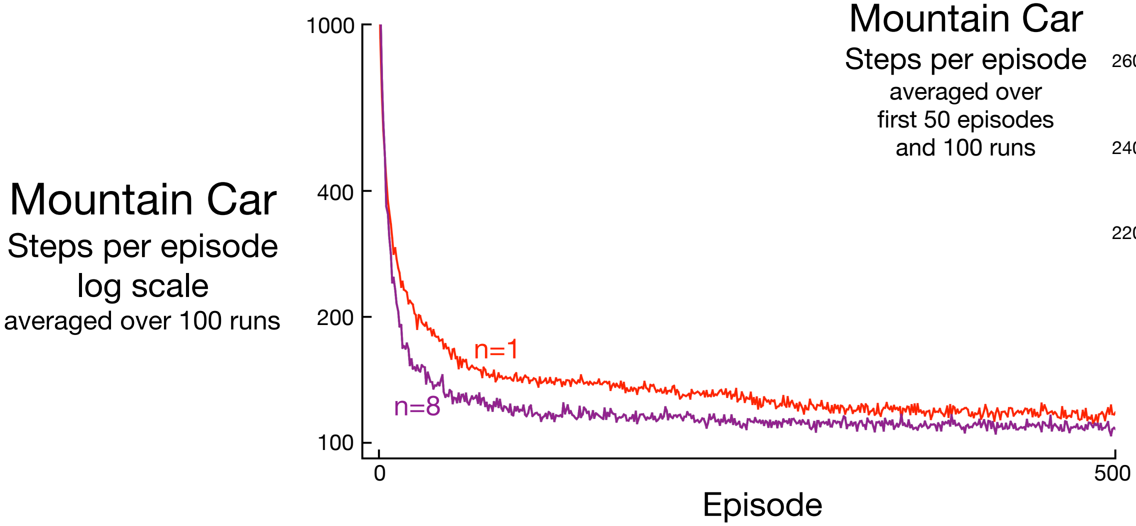
Episodic Semi-gradient Sarsa for Estimating $\hat{q} \approx q_*$

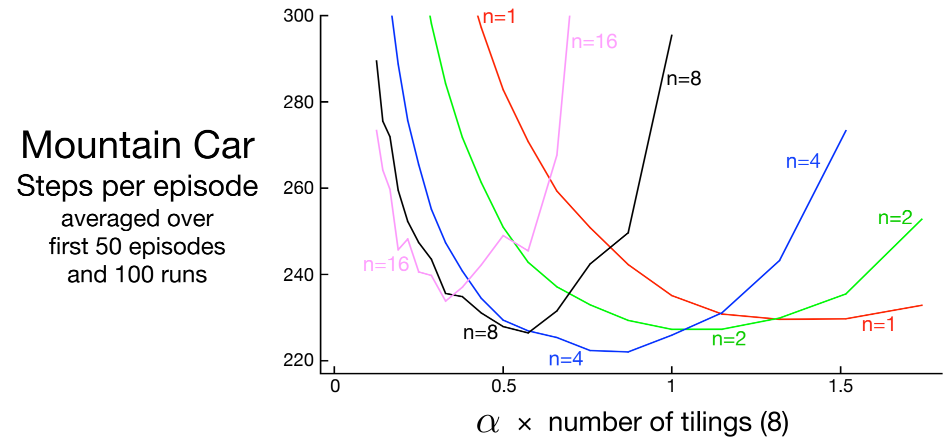
Input: a differentiable function $\hat{q} : \mathcal{S} \times \mathcal{A} \times \mathbb{R}^n \to \mathbb{R}$

Initialize value-function weights $\boldsymbol{\theta} \in \mathbb{R}^n$ arbitrarily (e.g., $\boldsymbol{\theta} = \mathbf{0}$)
Repeat (for each episode):
    $S, A \leftarrow$ initial state and action of episode (e.g., ε-greedy)
    Repeat (for each step of episode):
        Take action A, observe R, S'
        If S' is terminal:
            $\boldsymbol{\theta} \leftarrow \boldsymbol{\theta} + \alpha\big[R - \hat{q}(S, A, \boldsymbol{\theta})\big]\nabla\hat{q}(S, A, \boldsymbol{\theta})$
            Go to next episode
        Choose A' as a function of $\hat{q}(S', \cdot, \boldsymbol{\theta})$ (e.g., ε-greedy)
        $\boldsymbol{\theta} \leftarrow \boldsymbol{\theta} + \alpha\big[R + \gamma\hat{q}(S', A', \boldsymbol{\theta}) - \hat{q}(S, A, \boldsymbol{\theta})\big]\nabla\hat{q}(S, A, \boldsymbol{\theta})$
        $S \leftarrow S'$
        $A \leftarrow A'$

n-step semi-gradient Sarsa is better for *n*>1

$$\boldsymbol{\theta}_{t+n} \doteq \boldsymbol{\theta}_{t+n-1} + \alpha \left[G_t^{(n)} - \hat{q}(S_t, A_t, \boldsymbol{\theta}_{t+n-1}) \right] \nabla \hat{q}(S_t, A_t, \boldsymbol{\theta}_{t+n-1}), \qquad 0 \le t < T$$





Conclusions

- Control is straightforward in the on-policy case
- Formal results (bounds) exist for the linear, on-policy case (eg. Gordon, 2000, Perkins & Precup, 2003 and follow-up work)
    - we get chattering near a good solution, not convergence