COMP/MATH 533: Algorithmic Game Theory	September 13, 2014

Lecture 3

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NOTE: The content of these notes has not been formally reviewed by the lecturer. It is recommended that they are read critically.

1 Introduction

Last class we saw the basics of mechanism design. A successful mechanism has three desirable properties:

1. **Dominant Strategy Incentive Compatible (DSIC):** Truthful bidding should be the dominant strategy for all participants, and a participant will never have negative utility by bidding the true value.
2. **Performance Guarantee:** In the case of sponsored search. Our goal is to maximize social welfare, i.e. if bidder i has value v_i for an item, and he/she receives x_i quantity of the item, the mechanism should maximize $\sum x_i v_i$ if the bidders report their values truthfully.
3. **Polynomial Running Time:** The auction should be implementable in polynomial time.

We showed last class that Vickrey auction has all these properties. This class discusses mechanism design for sponsored search auction, and this motivates the introduction of Myerson's Lemma.

2 Sponsored Search Auction

This is the setup of sponsored search auction:

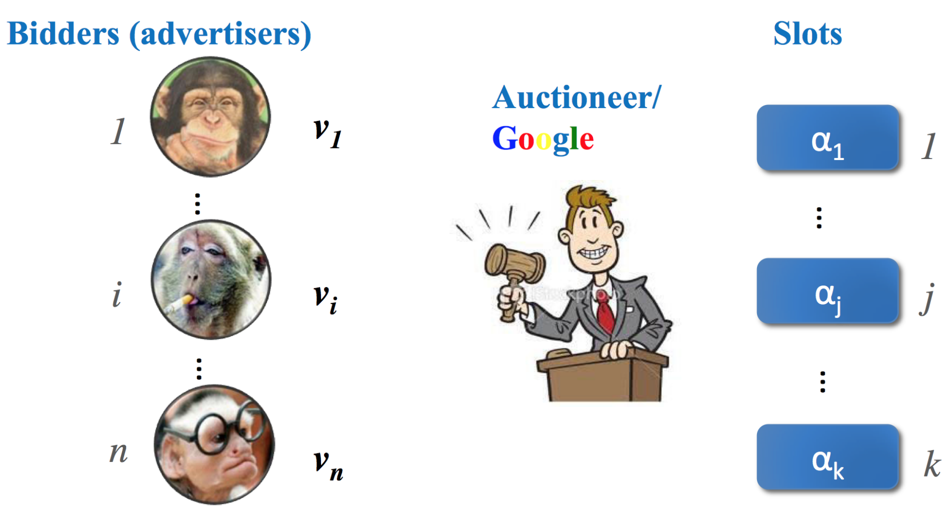

Figure 1: Sponsored search auction

- There are k slots for sale
- Slot j has CTR (Click-through rate) α_j

- Bidder i's value for slot j is $\alpha_j v_i$

We need to satisfy the three goals mentioned in the introduction.

Remark 1. *Why does a company care about social welfare? Because there are always competing companies, and if bidders are not happy in general, in the long run they would flock to other platforms.*

Coming up with a good mechanism that satisfies all these properties is non-trivial. We will take a two step approach:

1. Assuming the bidders bid truthfully, come up with an allocation rule to assign bidders to slots so that property (2) and (3) holds.
2. Given the allocation strategy, set a pricing strategy so that the dominant strategy of every bidder is to bid truthfully. This ensures the assumption of step 1 will hold.

Remark 2. *In general, when designing an auction there are two things to decide: allocation (who wins what), and pricing (who pays what).*

2.1 Allocation

Once we assume that all the bidders are bidding truthfully in the auction, the most natural allocation strategy is the greedy algorithm: assign the j-th highest bidder to the j-th highest slot.

Claim 1. *The greedy algorithm maximizes social welfare.*

Proof: Assume it does not. Then there exists a different allocation strategy A that provides the maximal welfare. Now in this new allocation, we must have two slots with CTR α_a and α_b assigned to bidder a and b who values the item v_a and v_b respectively, such that $\alpha_a > \alpha_b$ but $v_a < v_b$. Clearly, you can swap the slots for these two bidders and obtain a better social welfare, contradicting the maximality of A. $\square$

Thus the greedy allocation strategy satisfies property (2). Also, since sorting is polynomial time, clearly property (3) is satisfied.

2.2 Pricing

For single item auctions, we saw that the Vickrey aucitons satisfied all of our properties. The extension of Vickrey auctions to adopt to multiple item scenarios is known as the *generalized second-price auction.* In this auction, the j-th highest bidder gets the j-th highest slot (so it is the same as the allocation strategy we came up above), but the j-th highest bidder pays the price of the $(j+1)$-th highest bidder. (For example, the highest bidder will pay the price of the second highest bidder, and the second highest bidder will pay the price of the third highest bidder). Unfortunately this pricing strategy does not satisfy property (1).

Claim 2. *Generalized second-price auction is not DSIC.*

Proof: It suffices to provide a counterexample. Assume there are 3 bidders with values $v_1 = 7$, $v_2 = 6$, and $v_3 = 1$; and 2 slots with click-through rates $\alpha_1 = 1$ and $\alpha_2 = 0.4$. Let's assume for now bidder 2 and 3 bids truthfully, and analyze the behavior of bidder 1. If bidder 1 bids truthfully, she gets slot 1 and bidder 2 gets slot 2. Bidder 1 obtains value $v_1\alpha_1 = 7$, and pays 6, so her net utility is $7 - 6 = 1$. But if she decides to bid 5, then she gets the second slot and bidder 2 gets the first slot. She obtains value $v_1\alpha_2 = 2.4$, and pays 1. Her net utility becomes $2.4 - 1 = 1.4$, which is actually better than before. Bidder 1 is therefore no longer incentivized to bid truthfully. $\square$

To determine pricing strategy for sponsored search, we will resort to a fundamental result in Mechanism Design known as Myserson's Lemma.

3 Myerson's Lemma

We introduce some definitions before we jump into the statement of the Lemma.

Definition 1. *A* single dimensional environment *is defined as having n bidders, with each bidder i having a private valuation v_i representing its value "per unit of stuff". There is also a feasible set X. Each element of X is an n-dimensional vector $(x_1, x_2, \ldots, x_n)$, where x_i denotes the "amount of stuff" given to bidder i.*

A sponsored search auction is a single dimensional environment. If it has n bidders and m slots, the elements of feasible set X are n-dimensional vectors, and each coordinate of that vector is either 0 or an unique element of the set $\{\alpha_i : 1 \leq i \leq m\}$, where α_i is of course the CTR of slot i.

Single item sealed-bid auctions are also single dimensional. A *sealed bid* auction is an auction where every bidder bids only once and submits their bids in a sealed envelope, and thus each bidder is unaware of the bids of other bidders. Here, X is the set of 0-1 vectors, i.e. the set of vectors v such that at most one coordinate of v v is 1, and all other coordinates are 0.

Remark 3. *The environment is called* single dimensional *because each bidder only bids one amount. For example, in the sponsored search auction, each bidder bids one value b_i, instead of bidding separate amounts for each slot.*

Definition 2. *An allocation rule X for a single-dimensional environment is* implementable *if there is a payment rule P such that the sealed bid auction (X, P) is DSIC.*

Recall that you can consider a *mechanism* to be a two-dimensional vector (X, P), where X is the allocation rule and the P is the price rule. Thus the set of implementable allocation rules can be thought of as the set of projections (on the allocation rule axis) of all the viable mechanisms.

Definition 3. *An allocation rule X for a single-dimension environment is* monotone *if for every bidder i and bids b_{-i} by the other bidders, the allocation $x_i(z, b_{-i})$ to i is non-decreasing in its bid z, i.e. increasing one's bid only leads to getting more stuff.*

The Vickrey auction for one item has a monotone allocation rule: it allocates the item to the highest bidder, and it is impossible to win the item at a lower price, but lose it if you bid higher.

The greedy allocation rule for sponsored search is also monotone: x_i in the case of sponsored search is represented by the click-through rate of slot i, and by betting higher, you can only win a slot with a higher CTR.

Lemma 1 (Myerson's Lemma)**.** *The following holds for a single dimensional environment:*

(a) An allocation rule X is implementable if and only if *it is monotone.*

(b) If X is monotone, then there is a unique payment rule such that the sealed-bid mechanism (X, P) is DSIC [assuming the normalization that $b_i = 0$ implies $P_i(b) = 0$].

(c) The payment rule in (b), is given by an explicit formula.

Notations: Assume there are n bidders.

- X is allocation rule : set of bid vectors $\subseteq \mathbb{R}^n \to$ set of allocation vectors $\subseteq \mathbb{R}^n$
- P is pricing rule: set of bid vectors $\subseteq \mathbb{R}^n \to$ set of price vectors $\subseteq \mathbb{R}^n$
- v_i is bidder i's private value
- b_i is bidder i's bid
- b_{-i} is the bid of every bidder except i. Note $b_{-i} \in \mathbb{R}^{n-1}$
- $X(z) := X_i(z, b_{-i})$

- $P(z) := P_i(z, b_{-i})$

Proof: We first prove that if an allocation rule X is implementable, then it must be monotone.

By definition, if X is implementable, there exists a pricing rule P such that (X, P) is DSIC. We will analyze the behavior of particular bidder i. Fix b_{-i} (the bids of all the other bidders), and we will vary b_i (the bid of bidder i).

Pick any y and z such that $0 \le z \le y$. We analyze two separate cases: the one where bidder i bids more than his true value, and the one where bidder i bids less than his true value.

Case 1. Overbidding: Let $v_i = z$ and $b_i = y$, so that we have $b_i > v_i$. By DSIC, truthful bidding should yield maximum utility, and therefore,

$$\underbrace{X(z) \cdot z - P(z)}_{\substack{\text{Utility by bidding} \\ \text{true value } z}} \ge \underbrace{X(y) \cdot z - P(y)}_{\text{Utility by bidding } y} \tag{1}$$

Case 2. Underbidding: Let $v_i = y$ and $b_i = z$, so that we have $b_i < v_i$. Similarly by DSIC,

$$X(y) \cdot y - P(y) \ge X(z) \cdot y - P(z) \tag{2}$$

Manipulating and combining these two equations, we obtain:

$$(1) \iff P(y) - P(z) \ge z \cdot (X(y) - X(z))$$

$$(2) \iff X(y) \cdot y - X(z) \cdot y \ge P(y) - P(z) \iff y \cdot (X(y) - X(z)) \ge P(y) - P(z)$$

$$\therefore y \cdot (X(y) - X(z)) \ge P(y) - P(z) \ge z \cdot (X(y) - X(z)) \tag{3}$$

$$\therefore (y - z)(X(y) - X(z)) \ge 0$$

Therefore X is monotone.

Next we prove that under the reasonable assumption that $P(0) = 0$, if X is implementable (and therefore monotone), there exists at most one pricing rule P that makes (X, P) DSIC. In other words, we prove that X implementable $\Rightarrow$ P unique.

To make proof easier, assume $X(t)$ and $P(t)$ are differentiable, although the lemma can be proved without this assumption.

By dividing (3) by $y - z$, we get

$$y \cdot \frac{X(y) - X(z)}{y - z} \ge \frac{P(y) - P(z)}{y - z} \ge z \cdot \frac{X(y) - X(z)}{y - z}$$

then, take the limit as z goes to y

$$y \cdot X'(y) \ge P'(y) \ge y \cdot X'(y) \iff yX'(y) = P'(y)$$

We notice that

$$\int_0^y P'(t)dt = P(y) - P(0) = P(y) - 0 = P(y)$$

On the other hand,

$$\int_0^y P'(t)dt = \int_0^y t \cdot X'(t)dt = \int_0^y t \cdot d(X(t)) = t \cdot X(t)\Big|_0^y - \int_0^y X(t)dt = yX(y) - \int_0^y X(t)dt$$

Thus

$$P(y) = yX(y) - \int_0^y X(t)dt \tag{4}$$

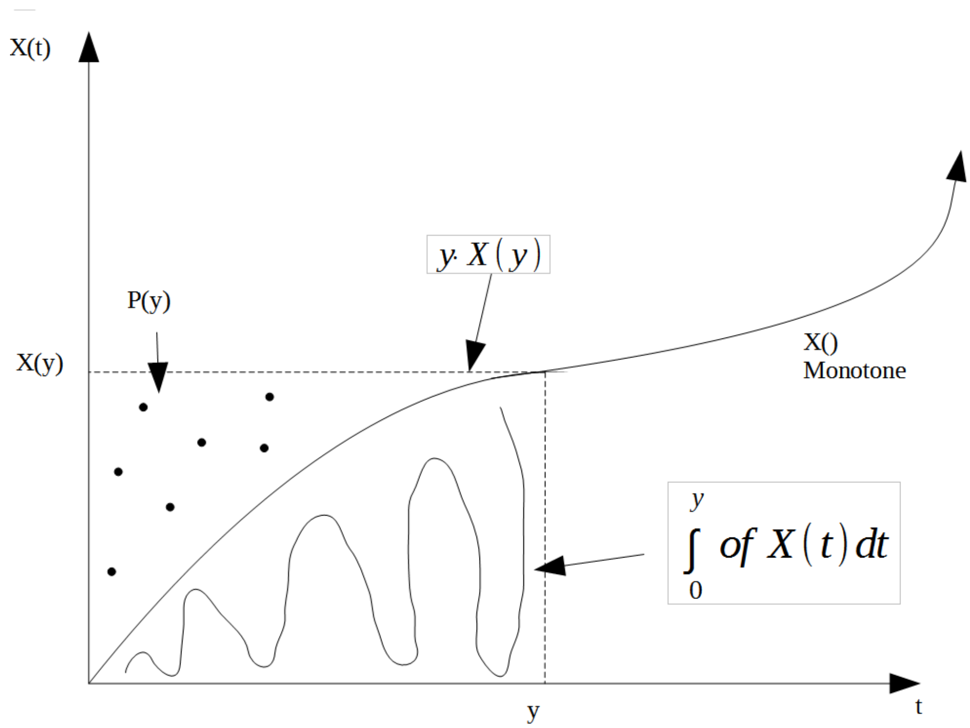

Figure 2: Graphical Illustration of Proof

which is clearly unique for any particular choice of X(t).

So far we have shown that if X is implementable, it must be monotone and it must have a unique pricing rule. It remains to show that if X monotone, X is implementable. We shall prove it by showing that (X, P) is DSIC, where P is the pricing given by equation (4). The proof is to be continued in the next lecture. $\square$