Note on the pigeonhole principle

Theorem 1 (Pigeonhole principle)**.** *If we put more than n objects into n boxes then there is a box containing at least 2 objects.*

Proof. Suppose the theorem is false. That means we can put more than n objects into n boxes and have at most one object per box. The total number of objects is the sum over all boxes i of the number object B_i in box i. Now,

$$\sum_{i=1}^{n} B_i \leq \sum_{i=1}^{n} 1 = n * 1$$

which is not more than n. Contradiction (to the number of objects we have). □

Example 1.

Theorem 2. *Every graph with at least 2 vertices contains 2 vertices of the same degree.*

Proof. First note that all vertices of a graph G on n vertices have degrees between 0 and n (inclusively). Second, note that no graph with at least 2 vertices has both a vertex u of degree 0 and a vertex v of degree $n-1$ (if they both existed, is there an edge between u and v?).

Thus, in any graph with at least 2 vertices, all degrees are either a subset of $\{0, 1, \ldots, n-2\}$ or $\{1, \ldots, n-1\}$. Both of these sets have size $n-1$. Therefore, since we have n vertices, by the pigeonhole principle, there are two vertices of the same degree. □

Example 2. (Rosen, p.348 example 4)

Theorem 3. *For every integer n, there is a positive multiple kn of k whose decimal representation contains only 0 and 1.*

Proof. Consider the number $1, 11, 111, \ldots, 11\ldots111$ where the last number contains $n+1$ digits. The remainder of these numbers (and in fact any number) when divided by n can take on values in $\{0, 1, 2, \ldots, n-1\}$. Thus, by the pigeonhole principle, there are two numbers which have the same remainder. Therefore, the difference between them has a remainder of 0 when divided by n. Furthermore, the difference between the larger of the two number and the smaller of the two contains only 0 and 1 in its decimal representation. □

Theorem 4 (Generalized pigeonhole principle)**.** *If we put k objects into n boxes then there is a box containing at least $\lceil \frac{n}{k} \rceil$ objects.*

Proof. Suppose the theorem is false. That means we can put k objects into n boxes and have at most $\lceil \frac{n}{k} \rceil - 1$ objects per box. The total number of objects is the sum over all boxes i of the number object B_i in box i. Now,

$$\sum_{i=1}^{n} B_i \leq \sum_{i=1}^{n} \left(\left\lceil \frac{n}{k} \right\rceil - 1 \right) = n \left(\left\lceil \frac{n}{k} \right\rceil - 1 \right)$$

which is less than n. Contradiction (to the number of objects we have). □