Beluga$^{\mu}$: Programming proofs in context ...

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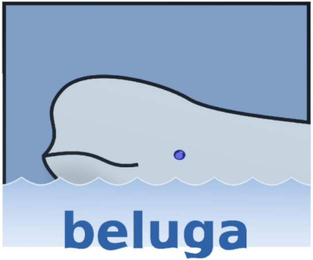

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- Formal systems (given via axioms and inference rules) play an important role when designing and implementing software.
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What are good meta-languages to program and reason with formal systems and proofs?

This talk

Design and implementation of Beluga

- Introduction
- Example: Simply typed lambda calculus
- Writing a proof in Beluga . . .
- Wanting more: . . .
  - Evaluation using closures
  - Normalization
- Conclusion

"The limits of my language mean the limits of my world."
- L. Wittgenstein

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Simply typed lambda-calculus

Types and Terms

$$\text{Types } T \; ::= \; \mathsf{nat} \mid \mathsf{arr}\, T_1 \; T_2 \qquad \text{Terms M} \; ::= \; x \mid \mathsf{lam}\; x{:}T.M \mid \mathsf{app}\; M \; N$$

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Typing rules (Gentzen-style, context-free)

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Context $\Gamma \;::=\; \cdot \mid \Gamma, x,\; \mathrm{oft}\;x\;T$ We are introducing the variable x together with the assumption $\mathrm{oft}\;x\;T$

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Typing Judgment: $\Gamma \vdash \text{oft } M\ T$ read as "M has type T in context Γ"

Typing rules

$$\frac{x, u :\ \text{oft } x\ T \in \Gamma}{\Gamma \vdash\ \text{oft } x\ T}\ u$$

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Any mechanization of proofs must deal with these issues; it is just a matter how much support one gets in a given meta-language.

Type uniqueness

Theorem

If $\mathcal{D} : \Gamma \vdash \mathsf{oft}\ M\ T$ and $\mathcal{C} : \Gamma \vdash \mathsf{oft}\ M\ S$ then $\mathcal{E} :\ \mathsf{eq}\ T\ S$.

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Induction on first typing derivation $\mathcal{D}$.

Case 1

$$\mathcal{D} = \dfrac{\overset{\mathcal{D}_1}{\Gamma, x, u{:}\ \mathsf{oft}\ x\ T \vdash \mathsf{oft}\ M\ S}}{\Gamma \vdash \mathsf{oft}\ (\mathsf{lam}\ x{:}T.M)\ (\mathsf{arr}\ T\ S)}\ \mathsf{t\_lam} \quad \mathcal{C} = \dfrac{\overset{\mathcal{C}_1}{\Gamma, x, u{:}\ \mathsf{oft}\ x\ T \vdash \mathsf{oft}\ M\ S'}}{\Gamma \vdash \mathsf{oft}\ (\mathsf{lam}\ x{:}T.M)\ (\mathsf{arr}\ T\ S')}\ \mathsf{t\_lam}$$

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Therefore there is a proof for eq (arr $T\ S$) (arr $T\ S'$) by reflexivity.

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Every variable x is associated with a unique typing assumption (property of the context), hence $v = u$ and $S = T$.

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Beluga$^\mu$: two level approach

Logical framework LF [HHP'93]

- Compact representation of formal systems and derivations
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Proof term language for first-order logic over a specifc domain (= contextual LF) together with domain-specific induction principle and recursive definitions

- Contextual LF: Contextual types characterize contextual objects [NPP'08]
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On paper proof	Proofs as functions in Beluga
Case analysis	Case analysis and pattern matching
Inversion	Pattern matching using let-expression
Induction Hypothesis	Recursive call

Step 1: Represent types and lambda-terms in LF

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| arr T_1 T_2

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LF representation in Beluga

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datatype tp:type =
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```
datatype oft: tm → tp → type =
| t_app: oft M (arr T S) → oft N T
    → oft (app M N) S
```

```
| t_lam: (Π x:tm.oft x T → oft (M x) S)
    → oft (lam T M) (arr T S);
```

Step 2a: Theorem as type

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Theorem

If $\mathcal{D} : \Gamma \vdash \mathsf{oft}\ M\ T$ and $\mathcal{C} : \Gamma \vdash \mathsf{oft}\ M\ S$ then $\mathcal{E} : \mathsf{eq}\ T\ S$.

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is represented as

Computation-level Type in Beluga

```
(Γ:ctx) [Γ ⊢ oft (M …) T] → [Γ ⊢ oft (M …) S] → [ ⊢ eq T S]
```

Read as: "For all contexts Γ of the schema `ctx`, ...

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- `[Γ ⊢ oft (M …)T]` and `[ ⊢ eq T S]` are contextual types [NPP'08].
- … describes dependency on context.
  `T` is a closed object `(M …)` is an object which may depend on context Γ.
- Contexts are structured sequences and are classified by context schemas

Step 2a: Theorem as type

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(Γ:ctx) [Γ ⊢ oft (M …) T] → [Γ ⊢ oft (M …) S] → [ ⊢ eq T S]
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- … describes dependency on context.
  `T` is a closed object `(M …)` is an object which may depend on context Γ.
- Contexts are structured sequences and are classified by context schemas
  schema `ctx = some [T:tp]` **block** `x:tm, u:oft x T.`

Step 2b: Proofs as Programs

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```
rec unique:(Γ:ctx)[Γ ⊢oft (M …)T] →[Γ ⊢oft (M …)S] →[ ⊢eq T S] =
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rec unique:(Γ:ctx)[Γ ⊢oft (M …)T] →[Γ ⊢oft (M …)S] →[ ⊢eq T S] =
fn d ⇒ fn c ⇒ case d of
```

Step 2b: Proofs as Programs

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fn  d  ⇒  fn  c  ⇒  case  d  of

| [Γ ⊢ t_app  (D1 …)  (D2 …)]  ⇒                    % Application Case
  let   [Γ ⊢ t_app  (C1 …)  (C2 …)]  = c  in
  let   [  ⊢  e_ref]  = unique  [Γ ⊢ D1 …]  [Γ ⊢ C1 …]  in
    [ ⊢  e_ref]
```

Step 2b: Proofs as Programs

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| [Γ ⊢ t_lam (λx.λu. D … x u) ⇒                  % Abstraction Case
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                            [Γ,b ⊢ C … b.1 b.2]  in
   [ ⊢ e_ref]

| [Γ ⊢ #q.2 …]  ⇒            % d  :  oft  (#q.1 …)  T       % Assumption Case
  let  [Γ ⊢ #r.2 …]  = c  in  % c  :  oft  (#r.1 …)  S
    [ ⊢ e_ref]  ;
```

Step 2b: Proofs as Programs

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    [ ⊢  e_ref]  ;
```

Recalll:

```
#q:block  x:tm,  u:oft  x  T
#r:block  x:tm,  u:oft  x  S
```

Step 2b: Proofs as Programs

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| [Γ ⊢#q.2 …] ⇒          % d : oft (#q.1 …) T       % Assumption Case
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    [ ⊢ e_ref] ;
```

Recalll:

`#q:`**`block`**` x:tm, u:oft x T`

`#r:`**`block`**` x:tm, u:oft x S`

We also know: `#r.1` = `#q.1`

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   [ ⊢ e_ref]

| [Γ ⊢#q.2 …] ⇒           % d : oft (#q.1 …) T        % Assumption Case
  let [Γ ⊢#r.2 …] = c in  % c : oft (#r.1 …) S
    [ ⊢ e_ref] ;
```

Recalll:

`#q:`**`block`**` x:tm, u:oft x T`

`#r:`**`block`**` x:tm, u:oft x S`

We also know: `#r.1` = `#q.1`

Therefore: `T` = `S`

Revisiting the design of Beluga

- Compact adequate representation of derivations and contexts

On paper proof	Implementation in Beluga [IJCAR'10]
Well-formed derivations	Dependent types
Renaming,Substitution	α-renaming, β-reduction in LF

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- Compact representation of proofs as functions [POPL'08,PPDP08]

Case analysis	Case analysis and pattern matching
Inversion	Pattern matching using let-expression
Induction Hypothesis	Recursive call

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Case analysis	Case analysis and pattern matching
Inversion	Pattern matching using let-expression
Induction Hypothesis	Recursive call

Comparison

- Twelf [Pf,Sch'99]: Encode proofs as relations
    - Requires lemma to prove injectivity of `arr` constructor.
    - No explicit contexts (cannot express types `T` and `S` and `eq T S` are closed)
    - Parameter case folded into abstraction case
- Delphin [Sch,Pos'08]: Encode proofs as functions
    - Requires lemma to prove injectivity of constructor
    - Cannot express that types `T` and `S` and `eq T S` are closed.
    - Variable carrying continuation as extra argument to handle context lookup
- Abella [Gacek'08], Tac[Baelde'10]: Proof assistants based on proof theory
    - Equality built-into the logic
    - Contexts are represented as lists
    - Requires lemmas about these lists (for example that all assumptions occur uniquely)

This talk

Design and implementation of Beluga

- Introduction
- Example: Simply typed lambda calculus
- Writing a proof in Beluga ...
- Wanting more ...
  - Evaluation using closures
  - Normalization
- Conclusion

Example: Evaluator using closures

- Lambda-terms and closures

$$\begin{array}{llll} \text{Terms} & M, N & := & x \mid \lambda x.M \mid M\ N \\ \text{Closures} & C & := & \mathsf{Cl}(x.M, \rho) \\ \text{Environment} & \rho & := & \cdot \mid \rho,\ (x, C) \end{array}$$

- Meaning of $\mathsf{Cl}(x.M, \rho)$: ρ provides instantiations for all the free variables in $x.M$.
- Environment ρ is a mapping from variables to closures

Example: Evaluator using closures

- Lambda-terms and closures

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- Meaning of $\mathsf{Cl}(x.M, \rho)$: ρ provides instantiations for all the free variables in $x.M$.
- Environment ρ is a mapping from variables to closures
- Evaluation : $(M,\ \rho) \Downarrow C$

$$\frac{\text{lookup } x\ \rho = C}{(x\ ,\ \rho) \Downarrow C} \qquad \frac{}{(\lambda x.M\ ,\ \rho) \Downarrow \mathsf{Cl}(x.M,\ \rho)}$$

$$\frac{(M_1,\ \rho) \Downarrow \mathsf{Cl}(x.N,\ \rho') \quad (M_2,\ \rho) \Downarrow C \quad (N,\ \rho', (x, C)) \Downarrow C'}{(M_1\ M_2,\ \rho) \Downarrow C'}$$

Representing terms, contexts and closures

LF representation in Beluga

```
 datatype tm: type =
   lam:  (tm → tm)  → tm
   app: tm → tm  → tm ;
schema ctx = tm;     % Define context schema
```

Representing terms, contexts and closures

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 datatype tm: type =
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   app: tm → tm   → tm ;
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```

Computation-level data types in Beluga

```
datatype Clos : ctype =
 Cl  : (ψ:ctx) [ψ, x:tm ⊢ tm] → ([ψ ⊢ tm] → Clos)→ Clos ;
```

Representing terms, contexts and closures

LF representation in Beluga

```
 datatype  tm: type  =
   lam:   (tm  →  tm)   →  tm
   app: tm  → tm   →  tm ;
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```

Computation-level data types in Beluga

```
datatype  Clos  :  ctype  =
 Cl   :  (ψ:ctx)  [ψ,  x:tm ⊢  tm]  →  ([ψ  ⊢  tm]  →  Clos)→  Clos  ;
```

Note: → is overloaded.

- `tm → tm` is the LF function space : binders in the object language are modelled by LF functions
- `[ψ ⊢tm] → Clos` is a computation-level function mapping variables of type `tm` in the context ψ to closures.

Representing terms, contexts and closures (revised)

LF representation in Beluga

```
 datatype  tm: type  =
   lam:   (tm → tm)   → tm
   app: tm → tm   → tm ;
schema ctx = tm;       % Define context schema
```

Computation-level data types in Beluga

```
datatype  Var :  {ψ:ctx} ctype =  V : {#p:[ψ ⊢ tm]} Var [ψ];
datatype  Clos : ctype  =
 Cl   : (ψ:ctx) [ψ, x:tm ⊢ tm] → (Var [ψ] → Clos)→ Clos ;
```

Representing terms, contexts and closures (revised)

LF representation in Beluga

```
 datatype tm: type =
   lam:  (tm → tm)  → tm
   app: tm → tm   → tm ;
schema ctx = tm;       % Define context schema
```

Computation-level data types in Beluga

```
datatype Var :  {ψ:ctx} ctype =  V : {#p:[ψ ⊢ tm]} Var [ψ];
datatype Clos : ctype =
 Cl  : (ψ:ctx) [ψ, x:tm ⊢ tm] → (Var [ψ] → Clos)→ Clos ;
```

Note: Index computation-level types [POPL'12]

- `Var [ψ]` is an indexed type
- `V : {#p:[ψ . tm]} Var [ψ]` defines a constructor `V` which takes variables of type `tm` in the context ψ as argument (Cast)

Evaluation using closures

Evaluation using closures

Define recursive program parametric in context

rec `eval:` $(\psi$`:ctx) [`$\psi \vdash$`tm]` $\rightarrow$ `(Var [`ψ`]` $\rightarrow$ `Clos)` $\rightarrow$ `Clos =`

Evaluation using closures

```
rec eval: (ψ:ctx) [ψ ⊢tm] → (Var [ψ] → Clos) → Clos =
fn e ⇒ fn env ⇒ case e of
```

Evaluation using closures

```
rec  eval: (ψ:ctx) [ψ ⊢tm] → (Var [ψ] → Clos)  → Clos =
fn  e ⇒ fn  env ⇒ case  e of
| [ψ ⊢#p …] ⇒env (V [ψ ⊢#p …])
```

Evaluation using closures

```
rec eval: (ψ:ctx) [ψ ⊢tm] → (Var [ψ] → Clos) → Clos =
fn e ⇒ fn env ⇒ case e of
| [ψ ⊢#p ...] ⇒env (V [ψ ⊢#p ...])
| [ψ ⊢lam λx. E ...x] ⇒ Cl [ψ,x:tm ⊢E ...x] env
```

Evaluation using closures

```
rec eval: (ψ:ctx) [ψ ⊢tm] → (Var [ψ] → Clos) → Clos =
fn e ⇒ fn env ⇒ case e of
| [ψ ⊢#p ...] ⇒env (V [ψ ⊢#p ...])
| [ψ ⊢lam λx. E ...x] ⇒ Cl [ψ,x:tm ⊢E ...x] env

| [ψ ⊢ app (E1 ...) (E2 ...)] ⇒
  let Cl [φ,x:tm ⊢ E ... x] env' = eval [ψ ⊢ E1 ...] env in
  let w                           = eval [ψ ⊢ E2 ...] env in
    eval [φ,x:tm ⊢ E ... x]
         (fn x ⇒ case x of
               | V [φ, x:tm ⊢ x]      ⇒ w
               | V [φ, x:tm ⊢ #p ... ] ⇒ env' (V [φ ⊢ #p ...])
         )
```

Evaluation using closures

```
rec eval: (ψ:ctx) [ψ ⊢tm] → (Var [ψ] → Clos)  → Clos =
fn e ⇒ fn env ⇒ case e of
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               | V [φ, x:tm ⊢ #p … ] ⇒ env' (V [φ ⊢ #p …])
         )
```

Features

- Pattern matching on contextual objects **and** computation-level data constructors
- Matching on contexts to lookup variables

Weak Normalization

- Good benchmark
  - Twelf, Delphin are too weak (to do it directly)
  - Coq/Agda lack support for substitutions and binders
  - Abella allows normalization proofs but lacks support for contexts

Weak Normalization

- Good benchmark
  - Twelf, Delphin are too weak (to do it directly)
  - Coq/Agda lack support for substitutions and binders
  - Abella allows normalization proofs but lacks support for contexts
- Weak normalization for simply typed lambda calculus

Theorem

If $\vdash M : A$ then M halts.

Proof.

1. Define reducibility candidate $\mathcal{R}_A$
2. If $M \in \mathcal{R}_A$ then M halts.
3. Backwards closed: If $M' \in \mathcal{R}_A$ and $M \longrightarrow M'$ then $M \in \mathcal{R}_A$.
4. Fundamental Lemma: If $\vdash M : A$ then $M \in \mathcal{R}_A$. (Requires a generalization)

□

Representing terms and evaluation in LF

Revisiting our definition of lambda-terms

```
datatype tm: tp → type =
| c   : tm i
| lam: (tm A → tm B) → tm (arr A B)
| app: tm (arr A B) → tm A → tm B;
```

Operational semantics

```
datatype mstep  : tm A → tm A → type =
| s/beta : mstep (app (lam M) N) (M N)
| s/app  : mstep M M’ → mstep (app M N) (app M’ N)
| s/refl : mstep M M
| s/trans: mstep M M’ → mstep M’ N → mstep M N;
```

A term M halts if there exists a value V s.t. $M \longrightarrow^* V$.

```
datatype halts : tm A → type =
| h/value : mstep M M’ → value M’ → halts M;
```

Reducibility Candidates

Reducibility candidates for terms $M \in \mathcal{R}_A$:

$$\begin{array}{lcl}
\mathcal{R}_{\mathbf{i}} & = & \{M \mid \texttt{halts}\ M\} \\
\mathcal{R}_{A \to B} & = & \{M \mid \texttt{halts}\ M \text{ and } \forall N \in \mathcal{R}_A, (M\ N) \in \mathcal{R}_B\}
\end{array}$$

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Computation-level data types in Beluga

```
datatype Reduce : {A:[ ⊢ tp]} {M:[ ⊢ tm A]} ctype =
| I   : [ ⊢ halts M] → Reduce [ ⊢ i] [⊢ M]
| Arr : [ ⊢ halts M] →
        ({N:[ ⊢ tm A]} Reduce [ ⊢ A] [ ⊢ N] → Reduce [ ⊢ B] [ ⊢ app M N])
     → Reduce [ ⊢ arr A B] [ ⊢ M];
```

- Not strictly positive definition, but stratified.

Reducibility Candidates

Reducibility candidates for terms $M \in \mathcal{R}_A$:

$$\begin{aligned} \mathcal{R}_{\mathbf{i}} &= \{M \mid \texttt{halts}\ M\} \\ \mathcal{R}_{A \to B} &= \{M \mid \texttt{halts}\ M \text{ and } \forall N \in \mathcal{R}_A, (M\ N) \in \mathcal{R}_B\} \end{aligned}$$

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| I   : [ ⊢ halts M] → Reduce [ ⊢ i] [⊢ M]
| Arr : [ ⊢ halts M] →
        ({N:[ ⊢ tm A]} Reduce [ ⊢ A] [ ⊢ N] → Reduce [ ⊢ B] [ ⊢ app M N])
     → Reduce [ ⊢ arr A B] [ ⊢ M];
```

- Not strictly positive definition, but stratified.

Reducibility candidates for substitutions $\sigma \in \mathcal{R}_\Gamma$:

```
datatype RedSub : (Γ:ctx){σ: ⊢ Γ} ctype =
| Nil : RedSub [ ⊢ ^ ]
| Cons : RedSub [ ⊢ σ] → Reduce [ ⊢ A] [ ⊢ M] → RedSub [ ⊢ σ M ];
```

Generalization of Fundamental Lemma

Lemma (Main lemma)

If $\Gamma \vdash M : A$ *and* $\sigma \in \mathcal{R}_\Gamma$ *then* $[\sigma]M \in \mathcal{R}_A$.

Proof.

Case $\dfrac{\Gamma, x{:}A \vdash M : B}{\Gamma \vdash \lambda x.M : A \to B}$

$[\sigma](\lambda x.M) = \lambda x.([\sigma, x/x]M)$ by properties of substitution

`halts` $(\lambda x.[\sigma, x/x]M)$ since it is a value

Suppose $N \in \mathcal{R}_A$.

$[\sigma, N/x]M \in \mathcal{R}_B$ by I.H. since $\sigma \in \mathcal{R}_\Gamma$

$[N/x][\sigma, x/x]M \in \mathcal{R}_B$ by properties of substitution

$(\lambda x.([\sigma, x/x]M))\ N \in \mathcal{R}_B$ by Backwards closure

Hence $[\sigma](\lambda x.M) \in \mathcal{R}_{A\to B}$ by definition

☐

Theorems as Computation-level Types

Lemma (Backward closed)

If $M \longrightarrow M'$ *and* $M' \in \mathcal{R}_A$ *then* $M \in \mathcal{R}_A$.

Computation-level Type in Beluga

```
rec closed : [⊢mstep M M'] → Reduce [⊢A] [⊢M'] → Reduce [⊢A] [⊢M] = ? ;
```

Lemma (Main lemma)

If $\Gamma \vdash M : A$ *and* $\sigma \in \mathcal{R}_\Gamma$ *then* $[\sigma]M \in \mathcal{R}_A$.

Computation-level Type in Beluga

```
rec main : {Γ:ctx}{M:[Γ ⊢ tm A]} RedSub [ ⊢ σ] → Reduce [ ⊢ A] [ ⊢ M σ] = ? ;
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Fundamental Lemma

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mlam Γ⇒mlam M ⇒fn rs ⇒ case [Γ ⊢ M …] of
| [Γ ⊢#p …] ⇒lookup [Γ] [Γ ⊢#p …] rs
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| [Γ ⊢#p ...] ⇒lookup [Γ] [Γ ⊢#p ...] rs
| [Γ ⊢ lam (λx. M1 ... x)] ⇒
  Arr [ ⊢ h/value s/refl v/lam]
   (mlam N ⇒ fn rN ⇒ closed [ ⊢ s/beta]
                                  (main [Γ,x:tm _] [Γ,x ⊢ M1 ... x] (Cons rs rN)))
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Fundamental Lemma

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| [Γ ⊢ app (M1 …) (M2 …)] ⇒
  let Arr ha f = main [Γ] [Γ ⊢ M1 …] rs in
  f [ ⊢ _ ] (main [Γ] [Γ ⊢ M2 …] rs)
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Fundamental Lemma

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| [Γ ⊢ c] ⇒ I [ ⊢ h/value s/refl v/c];
```

- Direct encoding of on-paper proof
- Equations about substitution properties automatically discharged (amounts to roughly a dozen lemmas about substitution and weakening)
- Total encoding about 75 lines of Beluga code

Other examples and comparison

- Other examples:
  - Weak normalization for which evaluates under lambda-abstraction
  - Algorithmic equality for LF (A. Cave) (draft available)

$\Longrightarrow$ Sufficient evidence that Beluga is ideally suited to support such advanced proofs

- Comparison (concentrating on the given weak normalization proof)
  - Coq/Agda formalization with well-scoped de Bruijn indices: dozen additional lemmas
  - Abella: 4 additional lemmas and diverges a bit from on-paper proof
  - Twelf: Too weak to for directly encoding such proofs; Implement auxiliary logic.

What have we achieved?

- Foundation for programming proofs in context (joint work with A. Cave [POPL'12])
  - Proof term language for first-order logic over contextual LF as domain
  - Uniform treatment of contextual types, context, ...
  - Modular foundation for dependently-typed programming with phase-distinction ⇒ Generalization of DML and ATS
  - Non-termination or effects are allowed, although we often want to concentrate on pure total programs.
- Extending contextual LF with first-class substitutions and their equational theory (joint work with A. Cave [LFMTP'13])
- Rich set of examples
  - Type-preserving compiler for simply typed lambda-calculus (joint work with O. Savary Belanger, S. Monnier [CPP'13])
  - (Weak) Normalization proofs (A. Cave)
- Latest release in Jan'14: Support for indexed data types, first-class substitutions, equational theory behind substitutions

This talk

Design and implementation of Beluga

- Introduction
- Example: Simply typed lambda calculus
- Writing a proof in Beluga . . .
- Wanting more . . .
    - Evaluation using closures
    - Normalization
- Conclusion

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Conclusion

Beluga$^{\mu}$: programming proofs in context

- Level 1: Contextual LF
  - Supports for specifying formal systems in LF
  - Embed contexts and contextual LF objects into computations and types
  - First-class substitution and contexts together with rich equational theory
- Level 2: Functional programming language supporting indexed types
  - Pattern match and manipulate contextual LF objects
  - Proof terms language for first-order logic over contextual LF
  - Supports indexed recursive types

$\Longrightarrow$ Elegant and compact framework for programming proofs.

"A language that doesn't affect the way you think about programming, is not worth knowing." *- Alan Perlis*

Current work

- Prototype in OCaml (ongoing)
  (providing an interactive programming mode)
- Structural recursion (S. S. Ruan, A. Abel)
  Develops a foundation of structural recursive functions for Beluga; proof of normalization; prototype implementation under way
- Coinduction in Beluga (D. Thibodeau)
  Extending work on simply-typed copatterns [POPL'13] to Beluga
- Case study:
  - Certified compiler (O. Savary Belanger, CPP'13)
  - Proof-carrying authorization with constraints (Tao Xue)
- Extending Beluga to full dependent types (A. Cave)
- Type reconstruction for dependently typed programs (F. Ferreira)

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- ORBI - Benchmarks for comparing systems supporting HOAS encodings (A. Felty, A. Momigliano)

The end

Thank you!

Download prototype and examples at

`http://complogic.cs.mcgill.ca/beluga/`

Current Belugians: Brigitte Pientka, Mathias Puech, Tao Xue, Olivier Savary Belanger, Andrew Cave, Francisco Ferreira, Stefan Monnier, David Thibodeau, Sherry Shanshan Ruan, Shawn Otis