COMP/MATH 533: Algorithmic Game Theory	September 15, 2014

Lecture 4

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NOTE: The content of these notes has not been formally reviewed by the lecturer. It is recommended that they are read critically.

1 Myerson's Lemma Proof continued

Last lecture ended with the proof of these two statement of Myerson's Lemma:

1. An allocation rule x is implementable **if and only if** it is **monotone**.
2. If x is **monotone**, then there is a **unique** payment rule such that the sealed-bid mechanism (x, p) is DSIC.

We will now show that item 1 and 2 imply a DSIC mechanism. We will do this by showing that both overbidding and underbidding can be no better than bidding your true value and can in fact be worse.

Claim 1. *Underbidding hurts the bidder.*

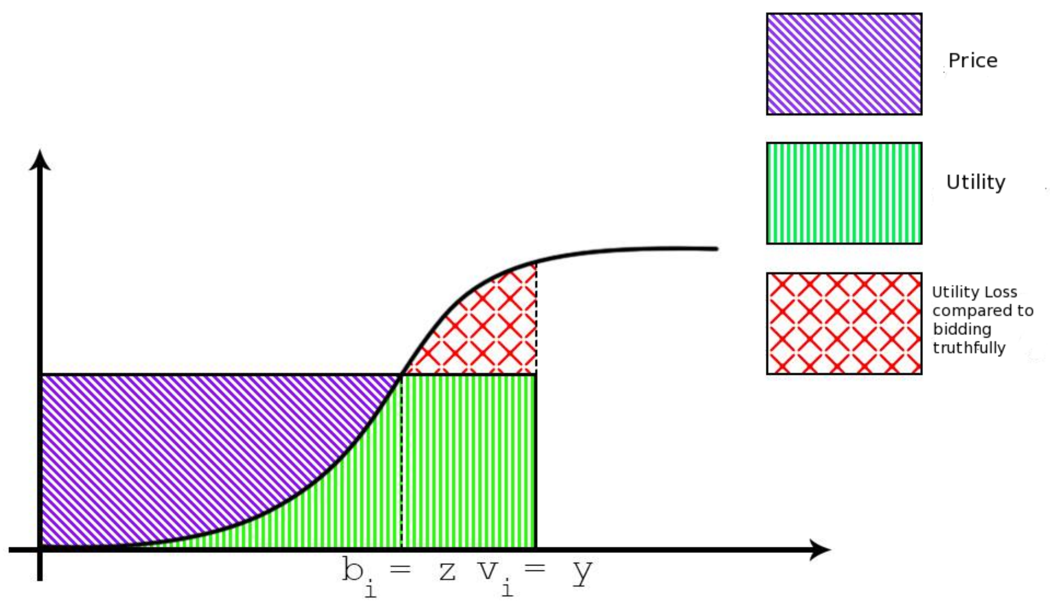

Figure 1: Underbidding case

As Figure 1 shows, the utility that the bidder receives when underbidding is at most the same as if they bid truthfully. In the above case since x() is monotone, there exists bids b_i, where if underbidding occurs, the utility is less than the utility if you were to bid your true value v_i

Claim 2. *Overbidding hurts the bidder.*

Figure 2 shows that the utility received from overbidding may be equal in special cases to the case when the bidder bids their true value. Since x() is monotone however, there exists bids b_i, where if overbid occurs, the utility is less than the utility if you were to bid your true value v_i.

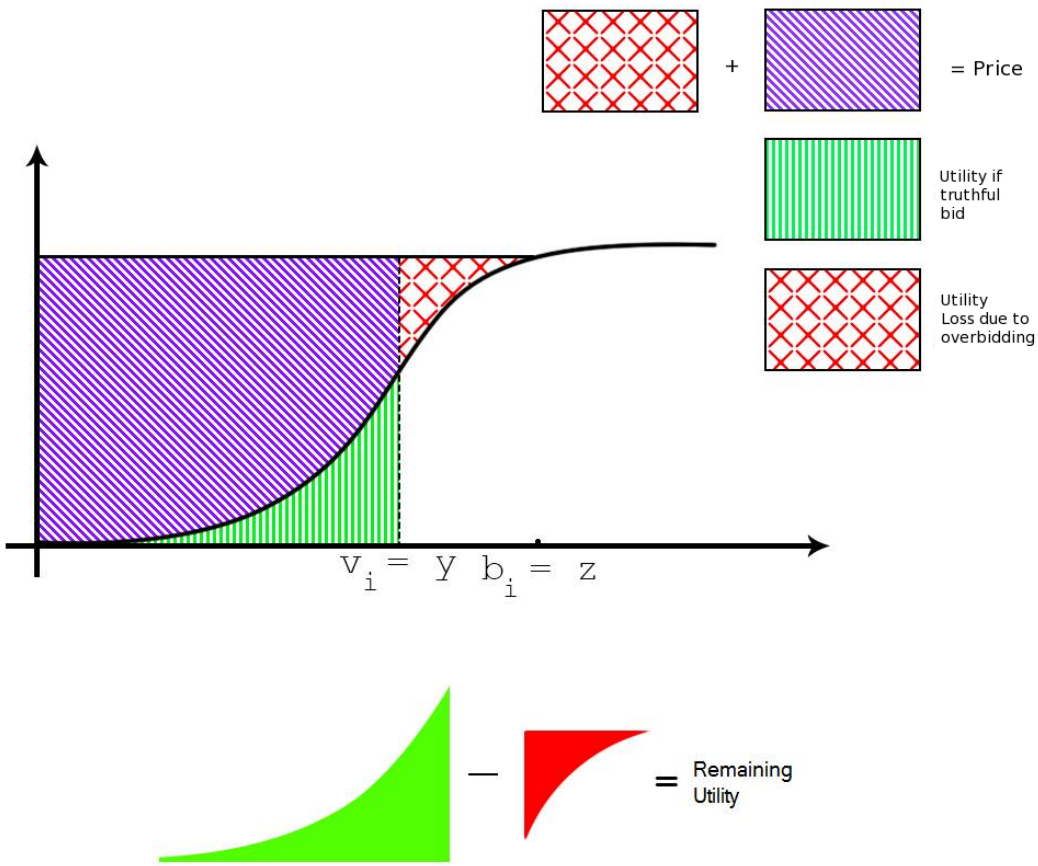

Figure 2: Overbidding case

As shown, if items 1 and 2 of the Lemma hold, both underbidding and overbidding are potentially harmful scenarios for the bidder in terms of maximizing the bidder's utility. That is, the cases described above are no better then just bidding the bidder's true value v_i. The utility and price of bidding truthfully is illustrated below.

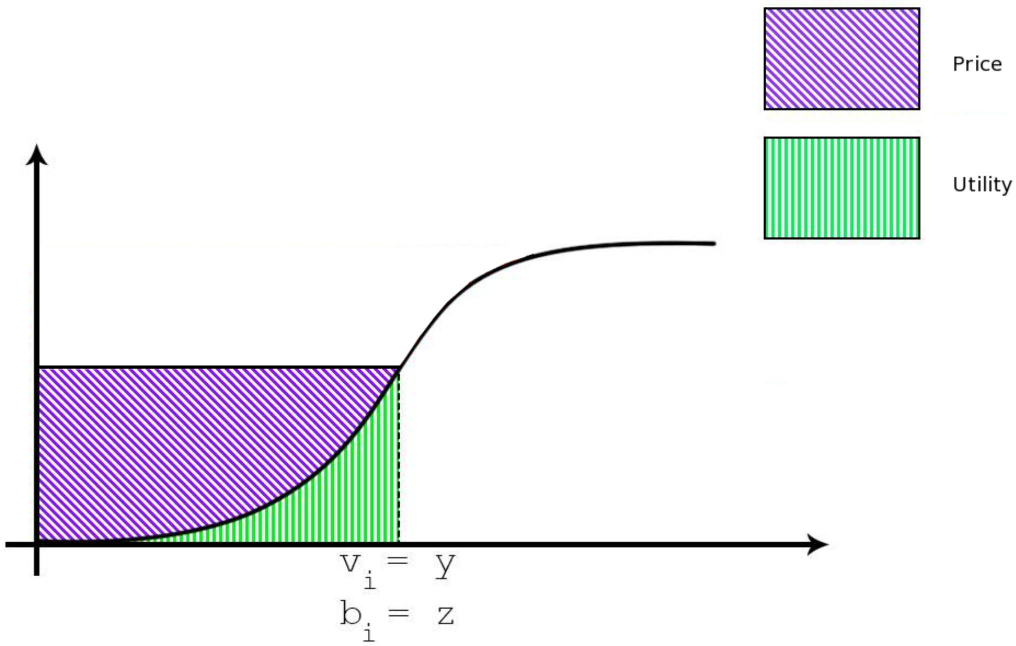

Figure 3: Bidding Truthfully

2 Applications of Myerson's Lemma

The following examples show how one can derive a payment rule from the allocation rule using Myerson's Lemma so that the mechanism is DSIC.

2.1 Single Item Auction

Allocation Rule: Give the item to the highest bid. Notice that the allocation rule is monotone: by fixing all other bids, your price in this auction can be illustrated using the graph in Figure 3 where B is the highest price among competitors. Given a monotone allocation function there are no incentives to lie about your true value: being the highest bidders secures you the item and maximizes your utility. The price you pay would be what is to the left of the graph and so the appropriate payment rule for such a setup is the second highest price for the item.

2.2 Sponsored Search

Allocation Rule: Give the highest slot to the bidder with the highest value. Remove the slot and bidder and iterate until all slots are filled. That is, we apply a greedy allocation rule. The rule is shown in the graph below. Again the function is monotone. The payment function is similar to the above Single Item Auction setup, but iterated and adjusted depending on position/item you have secured with your bid. That is,

$$if\ b \in (b_j, b_{j-1})$$

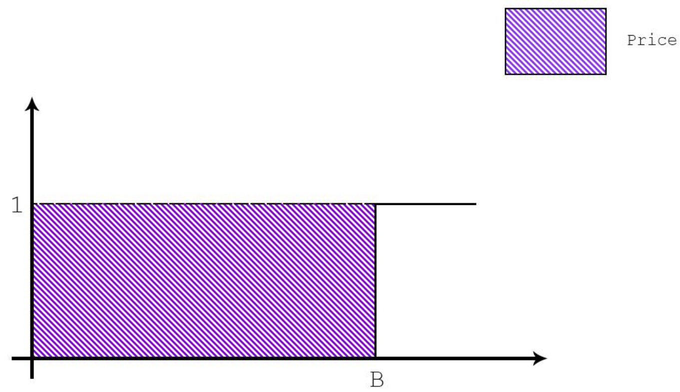

Figure 4: Single-item Auction

$$then\ P(b) = \sum_{j=1} (\alpha_j - \alpha_{j+1})$$

In turn, the above makes the allocation rule thruthful (i.e. ensures thruthful value reporting). The payment rule is an iterative version of the of the second price where the lowest bidder gets no items and the second-to last bidder pays the lowest bidder's price.

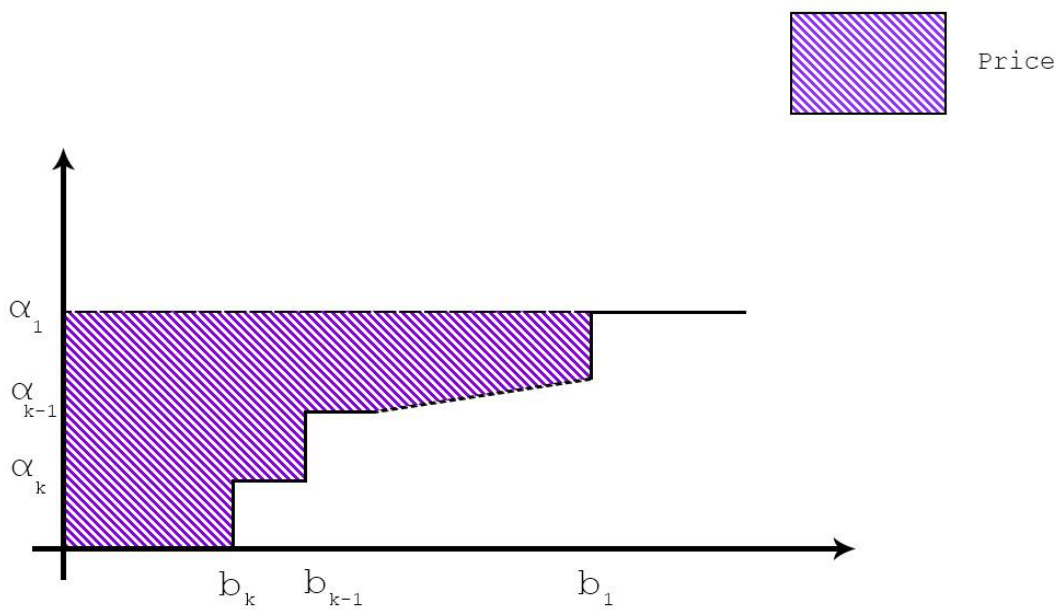

Figure 5: Sponsored Search Auctions

3 Revelation Principle

So far in class we've been discussing DSIC mechanisms. Why DSIC?

- They are easy for bidders to play as all they need to think about is their own value.
- They are easy for designers as the assumptions about player decision making are more obvious.

To think about: Can other mechanisms offer attractive features or characteristics that DSIC mechanisms cannot?

DSIC Mechanism Assumptions

1. There is a dominant strategy for everyone
2. Dominant Strategy is Direct Revelation: Players reveal all private information to the mechanism (eg. true value)

Remark 1. *There are mechanisms with 1, but not 2. (Take for example a Vickrey auction where the auctioneer takes players bids and multiplies them by 2 before choosing the highest. Bids here will be v/2).*

Assumption 1 Can we relax 1? In order to do so we need to make assumptions on bidder behaviour in order to be able to predict outcomes (eg. Nash Equilibrium). By relaxing 1 we may be able to get stronger results in certain settings that are not explored in class. With other mechanisms though, unlike DSIC, it is unclear if behaviour is predictable.

Assumption 2 Assumption 2 comes for "free".

Lemma 1. *Any mechanism with a dominant strategy M, there is an equivalent direct revelation DSIC mechanism M' where the best option for bids is to bid they true values.*

Proof: vi is bidder i's value. Si() is a dominant strategy (algorithm) for M.

Construct a wrapping mechanism that takes vi, applies Si(vi) and feeds that to M with the desired output. See the diagram for clarification. □

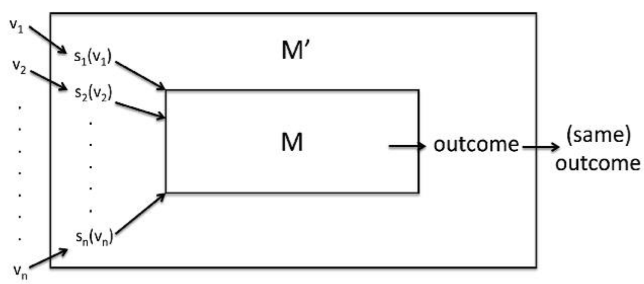

Figure 6: Proof by Simulation (Image taken from Tim Roughgarden notes on Game Theory)

Mechanism Design is not hard due to truth reporting but rather because it is hard to find an outcome in equilibrium. Each equilibrium concept leads to different mechanism design.

4 Revenue Maximization

We started with optimizing social welfare because it has many real world applications (government etc...). If you have a DSIC mechanism for optimizing welfare then it is as if you know the true values. This is not the case for other objective functions (eg. revenue).

4.1 One Bidder + One Item

In this setting, the only DSIC mechanism is to post a take it or leave it price R on the item. Depending on the bidder's value v, the revenue will be either R (if v is bigger than R) or 0. If the seller knows v, the obvious optimal price will be $R = v$. The fundamental problem here is that the there is no way for the seller to know v. Setting $R = 5$ will be optimal for $v = 5$ but performs poorly for $v = 15$. On the other hand, setting $R = 15$ fails terribly for the case of $v = 5$. This means there is no single mechanism that is optimal for all inputs. To argue about optimality, one has to be able to make tradeoffs between different inputs. The classical way to do this in Economics is to use Bayesian Analysis.

4.2 Bayesian Analysis

Assume a distribution F over the inputs (values) and using that make tradeoffs between different inputs, for the highest expected revenue. Note: The distribution for the inputs can be found using Machine Learning and statistical techniques in practice.

Expected revenue for a price r is

$$r(1 - F(r))$$

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eg. Assume F is uniform

$$r(1 - r) = 0$$

$$-2r + 1 = 0$$

$$r = 1/2$$

Note: The optimal r for the formula $r(1 - F(r))$ is referred to as the monopoly price.

4.3 Two Bidders + One Item

Assume data i.i.d from the $U[0, 1]$ distribution. Recall: We saw in Vickrey's Auction that r should be the expectation of the minimum of the two random variables which is 1/3. Can we do better?

$$r = 2/3$$

$$price = 2/3 \times (1 - (2/3)^2)$$

$$price = 10/27$$

Another Auction: Set a reserve at 1/2. The winning bid has to be greater then 1/2 and the price for the winning bid is $\max(1/2, \mathit{second\ bid})$.

$$r = 5/12 > 10/27 > 1/3$$

Is this the optimal one among all DSIC mechanisms? Yes! We will show how to design revenue-optimal auctions in the next lecture.