COMP/MATH 553 Algorithmic Game Theory
Lecture 5: Myerson's Optimal Auction

Sep 17, 2014

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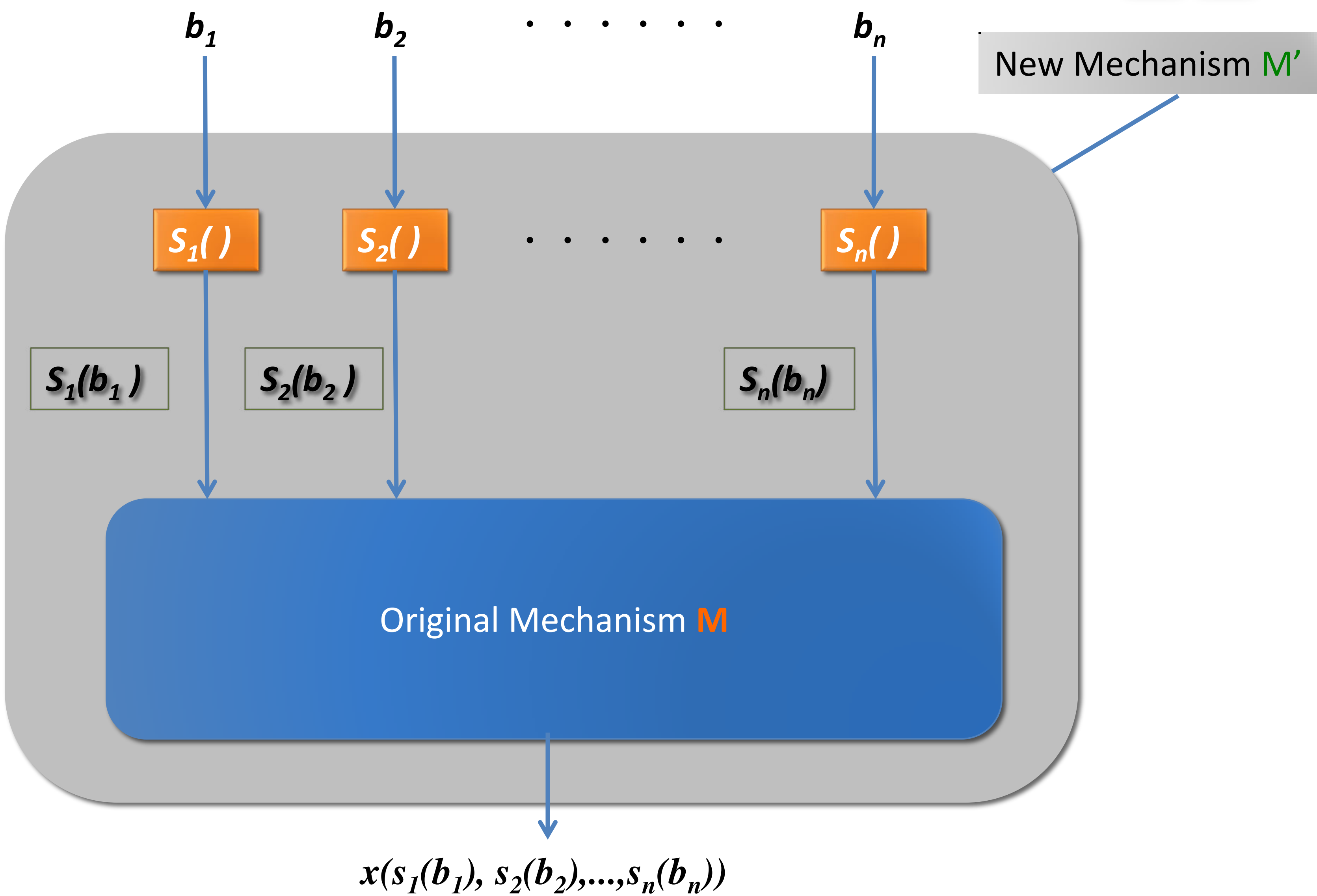
An overview of today's class

- *Expected Revenue = Expected Virtual Welfare*
- *2 Uniform [0,1] Bidders Example*
- *Optimal Auction*

Revelation Principle Recap



Revenue Maximization

Bayesian Analysis Model

- A single-dimensional environment, e.g. single-item

- The private valuation $\boldsymbol{v_i}$ of participant i is assumed to be drawn from a distribution $\boldsymbol{F_i}$ with density function $\boldsymbol{f_i}$ with support contained in $[0,\boldsymbol{v_{max}}]$.
  - We assume that the distributions $\boldsymbol{F_1, \ldots, F_n}$ are independent (not necessarily identical).
  - In practice, these distributions are typically derived from data, such as bids in past auctions.

- The distributions $\boldsymbol{F_1, \ldots, F_n}$ are known in advance to the mechanism designer. The realizations $\boldsymbol{v_1, \ldots, v_n}$ of bidders' valuations are private, as usual.

Revenue-Optimal Auctions

- **[Myerson ’81]**
  - Single-dimensional settings
  - Simple Revenue-Optimal auction

What do we mean by optimal?

- Step 0: What types of mechanism do we need to consider? In other words, optimal amongst what mechanisms?

- Consider the set of mechanisms that have a ***dominant strategy equilibrium***.

- Want to find the one whose revenue at the dominant strategy equilibrium is the ***highest***.

- A large set of mechanisms. How can we handle it?

- ***Revelation Principle*** comes to rescue! We only need to consider the direct-revelation DSIC mechanisms!

Expected Revenue = Expected Virtual Welfare

Revenue = Virtual Welfare

[Myerson '81  **]** For any single-dimensional environment.

Let $F = F_1 \times F_2 \times \dots \times F_n$ be the joint value distribution, and (x,p) be a DSIC mechanism. The expected revenue of this mechanism

$$E_{v\sim F}[\Sigma_i\ p_i(v)] = E_{v\sim F}[\Sigma_i\ x_i(v)\ \varphi_i\ (v_i)],$$

where $\varphi_i\ (v_i) := v_i - (1-F_i(v_i))/f_i(v_i)$ is called bidder i's virtual value (f_i is the density function for F_i).

Myerson's OPTIMAL AUCTION

Two Bidders + One Item

- Two bidders' values are drawn i.i.d. from U[0,1].

- Vickrey with reserve at ½
  - If the highest bidder is lower than ½, no one wins.
  - If the highest bidder is at least ½, he wins the item and pay max{1/2, the other bidder's bid}.

- Revenue 5/12.

- This is optimal. WHY???

Two Bidders + One Item

- Virtual value for v: $\varphi(v)= v- (1-F(v))/f(v) = v- (1-v)/1= 2v-1$

- Optimize expected revenue = Optimize expected ***virtual welfare***!!!

- Should optimize ***virtual welfare*** on ***every bid profile***.

- For any bid profile (v_1,v_2), what allocation rule optimizes virtual welfare? $(\varphi(v_1), \varphi(v_2))=(2v_1-1, 2v_2-1)$.
  - If $\max\{v_1,v_2\} \geq 1/2$, give the item to the highest bidder
  - Otherwise, $\varphi(v_1), \varphi(v_2) < 0$. Should not give it to either of the two.

- This allocation rule is monotone.

Revenue-optimal Single-item Auction

- Find the ***monotone*** allocation rule that optimizes expected ***virtual welfare***.

- Forget about ***monotonicity*** for a while. What allocation rule optimizes expected ***virtual welfare?***

- Should optimize ***virtual welfare*** on ***every bid profile*** ***v***.

  - max $\boldsymbol{\Sigma_i x_i(v)\ \varphi_i\ (v_i).}$, **s.t** $\boldsymbol{\Sigma_i x_i(v) \le 1}$

- Call this ***Virtual Welfare-Maximizing Rule***.

Revenue-optimal Single-item Auction

- Is the Virtual Welfare-Maximizing Rule ***monotone***?
- Depends on the distribution.
- **Definition 1 (Regular Distributions)**: A single-dimensional distribution F is ***regular*** if the corresponding virtual value function v- (1-F(v))/f(v) is non-decreasing.
- **Definition 2 (Monotone Hazard Rate (MHR))**: A single-dimensional distribution F has ***Monotone Hazard Rate***, if (1-F(v))/f(v) is non-increasing.

Revenue-optimal Single-item Auction

- What distributions are in these classes?
  - MHR: uniform, exponential and Gaussian distributions and many more.
  - Regular: MHR and Power-law...
  - Irregular: Multi-modal or distributions with very heavy tails.

- When all the F_i's are regular, the Virtual Welfare-Maximizing Rule is ***monotone!***

Two Extensions Myerson did (we won't teach)

- What if the distributions are irregular?
  - Point-wise optimizing virtual welfare is not monotone.
  - Need to find the allocation rule that maximizes expected virtual welfare among all monotone ones. Looks hard...
  - This can be done by "ironing" the virtual value functions to make them monotone, and at the same time preserving the virtual welfare.

- We restrict ourselves to DSIC mechanisms
  - Myerson's auction is optimal even amongst a much larger set of "Bayesian incentive compatible (BIC)" (essentially the largest set) mechanisms.
  - For example, this means first-price auction (at equilibrium) can't generate more revenue than Myerson's auction.

- Won't cover them in class.
  - Section 3.3.5 in "Mechanism Design and Approximation", book draft by Jason Hartline.
  - "Optimal auction design", the original paper by Roger Myerson.