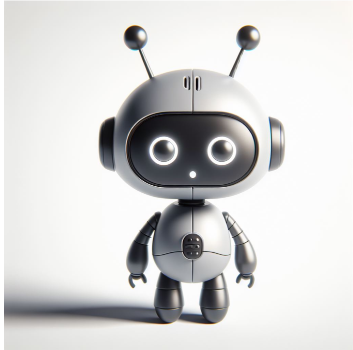
# Sequential Decision Making
# Markov Decision Processes
# Dynamic Programming

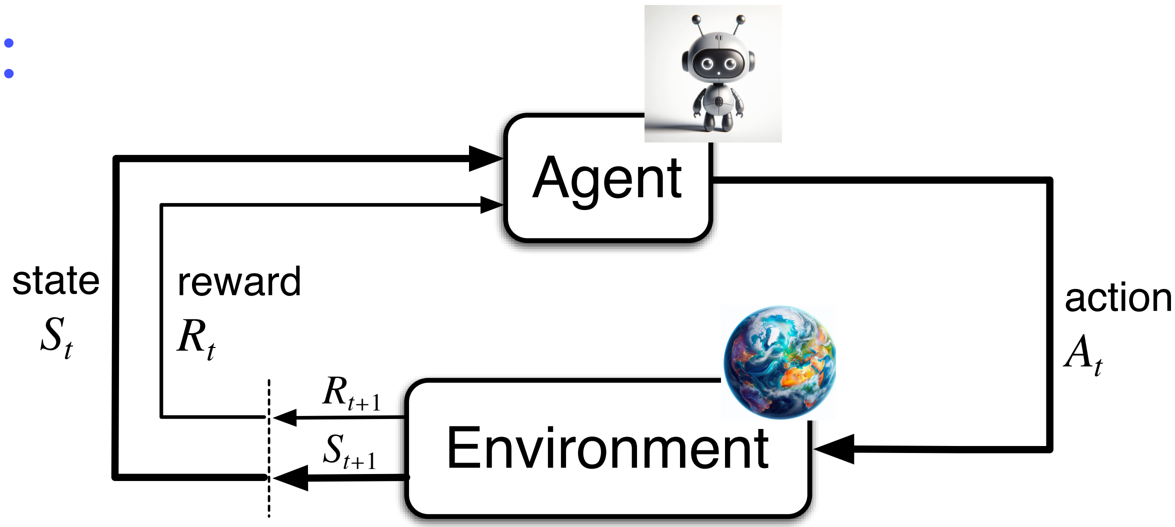
# RL Setting:

Environment:

Agent:



# RL Setting:




Agent and environment interact at discrete time steps: $t = 0, 1, 2, 3, \ldots$

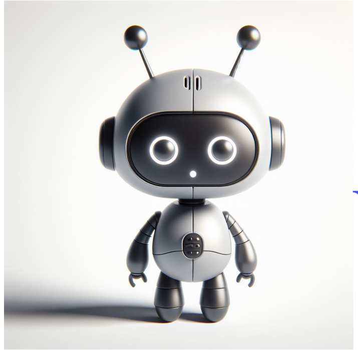
Agent observes state at step $t$: $S_t \in \mathcal{S}$

produces action at step $t$ : $A_t \in \mathcal{A}(S_t)$

gets resulting reward: $R_{t+1} \in \mathcal{R} \subset \mathbb{R}$

and resulting next state: $S_{t+1} \in \mathcal{S}^+$

# What is the difference between RL and Bandits?



Vs

# Recall: Markov Decision Processes

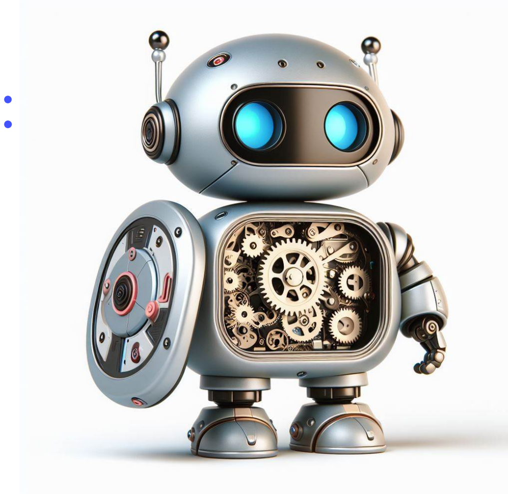
- If a reinforcement learning task has the Markov Property, it is basically a **Markov Decision Process (MDP)**.
- If state and action sets are finite, it is a **finite MDP**.
- To define a finite MDP, you need to give:
  - **state and action sets**
  - one-step “dynamics”

$$p(s', r|s, a) = \mathbf{Pr}\{S_{t+1} = s', R_{t+1} = r \mid S_t = s, A_t = a\}$$

$$p(s'|s, a) \doteq \Pr\{S_{t+1} = s' \mid S_t = s, A_t = a\} = \sum_{r \in \mathcal{R}} p(s', r|s, a)$$

$$r(s, a) \doteq \mathbb{E}[R_{t+1} \mid S_t = s, A_t = a] = \sum_{r \in \mathcal{R}} r \sum_{s' \in \mathcal{S}} p(s', r|s, a)$$

# Different Parts of an Agent:



- Value Functions :

- World Model:

- Policy:

# Recall: The Agent Learns a Policy

**Policy** at step $t$ = $\pi_t$ =

a mapping from states to action probabilities

$\pi_t(a \mid s) =$ probability that $A_t = a$ when $S_t = s$

Special case - *deterministic policies*:

$\pi_t(s)$ = the action taken with prob=1 when $S_t = s$

- Reinforcement learning methods specify how the agent changes its policy as a result of experience.
- Roughly, the agent's goal is to get as much reward as it can over the long run.

# What We Will See Today

- What is the Goal of the Agent?
  ((Discounted) Return G)

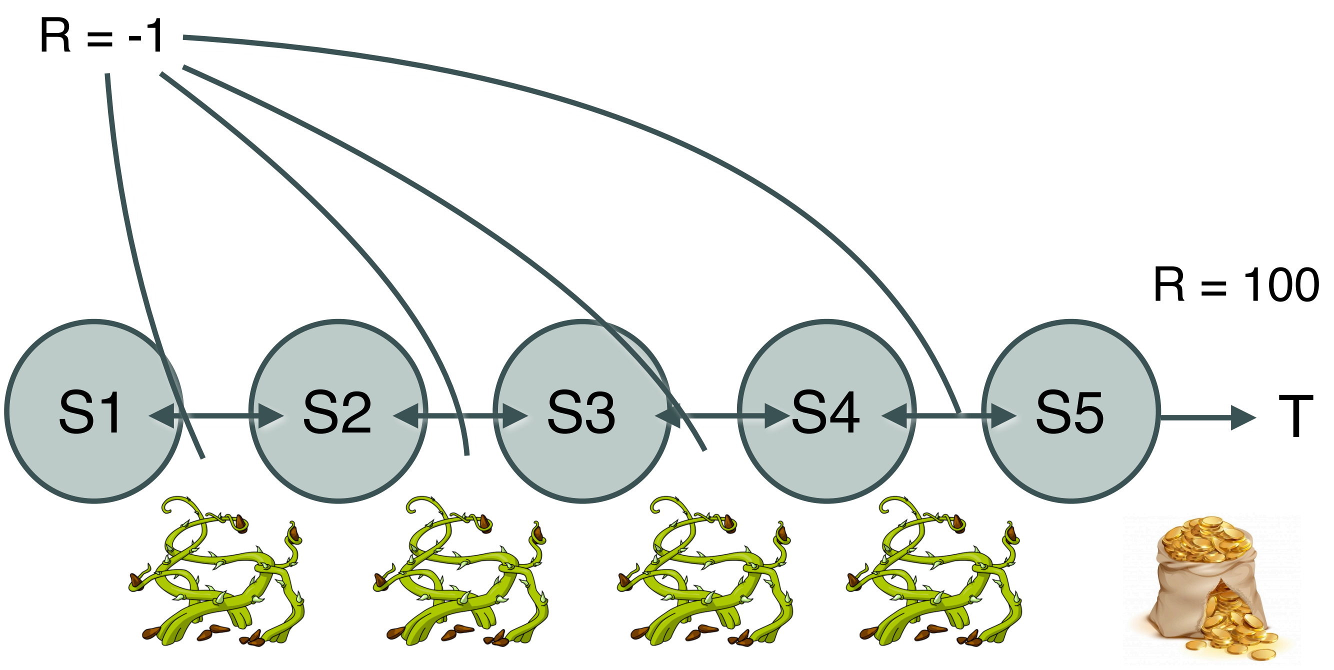
- How do we evaluate which state/actions are good?
  (Dynamic Programming, Value Fct V(s), Action-Value Q(a,s))

- How can we improve our policy π?
  (Bellman Eqn)

# What is the Goal of the Agent?

- Reward sequence A: 1, 0, 0, 0
- Reward sequence B: 0, 1, 0, 0
- Reward sequence C: 0, 0, 1.16, 0
- Reward sequence D: 0, 0, 0, 1.17

# How good are each states?



# The reward hypothesis

- That all of what we mean by goals and purposes can be well thought of as the maximization of the cumulative sum of a received scalar signal (reward).

- A sort of *null hypothesis*.
  - Possibly wrong, but very simple, and so far very successful.

# How can we convert the future sequence of rewards to a single number?

- Reward sequence A: 1, 0, 0, 0
- Reward sequence B: 0, 1, 0, 0
- Reward sequence C: 0, 0, 1.16, 0
- Reward sequence D: 0, 0, 0, 1.17

# Return

Suppose the sequence of rewards after step $t$ is:

$$R_{t+1}, R_{t+2}, R_{t+3}, \ldots$$

What do we want to maximize?

At least three cases, but in all of them,

we seek to maximize the **expected return**, $E\{G_t\}$, on each step $t$.

- Total reward, $G_t$ = sum of all future reward in the episode
- Discounted reward, $G_t$ = sum of all future *discounted* reward
- Average reward, $G_t$ = average reward per time step

# Discounted Return

**Continuing tasks**: interaction does not have natural episodes, but just goes on and on...

In this class, for continuing tasks we will always use *discounted return*:

$$G_t = R_{t+1} + \gamma R_{t+2} + \gamma^2 R_{t+3} + \cdots = \sum_{k=0}^{\infty} \gamma^k R_{t+k+1},$$

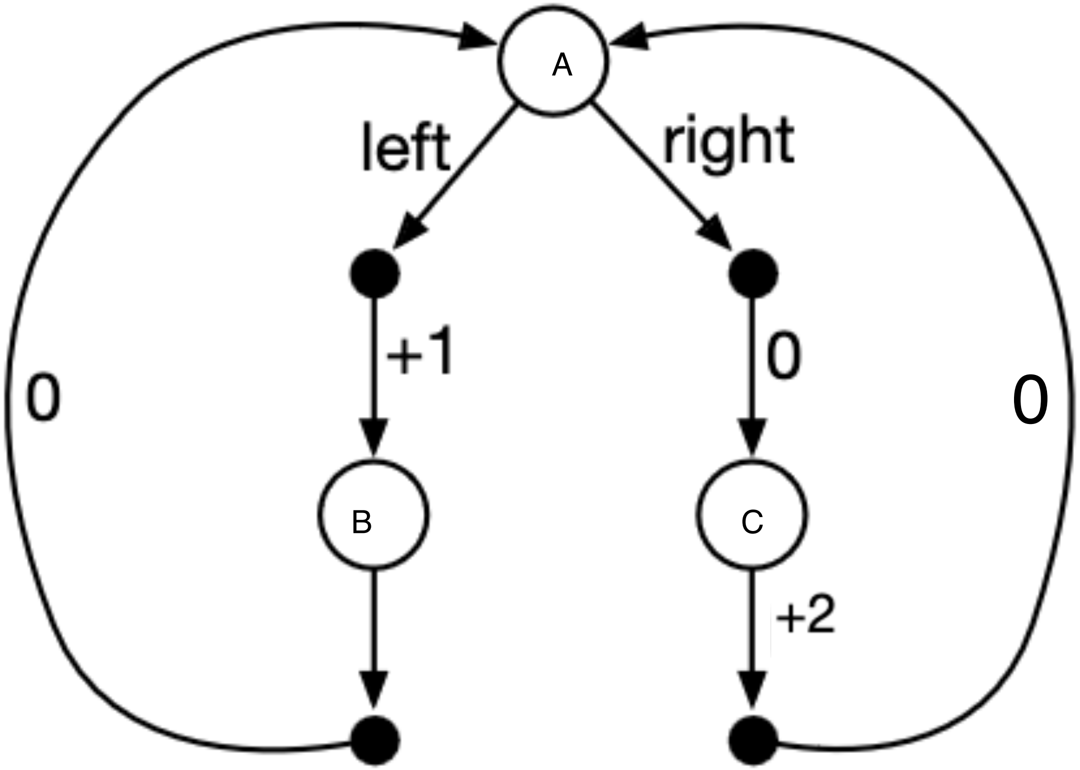
where $\gamma$, $0 \leq \gamma \leq 1$, is the **discount rate**.

shortsighted $0 \leftarrow \gamma \rightarrow 1$ farsighted

Typically, $\gamma = 0.9$

# Which one is the best?

- Reward sequence A: 1, 0, 0, 0
- Reward sequence B: 0, 1, 0, 0
- Reward sequence C: 0, 0, 1.16, 0
- Reward sequence D: 0, 0, 0, 1.17




What policy is optimal starting from A?

i) Going left.
ii) Going right.
iii) Something else.

If $\gamma = 0$?

If $\gamma = .99$

If $\gamma = \frac{1}{2}$?

# Episodic Tasks: Total Reward

**Episodic tasks**: interaction breaks naturally into episodes, e.g., plays of a game, trips through a maze

In episodic tasks, we often simply use *total reward*:

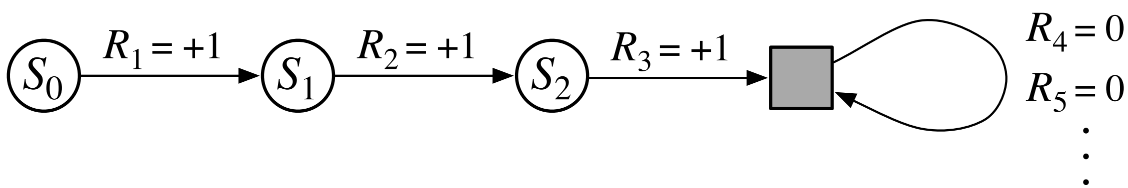
$$G_t = R_{t+1} + R_{t+2} + \cdots + R_T ,$$

where $T$ is a final time step at which a **terminal state** is reached, ending an episode.

# A Trick to Unify Notation for Returns

- In episodic tasks, we number the time steps of each episode starting from zero.

- Think of each episode as ending in an absorbing state that always produces reward of zero:




- We can cover <u>all</u> cases by writing $G_t = \sum_{k=0}^{\infty} \gamma^k R_{t+k+1}$,

where $\gamma$ can be 1 only if a zero reward absorbing state is always reached.

# Episodic and Continuing Tasks: Average Reward

In episodic tasks, we can also use *average reward*:

$$G_0 = (\sum_{t=0}^{T} R_t)/T$$

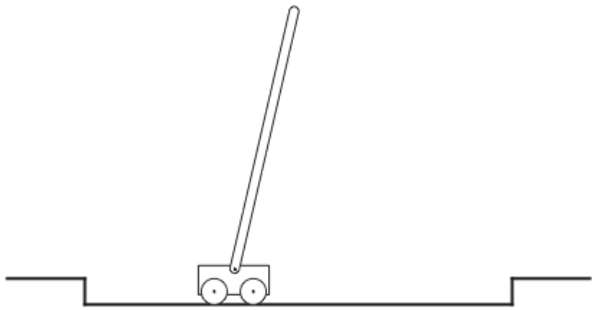
where $T$ is a final time step at which a **terminal state** is reached, ending an episode.

In continuing tasks, we can also define *average reward*:

$$G = \lim_{T\to\infty}\left( (\sum_{t=0}^{T} R_t)/T \right)$$

More on this later!

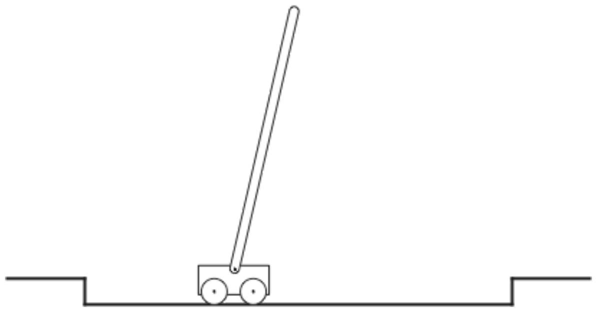
# An Example: Pole Balancing



Avoid **failure:** the pole falling beyond a critical angle or the cart hitting end of track

As an **episodic task** where episode ends upon failure:

# An Example: Pole Balancing



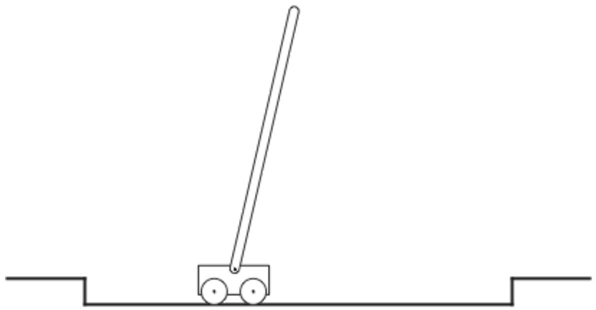
Avoid **failure:** the pole falling beyond a critical angle or the cart hitting end of track

As an **episodic task** where episode ends upon failure:

reward = +1 for each step before failure

$\Rightarrow$ return = number of steps before failure

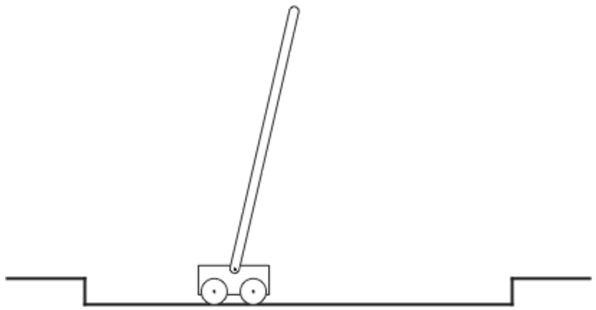
# An Example: Pole Balancing



Avoid **failure:** the pole falling beyond a critical angle or the cart hitting end of track

As a **continuing task** with discounted return:

# An Example: Pole Balancing



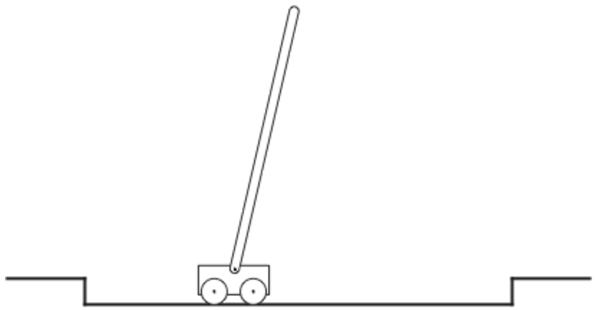
Avoid **failure:** the pole falling beyond a critical angle or the cart hitting end of track

As a **continuing task** with discounted return:

reward $= -1$ upon failure; 0 otherwise

$\Rightarrow$ return $= -\gamma^k$, for $k$ steps before failure

# An Example: Pole Balancing



Avoid **failure:** the pole falling beyond a critical angle or the cart hitting end of track

As an **episodic task** where episode ends upon failure:

reward = +1 for each step before failure

$\Rightarrow$ return = number of steps before failure

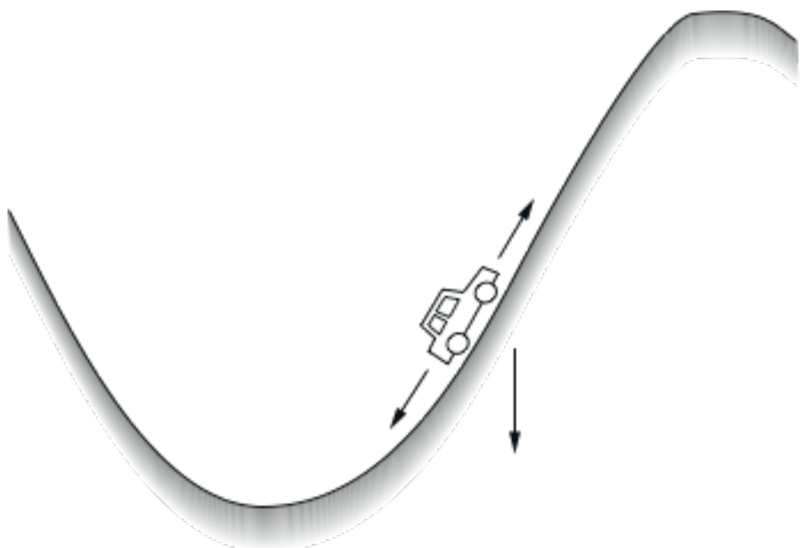
As a **continuing task** with discounted return:

reward = −1 upon failure; 0 otherwise

$\Rightarrow$ return = $-\gamma^k$, for $k$ steps before failure

In either case, return is maximized by avoiding failure for as long as possible.

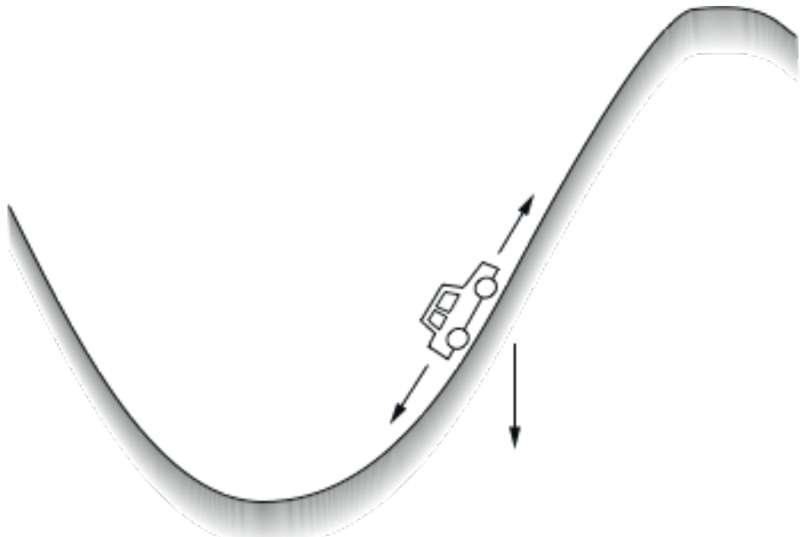
# Another Example: Mountain Car



Get to the top of the hill as quickly as possible.

Return is maximized by minimizing number of steps to reach the top of the hill.

# Mountain Car: Discounted

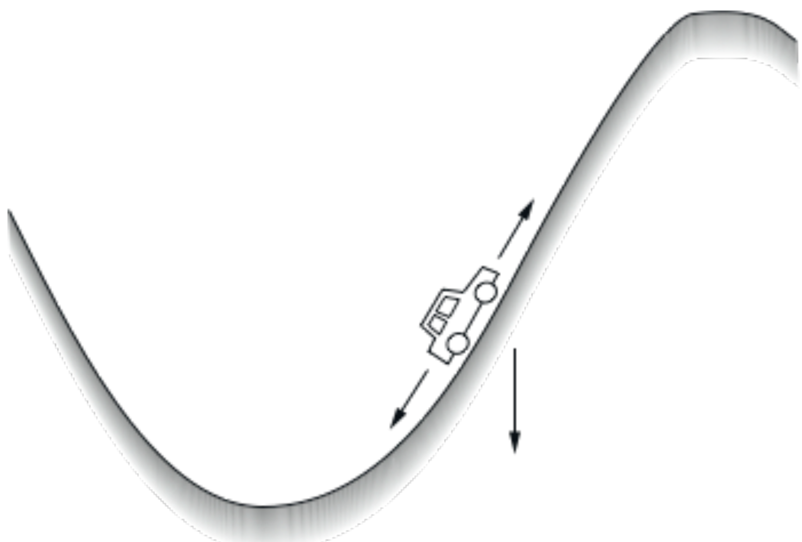


Get to the top of the hill as quickly as possible.

Reward: 1 at the top of the hill, 0 otherwise
Return: if discount <1, k=number of time steps, so return is $\gamma^k$

Return is maximized by minimizing number of steps to reach the top of the hill.

# Mountain Car: Episodic



Get to the top of the hill as quickly as possible.

reward = –1 for each step where **not** at top of hill

⇒ return = – number of steps before reaching top of hill

Return is maximized by minimizing number of steps to reach the top of the hill.

# 4 value functions

| | state values | action values |
|---|---|---|
| prediction | $v_\pi$ | $q_\pi$ |
| control | $v_*$ | $q_*$ |

- All theoretical objects, expected values
- Distinct from their estimates: $V_t(s)$ $\qquad Q_t(s,a)$

# Values are *expected* returns

- The value of a state, given a policy:

$$v_\pi(s) = \mathbb{E}\{G_t \mid S_t = s, A_{t:\infty} \sim \pi\} \qquad v_\pi : \mathcal{S} \to \Re$$

- The value of a state-action pair, given a policy:

$$q_\pi(s,a) = \mathbb{E}\{G_t \mid S_t = s, A_t = a, A_{t+1:\infty} \sim \pi\} \qquad q_\pi : \mathcal{S} \times \mathcal{A} \to \Re$$

- The optimal value of a state:

$$v_*(s) = \max_\pi v_\pi(s) \qquad v_* : \mathcal{S} \to \Re$$

- The optimal value of a state-action pair:

$$q_*(s,a) = \max_\pi q_\pi(s,a) \qquad q_* : \mathcal{S} \times \mathcal{A} \to \Re$$

- Optimal policy: $\pi_*$ is an optimal policy if and only if

$$\pi_*(a|s) > 0 \text{ only where } q_*(s,a) = \max_b q_*(s,b) \qquad \forall s \in \mathcal{S}$$

  - in other words, $\pi_*$ is optimal iff it is *greedy* wrt $q_*$

# Value Functions

- The **value of a state** is the expected return starting from that state; depends on the agent's policy:

**State - value function for policy $\pi$ :**

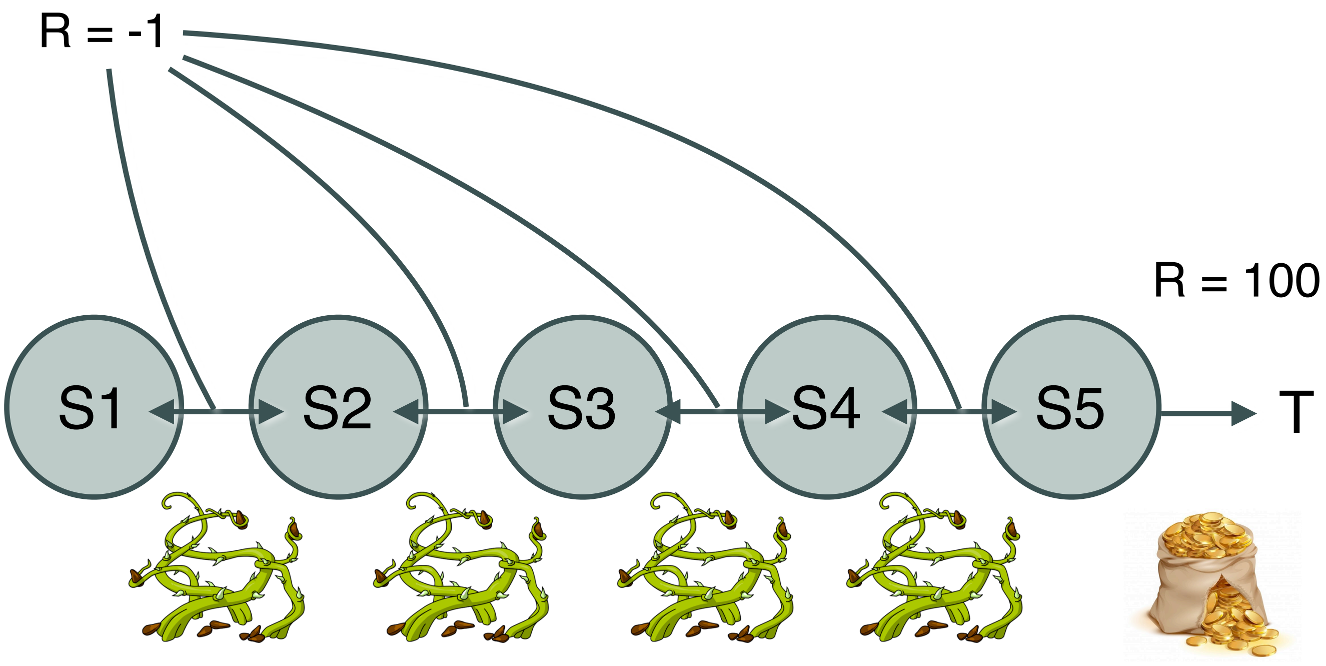
$$v_\pi(s) = E_\pi\left\{G_t \mid S_t = s\right\} = E_\pi\left\{\sum_{k=0}^{\infty} \gamma^k R_{t+k+1} \mid S_t = s\right\}$$

- The **value of an action (in a state)** is the expected return starting after taking that action from that state; depends on the agent's policy:

**Action - value function for policy $\pi$ :**

$$q_\pi(s,a) = E_\pi\left\{G_t \mid S_t = s, A_t = a\right\} = E_\pi\left\{\sum_{k=0}^{\infty} \gamma^k R_{t+k+1} \mid S_t = s, A_t = a\right\}$$

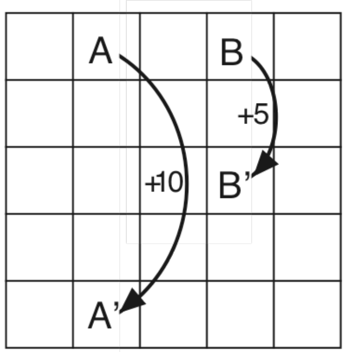
# How good are each states?



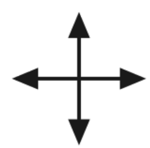

If $\gamma=1$, $V^* = ?$

# Gridworld

- Actions: `north`, `south`, `east`, `west`; deterministic.
- If would take agent off the grid: no move but reward = –1
- Other actions produce reward = 0, except actions that move agent out of special states A and B as shown.




(a)




| | | | | |
|---|---|---|---|---|
| 3.3 | 8.8 | 4.4 | 5.3 | 1.5 |
| 1.5 | 3.0 | 2.3 | 1.9 | 0.5 |
| 0.1 | 0.7 | 0.7 | 0.4 | -0.4 |
| -1.0 | -0.4 | -0.4 | -0.6 | -1.2 |
| -1.9 | -1.3 | -1.2 | -1.4 | -2.0 |

(b)

State-value function for equiprobable random policy; $\gamma = 0.9$

# Policy Evaluation

**Policy Evaluation**: for a given policy $\pi$, compute the state-value function $v_\pi$

Recall: **State-value function for policy $\pi$**

$$v_\pi(s) \doteq \mathbb{E}_\pi[G_t \mid S_t = s] = \mathbb{E}_\pi\left[\sum_{k=0}^{\infty} \gamma^k R_{t+k+1} \,\middle|\, S_t = s\right]$$

# Bellman Equation for a Policy π

The basic idea:

$$\begin{aligned} G_t &= R_{t+1} + \gamma R_{t+2} + \gamma^2 R_{t+3} + \gamma^3 R_{t+4} + \cdots \\ &= R_{t+1} + \gamma \left( R_{t+2} + \gamma R_{t+3} + \gamma^2 R_{t+4} + \cdots \right) \\ &= R_{t+1} + \gamma G_{t+1} \end{aligned}$$

So:

$$\begin{aligned} v_\pi(s) &= E_\pi \left\{ G_t \mid S_t = s \right\} \\ &= E_\pi \left\{ R_{t+1} + \gamma v_\pi\left(S_{t+1}\right) \mid S_t = s \right\} \end{aligned}$$

Or, explicitly writing out the expectation:

$$v_\pi(s) = \sum_a \pi(a|s) \sum_{s',r} p(s', r|s, a) \Big[ r + \gamma v_\pi(s') \Big]$$

# More on the Bellman Equation

$$v_\pi(s) = \sum_a \pi(a|s) \sum_{s',r} p(s', r|s, a)\Big[r + \gamma v_\pi(s')\Big]$$

This is a set of equations (in fact, linear), one for each state.
The value function for $\pi$ is its unique solution*.

* In the usual case where the system of equations is invertible, but in the current context you would really need to work hard to make it non-invertible.

# Q-Function

$$
\begin{aligned}
q_\pi(s,a) &= \mathbb{E}_\pi[R_{t+1} + \gamma v_\pi(S_{t+1}) \mid S_t = s, A_t = a] \\
&= \sum_{s',r} p(s',r|s,a)\Big[r + \gamma v_\pi(s')\Big].
\end{aligned}
$$

# Policy Evaluation

**Policy Evaluation**: for a given policy $\pi$, compute the state-value function $v_\pi$

Recall: **State-value function for policy $\pi$**

$$v_\pi(s) \doteq \mathbb{E}_\pi[G_t \mid S_t = s] = \mathbb{E}_\pi\left[\sum_{k=0}^{\infty} \gamma^k R_{t+k+1} \,\middle|\, S_t = s\right]$$

Recall: **Bellman equation for $v_\pi$**

$$v_\pi(s) = \sum_a \pi(a|s) \sum_{s',r} p(s',r|s,a)\Big[r + \gamma v_\pi(s')\Big]$$

—a system of $|\mathcal{S}|$ simultaneous equations

# Iterative Methods

$$v_0 \rightarrow v_1 \rightarrow \cdots \rightarrow v_k \rightarrow v_{k+1} \rightarrow \cdots \rightarrow v_\pi$$

a "sweep"

A sweep consists of applying a **backup operation** to each state.

A **full policy-evaluation backup**:

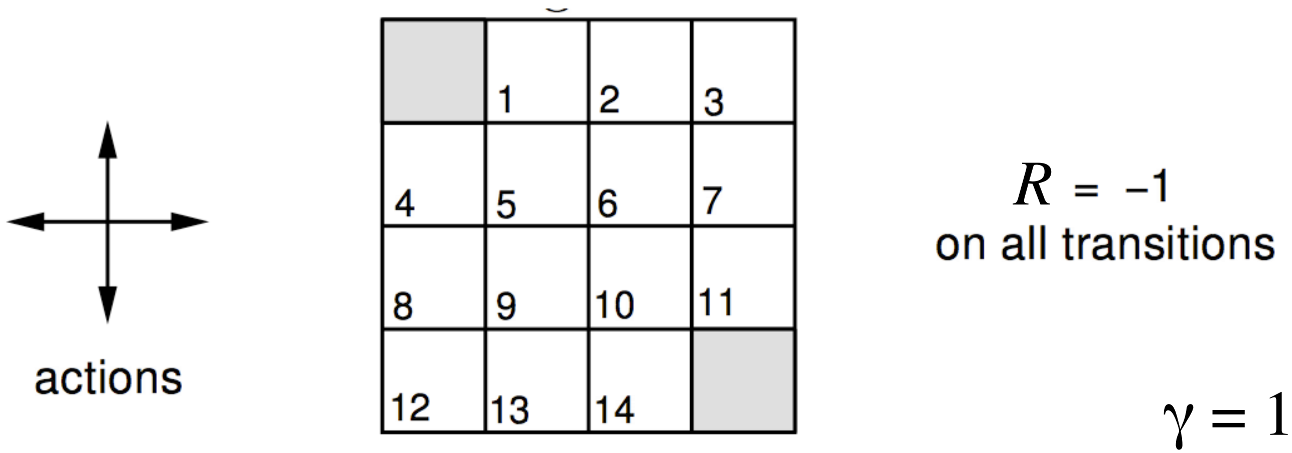
$$v_{k+1}(s) = \sum_a \pi(a|s) \sum_{s',r} p(s', r|s, a)\Big[r + \gamma v_k(s')\Big] \qquad \forall s \in \mathcal{S}$$

* This works because of Banach's fixed-point theorem, and that probabilities sum to 1 and γ<1.

# Iterative Policy Evaluation – One array version

Input $\pi$, the policy to be evaluated
Initialize an array $V(s) = 0$, for all $s \in \mathcal{S}^+$
Repeat
    $\Delta \leftarrow 0$
    For each $s \in \mathcal{S}$:
        $v \leftarrow V(s)$
        $V(s) \leftarrow \sum_a \pi(a|s) \sum_{s',r} p(s',r|s,a)\big[r + \gamma V(s')\big]$
        $\Delta \leftarrow \max(\Delta, |v - V(s)|)$
until $\Delta < \theta$ (a small positive number)
Output $V \approx v_\pi$

# A Small Gridworld




- An undiscounted episodic task
- Nonterminal states: 1, 2, . . ., 14;
- One terminal state (shown twice as shaded squares)
- Actions that would take agent off the grid leave state unchanged
- Reward is –1 until the terminal state is reached

# Iterative Policy Eval for the Small Gridworld

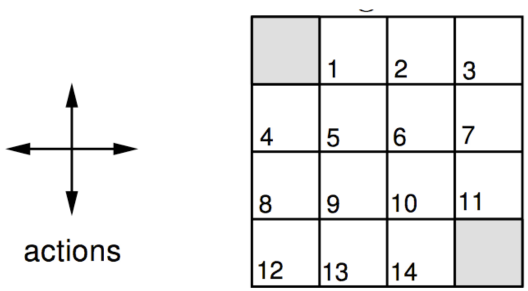
$V_k$ for the Random Policy

$k = 0$

| | | | |
|---|---|---|---|
| 0.0 | 0.0 | 0.0 | 0.0 |
| 0.0 | 0.0 | 0.0 | 0.0 |
| 0.0 | 0.0 | 0.0 | 0.0 |
| 0.0 | 0.0 | 0.0 | 0.0 |

$\pi =$ equiprobable random action choices

$k = 1$




$R = -1$
on all transitions

$k = 2$

$\gamma = 1$

$k = 3$

- An undiscounted episodic task
- Nonterminal states: 1, 2, . . ., 14;
- One terminal state (shown twice as shaded squares)
- Actions that would take agent off the grid leave state unchanged
- Reward is –1 until the terminal state is reached

$k = 10$

$k = \infty$

# Iterative Policy Eval for the Small Gridworld

$V_k$ for the Random Policy

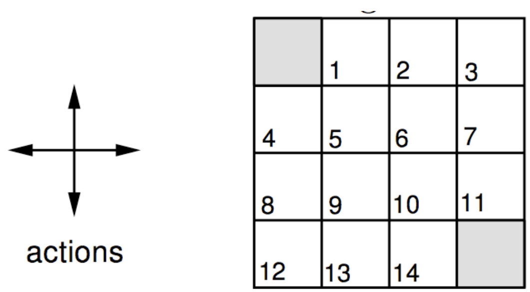
$k = 0$

| | | | |
|---|---|---|---|
| 0.0 | 0.0 | 0.0 | 0.0 |
| 0.0 | 0.0 | 0.0 | 0.0 |
| 0.0 | 0.0 | 0.0 | 0.0 |
| 0.0 | 0.0 | 0.0 | 0.0 |

$\pi =$ equiprobable random action choices

$k = 1$

| | | | |
|---|---|---|---|
| 0.0 | -1.0 | -1.0 | -1.0 |
| -1.0 | -1.0 | -1.0 | -1.0 |
| -1.0 | -1.0 | -1.0 | -1.0 |
| -1.0 | -1.0 | -1.0 | 0.0 |




$R = -1$
on all transitions

$\gamma = 1$

$k = 2$

$k = 3$

- An undiscounted episodic task
- Nonterminal states: 1, 2, . . ., 14;
- One terminal state (shown twice as shaded squares)
- Actions that would take agent off the grid leave state unchanged
- Reward is –1 until the terminal state is reached

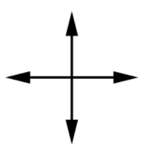
$k = 10$

$k = \infty$

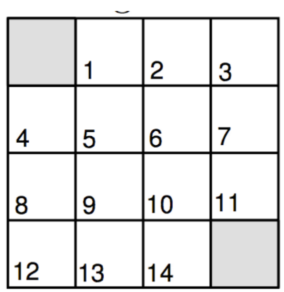
# Iterative Policy Eval for the Small Gridworld

$V_k$ for the Random Policy

$\pi =$ equiprobable random action choices



actions




$R = -1$ on all transitions

$\gamma = 1$

$k = 0$

| | | | |
|---|---|---|---|
| 0.0 | 0.0 | 0.0 | 0.0 |
| 0.0 | 0.0 | 0.0 | 0.0 |
| 0.0 | 0.0 | 0.0 | 0.0 |
| 0.0 | 0.0 | 0.0 | 0.0 |

$k = 1$

| | | | |
|---|---|---|---|
| 0.0 | -1.0 | -1.0 | -1.0 |
| -1.0 | -1.0 | -1.0 | -1.0 |
| -1.0 | -1.0 | -1.0 | -1.0 |
| -1.0 | -1.0 | -1.0 | 0.0 |

$k = 2$

| | | | |
|---|---|---|---|
| 0.0 | -1.7 | -2.0 | -2.0 |
| -1.7 | -2.0 | -2.0 | -2.0 |
| -2.0 | -2.0 | -2.0 | -1.7 |
| -2.0 | -2.0 | -1.7 | 0.0 |

$k = 3$

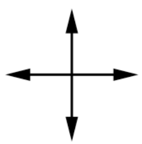
- An undiscounted episodic task
- Nonterminal states: 1, 2, . . ., 14;
- One terminal state (shown twice as shaded squares)
- Actions that would take agent off the grid leave state unchanged
- Reward is –1 until the terminal state is reached

$k = 10$

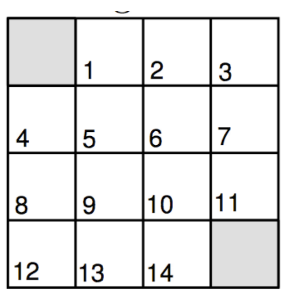
$k = \infty$

# Iterative Policy Eval for the Small Gridworld

$\pi =$ equiprobable random action choices



actions




$R = -1$
on all transitions

$\gamma = 1$

- An undiscounted episodic task
- Nonterminal states: 1, 2, . . ., 14;
- One terminal state (shown twice as shaded squares)
- Actions that would take agent off the grid leave state unchanged
- Reward is –1 until the terminal state is reached

$V_k$ for the Random Policy

$k = 0$

| | | | |
|---|---|---|---|
| 0.0 | 0.0 | 0.0 | 0.0 |
| 0.0 | 0.0 | 0.0 | 0.0 |
| 0.0 | 0.0 | 0.0 | 0.0 |
| 0.0 | 0.0 | 0.0 | 0.0 |

$k = 1$

| | | | |
|---|---|---|---|
| 0.0 | -1.0 | -1.0 | -1.0 |
| -1.0 | -1.0 | -1.0 | -1.0 |
| -1.0 | -1.0 | -1.0 | -1.0 |
| -1.0 | -1.0 | -1.0 | 0.0 |

$k = 2$

| | | | |
|---|---|---|---|
| 0.0 | -1.7 | -2.0 | -2.0 |
| -1.7 | -2.0 | -2.0 | -2.0 |
| -2.0 | -2.0 | -2.0 | -1.7 |
| -2.0 | -2.0 | -1.7 | 0.0 |

$k = 3$

| | | | |
|---|---|---|---|
| 0.0 | -2.4 | -2.9 | -3.0 |
| -2.4 | -2.9 | -3.0 | -2.9 |
| -2.9 | -3.0 | -2.9 | -2.4 |
| -3.0 | -2.9 | -2.4 | 0.0 |

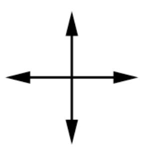
$k = 10$

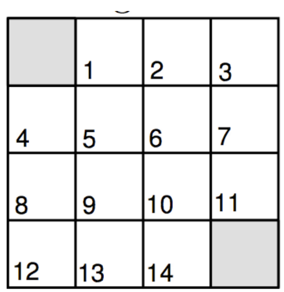
$k = \infty$

# Iterative Policy Eval for the Small Gridworld

$V_k$ for the Random Policy

$\pi =$ equiprobable random action choices




$R = -1$
on all transitions

$\gamma = 1$

- An undiscounted episodic task
- Nonterminal states: 1, 2, . . ., 14;
- One terminal state (shown twice as shaded squares)
- Actions that would take agent off the grid leave state unchanged
- Reward is –1 until the terminal state is reached

$k = 0$

| | | | |
|---|---|---|---|
| 0.0 | 0.0 | 0.0 | 0.0 |
| 0.0 | 0.0 | 0.0 | 0.0 |
| 0.0 | 0.0 | 0.0 | 0.0 |
| 0.0 | 0.0 | 0.0 | 0.0 |

$k = 1$

| | | | |
|---|---|---|---|
| 0.0 | -1.0 | -1.0 | -1.0 |
| -1.0 | -1.0 | -1.0 | -1.0 |
| -1.0 | -1.0 | -1.0 | -1.0 |
| -1.0 | -1.0 | -1.0 | 0.0 |

$k = 2$

| | | | |
|---|---|---|---|
| 0.0 | -1.7 | -2.0 | -2.0 |
| -1.7 | -2.0 | -2.0 | -2.0 |
| -2.0 | -2.0 | -2.0 | -1.7 |
| -2.0 | -2.0 | -1.7 | 0.0 |

$k = 3$

| | | | |
|---|---|---|---|
| 0.0 | -2.4 | -2.9 | -3.0 |
| -2.4 | -2.9 | -3.0 | -2.9 |
| -2.9 | -3.0 | -2.9 | -2.4 |
| -3.0 | -2.9 | -2.4 | 0.0 |

$k = 10$

| | | | |
|---|---|---|---|
| 0.0 | -6.1 | -8.4 | -9.0 |
| -6.1 | -7.7 | -8.4 | -8.4 |
| -8.4 | -8.4 | -7.7 | -6.1 |
| -9.0 | -8.4 | -6.1 | 0.0 |

$k = \infty$

| | | | |
|---|---|---|---|
| 0.0 | -14. | -20. | -22. |
| -14. | -18. | -20. | -20. |
| -20. | -20. | -18. | -14. |
| -22. | -20. | -14. | 0.0 |

# Bellman Optimality Eqn

$$v_\pi(s) = \sum_a \pi(a|s) \overbrace{\sum_{s',r} p(s', r|s, a)\Big[r + \gamma v_\pi(s')\Big]}^{q_\pi(s,a)}$$

# Bellman Optimality Eqn

$$v_\pi(s) = \sum_a \pi(a|s) \overbrace{\sum_{s',r} p(s', r|s, a)\Big[r + \gamma v_\pi(s')\Big]}^{q_\pi(s,a)}$$

$$v_*(s) = \max_{a \in \mathcal{A}(s)} q_{\pi_*}(s, a)$$

# Bellman Optimality Eqn

$$v_\pi(s) = \sum_a \pi(a|s) \overbrace{\sum_{s',r} p(s',r|s,a)\Big[r + \gamma v_\pi(s')\Big]}^{q_\pi(s,a)}$$

$$\begin{aligned} v_*(s) &= \max_{a \in \mathcal{A}(s)} q_{\pi_*}(s,a) \\ &= \max_a \mathbb{E}_{\pi_*}[G_t \mid S_t = s, A_t = a] \end{aligned}$$

# Bellman Optimality Eqn

$$v_\pi(s) = \sum_a \pi(a|s) \overbrace{\sum_{s',r} p(s',r|s,a)\Big[r + \gamma v_\pi(s')\Big]}^{q_\pi(s,a)}$$

$$\begin{aligned} v_*(s) &= \max_{a \in \mathcal{A}(s)} q_{\pi_*}(s,a) \\ &= \max_a \mathbb{E}_{\pi_*}[G_t \mid S_t = s, A_t = a] \\ &= \max_a \mathbb{E}_{\pi_*}[R_{t+1} + \gamma G_{t+1} \mid S_t = s, A_t = a] \end{aligned}$$

# Bellman Optimality Eqn

$$v_\pi(s) = \sum_a \pi(a|s) \overbrace{\sum_{s',r} p(s', r|s, a)\Big[r + \gamma v_\pi(s')\Big]}^{q_\pi(s,a)}$$

$$\begin{aligned}
v_*(s) &= \max_{a \in \mathcal{A}(s)} q_{\pi_*}(s, a) \\
&= \max_a \mathbb{E}_{\pi_*}[G_t \mid S_t = s, A_t = a] \\
&= \max_a \mathbb{E}_{\pi_*}[R_{t+1} + \gamma G_{t+1} \mid S_t = s, A_t = a] \\
&= \max_a \mathbb{E}[R_{t+1} + \gamma v_*(S_{t+1}) \mid S_t = s, A_t = a]
\end{aligned}$$

# Bellman Optimality Eqn

$$v_\pi(s) = \sum_a \pi(a|s) \overbrace{\sum_{s',r} p(s', r|s, a)\Big[r + \gamma v_\pi(s')\Big]}^{q_\pi(s,a)}$$

$$\begin{aligned}
v_*(s) &= \max_{a \in \mathcal{A}(s)} q_{\pi_*}(s, a) \\
&= \max_a \mathbb{E}_{\pi_*}[G_t \mid S_t = s, A_t = a] \\
&= \max_a \mathbb{E}_{\pi_*}[R_{t+1} + \gamma G_{t+1} \mid S_t = s, A_t = a] \\
&= \max_a \mathbb{E}[R_{t+1} + \gamma v_*(S_{t+1}) \mid S_t = s, A_t = a] \\
&= \max_a \sum_{s',r} p(s', r | s, a)\big[r + \gamma v_*(s')\big].
\end{aligned}$$

# Bellman Optimality Eqn

$$v_\pi(s) = \sum_a \pi(a|s) \sum_{s',r} p(s', r|s, a)\Big[r + \gamma v_\pi(s')\Big]$$

$$v_*(s) = \max_a \sum_{s',r} p(s', r | s, a)\left[r + \gamma v_*(s')\right]$$

Also as many equations as unknowns (non-linear, this time though).

# Policy Iteration

$$\pi_0 \xrightarrow{\text{E}} v_{\pi_0} \xrightarrow{\text{I}} \pi_1 \xrightarrow{\text{E}} v_{\pi_1} \xrightarrow{\text{I}} \pi_2 \xrightarrow{\text{E}} \cdots \xrightarrow{\text{I}} \pi_* \xrightarrow{\text{E}} v_*$$

policy evaluation

policy improvement
“greedification”

# Policy Improvement

Suppose we have computed $v_\pi$ for a deterministic policy π.

For a given state $s$,
would it be better to do an action $a \neq \pi(s)$?

It is better to switch to action $a$ for state $s$ if

$$q_\pi(s,a) > v_\pi(s)$$

# Policy Improvement Cont.

Do this for all states to get a new policy $\pi' \geq \pi$ that is **greedy** with respect to $v_\pi$ :

$$
\begin{aligned}
\pi'(s) &= \arg\max_a q_\pi(s,a) \\
&= \arg\max_a \mathbb{E}[R_{t+1} + \gamma v_\pi(S_{t+1}) \mid S_t = s, A_t = a] \\
&= \arg\max_a \sum_{s',r} p(s',r|s,a)\Big[r + \gamma v_\pi(s')\Big],
\end{aligned}
$$

What if the policy is unchanged by this?

Then the policy must be optimal!

# Policy Iteration

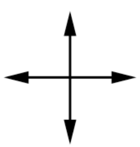
$$\pi_0 \xrightarrow{\text{E}} v_{\pi_0} \xrightarrow{\text{I}} \pi_1 \xrightarrow{\text{E}} v_{\pi_1} \xrightarrow{\text{I}} \pi_2 \xrightarrow{\text{E}} \cdots \xrightarrow{\text{I}} \pi_* \xrightarrow{\text{E}} v_*$$

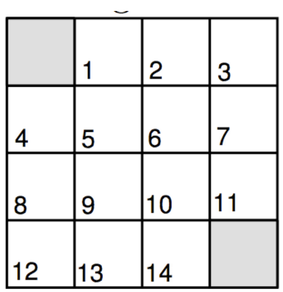
policy evaluation

policy improvement
“greedification”

# Greedy Policies for the Small Gridworld

$\pi =$ equiprobable random action choices




$R = -1$ on all transitions

$\gamma = 1$

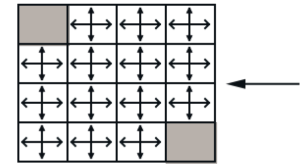
- An undiscounted episodic task
- Nonterminal states: 1, 2, . . ., 14;
- One terminal state (shown twice as shaded squares)
- Actions that would take agent off the grid leave state unchanged
- Reward is –1 until the terminal state is reached

$V_k$ for the Random Policy

Greedy Policy w.r.t. $V_k$

$k = 0$

| | | | |
|---|---|---|---|
| 0.0 | 0.0 | 0.0 | 0.0 |
| 0.0 | 0.0 | 0.0 | 0.0 |
| 0.0 | 0.0 | 0.0 | 0.0 |
| 0.0 | 0.0 | 0.0 | 0.0 |




$k = 1$

| | | | |
|---|---|---|---|
| 0.0 | -1.0 | -1.0 | -1.0 |
| -1.0 | -1.0 | -1.0 | -1.0 |
| -1.0 | -1.0 | -1.0 | -1.0 |
| -1.0 | -1.0 | -1.0 | 0.0 |

$k = 2$

| | | | |
|---|---|---|---|
| 0.0 | -1.7 | -2.0 | -2.0 |
| -1.7 | -2.0 | -2.0 | -2.0 |
| -2.0 | -2.0 | -2.0 | -1.7 |
| -2.0 | -2.0 | -1.7 | 0.0 |

$k = 3$

| | | | |
|---|---|---|---|
| 0.0 | -2.4 | -2.9 | -3.0 |
| -2.4 | -2.9 | -3.0 | -2.9 |
| -2.9 | -3.0 | -2.9 | -2.4 |
| -3.0 | -2.9 | -2.4 | 0.0 |

$k = 10$

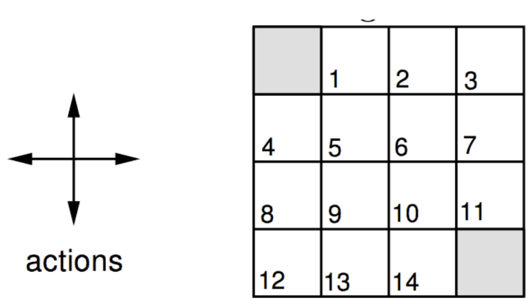
| | | | |
|---|---|---|---|
| 0.0 | -6.1 | -8.4 | -9.0 |
| -6.1 | -7.7 | -8.4 | -8.4 |
| -8.4 | -8.4 | -7.7 | -6.1 |
| -9.0 | -8.4 | -6.1 | 0.0 |

$k = \infty$

| | | | |
|---|---|---|---|
| 0.0 | -14. | -20. | -22. |
| -14. | -18. | -20. | -20. |
| -20. | -20. | -18. | -14. |
| -22. | -20. | -14. | 0.0 |

# Greedy Policies for the Small Gridworld

$\pi =$ equiprobable random action choices




$R = -1$
on all transitions

$\gamma = 1$

- An undiscounted episodic task
- Nonterminal states: 1, 2, . . ., 14;
- One terminal state (shown twice as shaded squares)
- Actions that would take agent off the grid leave state unchanged
- Reward is –1 until the terminal state is reached

$V_k$ for the Random Policy

$k = 0$

| | | | |
|---|---|---|---|
| 0.0 | 0.0 | 0.0 | 0.0 |
| 0.0 | 0.0 | 0.0 | 0.0 |
| 0.0 | 0.0 | 0.0 | 0.0 |
| 0.0 | 0.0 | 0.0 | 0.0 |

$k = 1$

| | | | |
|---|---|---|---|
| 0.0 | -1.0 | -1.0 | -1.0 |
| -1.0 | -1.0 | -1.0 | -1.0 |
| -1.0 | -1.0 | -1.0 | -1.0 |
| -1.0 | -1.0 | -1.0 | 0.0 |

$k = 2$

| | | | |
|---|---|---|---|
| 0.0 | -1.7 | -2.0 | -2.0 |
| -1.7 | -2.0 | -2.0 | -2.0 |
| -2.0 | -2.0 | -2.0 | -1.7 |
| -2.0 | -2.0 | -1.7 | 0.0 |

$k = 3$

| | | | |
|---|---|---|---|
| 0.0 | -2.4 | -2.9 | -3.0 |
| -2.4 | -2.9 | -3.0 | -2.9 |
| -2.9 | -3.0 | -2.9 | -2.4 |
| -3.0 | -2.9 | -2.4 | 0.0 |

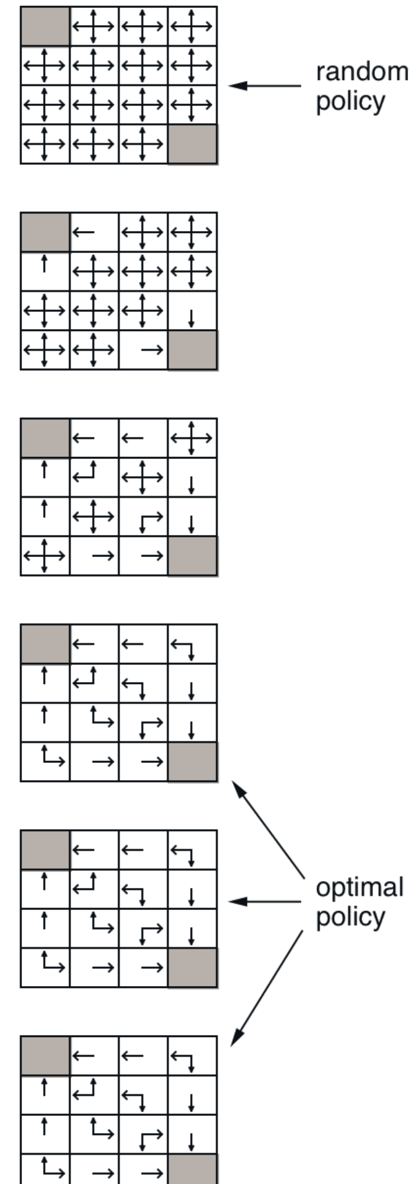
$k = 10$

| | | | |
|---|---|---|---|
| 0.0 | -6.1 | -8.4 | -9.0 |
| -6.1 | -7.7 | -8.4 | -8.4 |
| -8.4 | -8.4 | -7.7 | -6.1 |
| -9.0 | -8.4 | -6.1 | 0.0 |

$k = \infty$

| | | | |
|---|---|---|---|
| 0.0 | -14. | -20. | -22. |
| -14. | -18. | -20. | -20. |
| -20. | -20. | -18. | -14. |
| -22. | -20. | -14. | 0.0 |

Greedy Policy w.r.t. $V_k$



# Policy Iteration – One array version (+ policy)

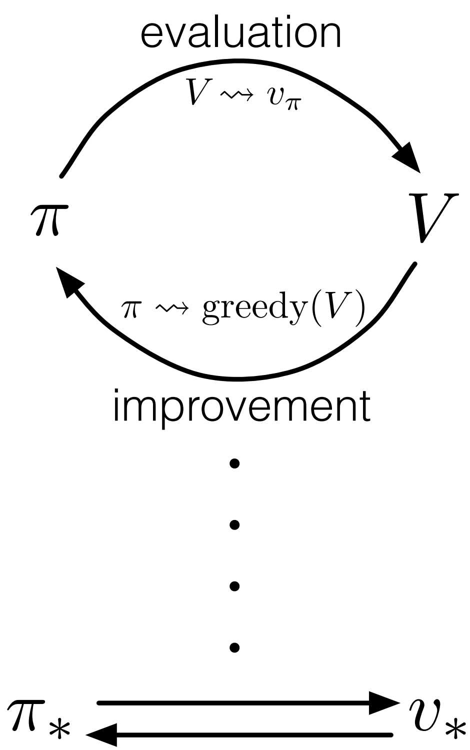
1. Initialization
   $V(s) \in \mathbb{R}$ and $\pi(s) \in \mathcal{A}(s)$ arbitrarily for all $s \in \mathcal{S}$

2. Policy Evaluation
   Repeat
   $\Delta \leftarrow 0$
   For each $s \in \mathcal{S}$:
   $v \leftarrow V(s)$
   $V(s) \leftarrow \sum_{s',r} p(s', r|s, \pi(s))\big[r + \gamma V(s')\big]$
   $\Delta \leftarrow \max(\Delta, |v - V(s)|)$
   until $\Delta < \theta$ (a small positive number)

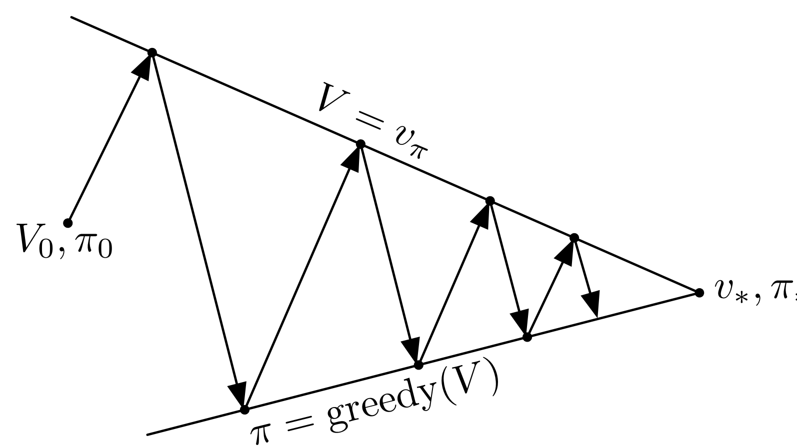
3. Policy Improvement
   *policy-stable* $\leftarrow$ *true*
   For each $s \in \mathcal{S}$:
   $a \leftarrow \pi(s)$
   $\pi(s) \leftarrow \arg\max_a \sum_{s',r} p(s', r|s, a)\big[r + \gamma V(s')\big]$
   If $a \neq \pi(s)$, then *policy-stable* $\leftarrow$ *false*
   If *policy-stable*, then stop and return $V$ and $\pi$; else go to 2

# Generalized Policy Iteration

**Generalized Policy Iteration** (GPI):
any interaction of policy evaluation and policy improvement, independent of their granularity.




A geometric metaphor for convergence of GPI:



# Value Iteration

Recall the **full policy-evaluation backup**:

$$v_{k+1}(s) = \sum_a \pi(a|s) \sum_{s',r} p(s',r|s,a)\Big[r + \gamma v_k(s')\Big] \qquad \forall s \in \mathcal{S}$$

Here is the **full value-iteration backup**:

$$v_{k+1}(s) = \max_a \sum_{s',r} p(s',r|s,a)\Big[r + \gamma v_k(s')\Big] \qquad \forall s \in \mathcal{S}$$

# Value Iteration – One array version

Initialize array $V$ arbitrarily (e.g., $V(s) = 0$ for all $s \in \mathcal{S}^+$)

Repeat
$\quad \Delta \leftarrow 0$
$\quad$ For each $s \in \mathcal{S}$:
$\quad\quad v \leftarrow V(s)$
$\quad\quad V(s) \leftarrow \max_a \sum_{s',r} p(s', r|s, a)\big[r + \gamma V(s')\big]$
$\quad\quad \Delta \leftarrow \max(\Delta, |v - V(s)|)$
until $\Delta < \theta$ (a small positive number)

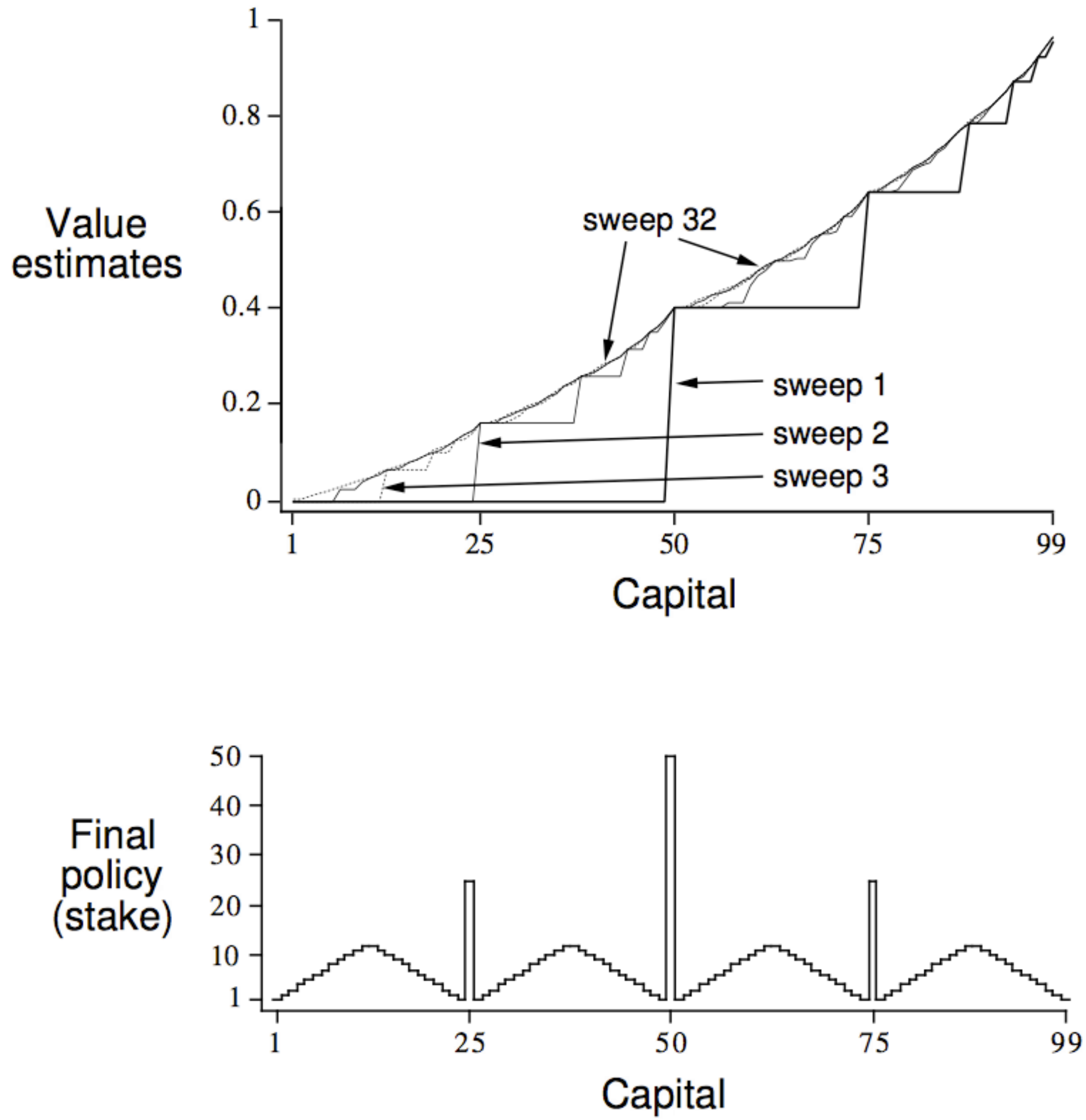
Output a deterministic policy, $\pi$, such that
$\quad \pi(s) = \arg\max_a \sum_{s',r} p(s', r|s, a)\big[r + \gamma V(s')\big]$

# Gambler's Problem

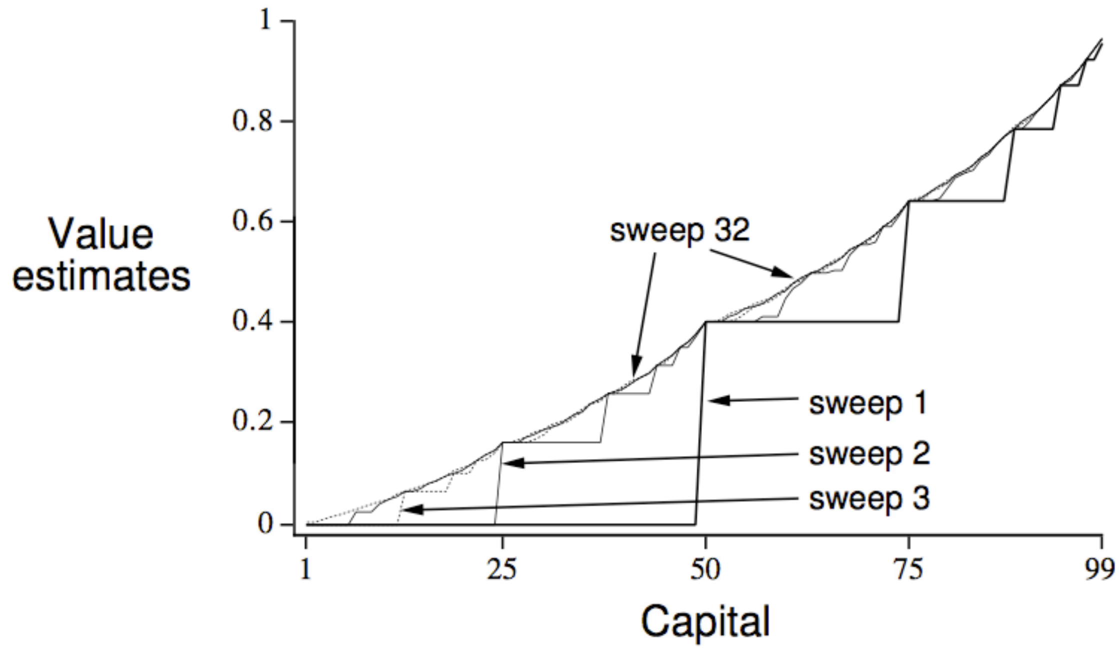
- Gambler can repeatedly bet $ on a coin flip
- Heads he wins his stake, tails he loses it
- Initial capital $\in$ {$1, $2, … $99}
- Gambler wins if his capital becomes $100
  loses if it becomes $0
- Coin is unfair
  - Heads (gambler wins) with probability $p = .4$

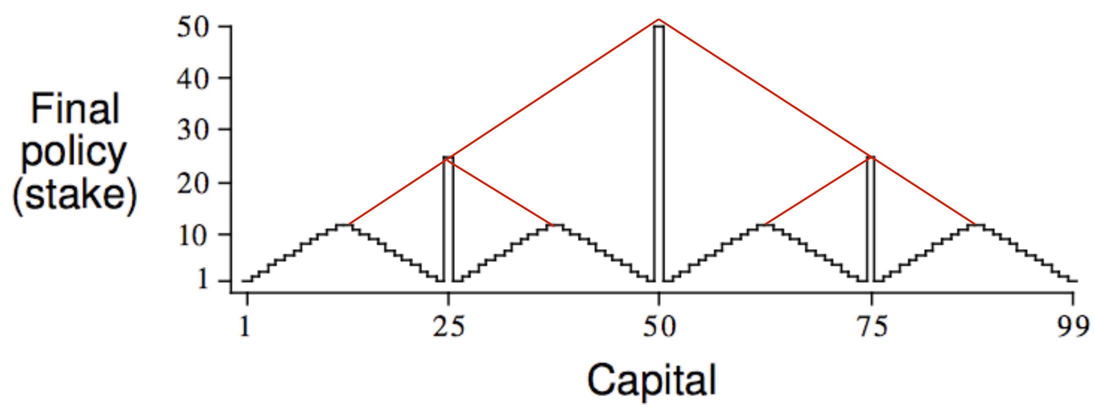
- States, Actions, Rewards? Discounting?

# Gambler's Problem Solution



# Gambler's Problem Solution



# Asynchronous Dynamic Programming

- All the DP methods described so far require exhaustive sweeps of the entire state set.
- Asynchronous DP does not use sweeps. Instead it works like this:
  - Repeat until convergence criterion is met:
    - Pick a state at random and apply the appropriate backup
- Still need lots of computation, but does not get locked into hopelessly long sweeps
- Can you select states to backup intelligently? YES: an agent's experience can act as a guide.

# Efficiency of DP

- To find an optimal policy is polynomial in the number of states…
- BUT, the number of states is often astronomical, e.g., often growing exponentially with the number of state variables (what Bellman called "the curse of dimensionality").
- In practice, classical DP can be applied to problems with a few millions of states.
- Asynchronous DP can be applied to larger problems, and is appropriate for parallel computation.
- It is surprisingly easy to come up with MDPs for which DP methods are not practical.

# Summary

- Policy evaluation: backups without a max
- Policy improvement: form a greedy policy, if only locally
- Policy iteration: alternate the above two processes
- Value iteration: backups with a max
- Full backups (to be contrasted later with sample backups)
- Generalized Policy Iteration (GPI)
- Asynchronous DP: a way to avoid exhaustive sweeps
- **Bootstrapping**: updating estimates based on other estimates
- Biggest limitation of DP is that it requires a *probability model* (as opposed to a generative or simulation model)