Common mistakes in proofs

Here are a list of the most common mistakes seen in proofs (at this level). I am posting this here in hopes that it will prevent people from making them early on. I will add more when they come to mind. There is no shame in making these mistakes at first. They are common mistakes after all. Just make sure you replace a faulty proof with a correct one in that case.

1. **Assuming the conclusion.** Suppose we want to infer Q from the premise P. A common mistake is to assume Q (either at the beginning of the proof or somewhere in the middle). Another related mistake is to assume Q, prove some statements and arrive at Q (again). Of course, this does not prove $P \vdash Q$ (and it does not help in the proof).

   The only case where this could possibly be useful is when we are trying to find (an easier to prove) equivalent statement R to Q so that we can prove R first and then prove Q. A conclusion is also sometimes assumed to help determine if the argument is valid. But again, such an assumption would never go into the final proof (in the idea of the proof, maybe).

2. **Proving T.** We have seen a proof technique called "proofs by contradiction". Such a proof consists of assuming the negation of a statement we are trying to prove to be true and arriving at a contradiction ($\mathbf{F}$). This is a valid proof.

   However, assuming the statement we are trying to prove and not arriving at a contradiction does not constitute a valid proof. In fact, if we think about this in terms of logic, this method would be summarized as $P \to \mathbf{T} \vdash P$. This is absurd since $P \to \mathbf{T}$ is always true (just apply the proof of $P \to Q, \neg Q \vdash \neg P$ to $P = P$, $Q = \mathbf{T}$). Thus, $P \to \mathbf{T} \vdash P$ is the same as saying $\vdash P$. In other words, we could prove *any* statement we wanted using this method. This is clearly unsound.

   A variant of this mistake involves "multiple conclusions" where all conclusions are assumed and are then "check against each other" to see that no immediate contradiction arises.

3. **"Upwards" induction.** Say we want to prove some statement (predicate) holds for all graphs. Suppose we want to show this by induction on the number of vertices of a graph G. We start by proving some base case (say, when G has no vertices).

   The correct method then assumes that the property (we want to prove) holds for all graphs on $n-1$ vertices and from this, proves that an arbitrary graph G on n vertices has the property. The hypothesis is used by constructing graphs on $n-1$ vertices from G in some way (e.g., using vertex deletion, edge contraction, vertex deletion while adding some edges, etc).

   The incorrect method also assumes that the property holds for all graphs on $n-1$ vertices. But then a graph G on $n-1$ vertices is taken (and assumed to have the property, which is correct) and augmented to a graph H on n vertices (e.g., by adding vertex, splitting a vertex, subdividing an edge, etc). Then H is show to have the property (we are trying to show is true for all graphs).

   This method is incorrect since it does not necessarily prove that every graph on n vertices has the property. We could "miss" a graph because we did not obtain it from a smaller graph via whichever augmentation we decided to use.

   In the above, we could replace "graph" with any object and induction on the number of vertices can be replaced by induction on any (integer valued) parameter on these objects.