Note on Ramsey theory

Theorem 1. *Amongst 6 people, there are always either*

- *3 mutual acquaintances, or*
- *3 mutual non-acquaintances.*

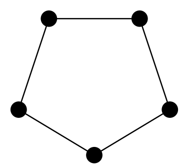
Here, we are assuming that if A is acquainted with B then B is also acquainted with A.
We give an informal proof.

Proof. Let A, B, C, D, E, F be 6 people. Then either A knows at least 3 other people or A doesn't know at least 3 of the other people.

In the first case, if any two people that A knows also know each other then these two people together with A are 3 mutual acquaintances. Otherwise, no two people that A knows are acquainted with each other. Since there are at least 3 people that A doesn't know, we have (at least) 3 mutual non-acquaintances.

The second case is the same as the first case by switching the notion of acquaintance and non-acquaintance. $\square$

Note that in the above theorem, we cannot replace 6 by 5 because of the following counter-example.



We would like to reformulate this last theorem in terms of graphs.

Definition 1. Let $G = (V, E)$ be a graph.
A *clique* in G is a subset U of V such that there is an edge between every pair of vertices in U.
A *stable set* in G is a subset U of V such that there are no edge between any pair of vertices in U.

Thus, our original theorem now becomes the following.

Theorem 2. *A graph with 6 vertices contains either a clique of size 3 or a stable set of size 3 (or both).*

We could ask a more general question. How many vertices do we need if we want to be guarantee a clique of size s or a stable set of size t?

Definition 2. The *Ramsey number* $R(s, t)$ is the minimum number such that every graph with (at least) $R(s, t)$ vertices contains either a clique of size s or a stable set of size t.

At first it is not even clear that $R(s, t)$ is well defined for all s and t. It could be that for some values, there are graphs as large as we want that do not contain a clique of size s or a stable set of size t.
Amazingly, these numbers do exists.
First we verify this for some easy cases.

Lemma 1. *For $s \geq 2$, $R(s, 2) = s$ and for $t \geq 2$, $R(2, t) = t$.*

Proof. A graph G with no cliques of size 2 has no edges. So it contains a stable set of size $|V(G)|$ consisting of all of its vertices. Therefore, a graph with at least t vertices contains a clique of size 2 or stable set of size t. So, $R(2, t) = t$.

A graph G with no stable set of size 2 has all edges. So it contains a clique of size $|V(G)|$ consisting of all of its vertices. Therefore, a graph with at least s vertices contains a stable set of size 2 or clique of size s. So, $R(s, 2) = s$. $\square$

Lemma 2. *If $s \geq 3, t \geq 3$ then $R(s,t) \leq R(s-1,t) + R(s,t-1)$.*

Proof. We follow the idea of the proof for $s = 3$ and $t = 3$. That is, if a person A knows at least half the other people then we can complete the proof of the theorem if amongst the people that A know, there are either

- 2 mutual acquaintances, or
- 3 mutual non-acquaintances.

Let G be a graph with $R(s-1,t) + R(s,t-1)$ vertices. We will show that G contains either a clique of size s or a stable set of size t.

Let v be any vertex in G. There are $R(s-1,t) + R(s,t-1) - 1$ vertices other than v. Therefore, either

- there are $R(s-1,t)$ vertices that v is adjacent to or
- there are $R(s,t-1)$ that v is not adjacent to.
- In the first case, by definition of $R(s-1,t)$, $N(v)$ contains either a clique of size $s-1$ or a stable set of size t.
  - If $N(v)$ contains a clique S of size $s-1$ then $S \cup \{v\}$ is a clique of size s since v is adjacent to every vertex in $C \subseteq N(v)$.
  - If $N(v)$ contains a stable set T of size t then we have found a stable set of size t in G (namely, T).
- In the second case, by definition of $R(s,t-1)$, $V \setminus N(v)$ contains either a clique of size s or a stable set of size $t-1$.
  - If $V \setminus N(v)$ contains a clique S of size s then we have found a clique of size s in G (namely, S).
  - If $V \setminus N(v)$ contains a stable set of size t then $T \cup \{v\}$ is a stable set of size t since v is not adjacent to any vertex in $S \subseteq (V \setminus N(v))$.

Thus in all cases, we have found a clique of size s or a stable set of size t in G. □

Theorem 3. *If $s \geq 2, t \geq 2$ then $R(s,t) \leq 2^{s+t}$.*

Proof. We prove this by induction on $s + t$.

By the previous lemma, $R(s,2) = s \leq 2^{s+2}$ for all $s \geq 2$ and $R(2,t) = s \leq 2^{2+t}$ for all $t \geq 2$.

Now suppose the lemma is true when $s + t = k - 1$. We want to prove the theorem for $s \geq 2, t \geq 2$ where $s + t = k$. If $s = 2$ or $t = 2$, we have just checked that the theorem is true in that case. If $s \geq 3$ and $t \geq 3$, then by Lemma 2,

$$R(s,t) \leq R(s-1,t) + R(s,t-1)$$

. By induction,

$$R(s-1,t) + R(s,t-1) \leq 2^{s-1+t} + 2^{s+t-1} = 2 * 2^{s+t-1} = 2^{s+t}$$

Thus, we have proven the theorem by induction. □

Theorem 4. *$R(k,k) \geq 2^{k/2}$ for any $k \geq 3$.*

Proof. In some sense, we will count all stable set and all cliques in graphs one $n < 2^{s/2}$ vertices and show that there is not "enough" stable set and cliques for all graphs to contain at least one.

Define a bipartite graph H with parts A and B as follows. A consist of vertices labelled by all labelled graphs on n vertices (labelled $\{1, 2, \ldots, n\}$. B consist of vertices labelled by all subsets of size k of $\{1, 2, \ldots, n\}$.

Thus, $|A| = 2^{\binom{n}{2}}$ and $|B| = \binom{n}{k}$.

There is an edge between a vertex labelled by G in A and a vertex labelled by the subset S in B if and only if S is a clique or stable set in G.

Note that the degree of every vertex in B is the same (because we can permute the elements $\{1, 2, \ldots, n\}$. For a vertex labelled by S in B, we can count all graph containing S as a clique. We know that all such graphs contain all edges between vertices of S. And then the between any other pair of vertices (as long as both are not in S), the graph could either have an edge or no edge. There are $2^{\binom{n}{2}-\binom{k}{2}}$ such "free" pairs (since there are $\binom{n}{2}$ pairs in total and $\binom{k}{2}$ pairs with both vertices in S).

We can make a similar count for graphs containing S as a stable set. Furthermore, since $k \geq 2$, no graph contains S as a stable set and a clique.

Thus, the degree of every vertex in B is $2 * 2^{\binom{n}{2}-\binom{k}{2}}$. By a lemma proven earlier,

$$\sum_{v \in A} \deg(v) = \sum_{v \in B} \deg(v) = 2 * 2^{\binom{n}{2}-\binom{k}{2}} * \binom{n}{k}$$

This is the total degree of vertices in A. Therefore, the average degree of a vertex in A is

$$\frac{2 * 2^{\binom{n}{2}-\binom{k}{2}} * \binom{n}{k}}{2^{\binom{n}{2}}} = 2 * 2^{-\binom{k}{2}} \binom{n}{k}$$

We will prove in a lemma that this number is less than 1. But first, we complete the proof of the theorem given this lemma.

Since the average degree of vertices in A is less than one, there exists a vertex of degree 0 in A. By our definition of the bipartite graph H, the graph labelling a vertex of degree 0 contains no clique of size k or stable set of size k.

We have therefore proven the theorem since this graph exists. $\square$

Lemma 3. *If $n < 2^{k/2}$ then*

$$2 * 2^{-\binom{k}{2}} \binom{n}{k} < 1$$

Proof.

$$\binom{n}{k} = \frac{n(n-1)(n-2)\ldots(n-k+2)(n-k+1)}{k(k-1)(k-2)\ldots 2 * 1} \leq \frac{n * n * n * \ldots * n * n}{2 * 2 * 2 * \ldots * 2 * 1} = \frac{n^k}{2^{k-1}}$$

Therefore,

$$\begin{aligned} 2 * 2^{-\binom{k}{2}} \binom{n}{k} &\leq 2 * 2^{-\binom{k}{2}} \frac{n^k}{2^{k-1}} \\ &< 2 * 2^{-\binom{k}{2}} \frac{\left(2^{k/2}\right)^k}{2^{k-1}} \\ &= 2^{1-\frac{k(k-1)}{2}+\frac{k^2}{2}-(k-1)} \\ &= 2^{1+\frac{k}{2}-(k-1)} \\ &= 2^{2-\frac{k}{2}} \end{aligned}$$

This is at most 1 when $k \geq 4$. But one of these inequalities is strict so the lemma is true when $k \geq 4$. For $k = 3$, $n \leq 3$ so $\binom{n}{k} \leq 1$. Now, $2 * 2^{-3} = 2^{-2} < 1$. $\square$