Note on the inclusion-exclusion principle

Problem 1 (Hats problem)**.** 8 people enter a restaurant leaving their hats at the front. When leaving, each person takes a random hat.

Q1 How many "ways" can they leave in (looking only at person/hats combinations)?

Q2 In how many ways could they all end up with someone else's hat?

We can think of each "way" (or set of choices) the 8 people can make as a function f from the 8 people P to themselves. We interpret $f(p) = q$ as "p took q's hat".

Of course, two people cannot be wearing the same hat and every hat is taken (since there are 8 hats). Therefore, question 1 is just asking for the number of bijections from P to P.

Theorem 1. *The number of bijections from A to A is $n!$ where $|A| = n$.*

Proof. We can obtain a bijection from $A = \{a_1, a_2, \ldots, a_n\}$ to A by

- first choosing $f(a_1)$ (we have n choices here, namely all elements of A), and
- then choosing $f(a_2)$ (we have $n-1$ choices here, namely all elements of A except $f(a_1)$), and
- then choosing $f(a_3)$,
- and so on.

The number of choices we have is the product of the number of choices we had at each step. This is $n!$.

We now need to show that our procedure chooses every bijection and every bijection exactly once.

Given a bijection g, we can simply look at $g(a_1)$ and make that as our first choice. Then look at $g(a_2)$ and make that as our second choice. And so on. Until the function we have chosen is exactly g.

If we make two different sets of choices to build function f_1 and f_2 then there is a first step (say, step i), where we make a different choice. Thus, $f_1(a_i) \neq f_2(a_i)$ since we do not change our decision about $f(a_i)$ after the ith step. So $f_1 \neq f_2$ as functions.

Therefore our procedure chooses every bijection exactly once and thus the number of bijections is $n!$. □

Now to answer the second question. The second question asks for the number of functions where we do not have $f(p) = p$ for any person p. We can define these notions more formally as follows.

Definition 1. A *fixed point* of a function $f : A \to A$ is an element $a \in A$ such that $f(a) = a$.

Definition 2. A *derangement* is a function $f : A \to A$ with no fixed points.

Thus, we want to know the number of derangements from A to A. However, this answer is not as simple. To answer this question, we make use of the inclusion-exclusion principle.

Theorem 2. (Inclusion-exclusion principle) *For any n sets $S_1, S_2, \ldots, S_n$,*

$$|S_1 \cup S_2 \cup \ldots \cup S_n| = \sum_{i=1}^{n} |S_i| - \sum_{1 \leq i < j \leq n} |S_i \cap S_j| + \sum_{1 \leq i < j < k \leq n} |S_i \cap S_j \cap S_k| - \ldots + (-1)^{n+1} |S_1 \cap S_2 \cap \ldots \cap S_n|$$

Proof. We prove this theorem by induction on n.

For $n = 2$, this formula is simply $|S_1 \cup S_2| = |S_1| + |S_2| - |S_1 \cap S_2|$. To prove this, we note that any element is in exactly one of

1. S_1 and S_2,
2. S_1 but not S_2,

3. S_2 but not S_1, or
4. Neither S_1 nor S_2

(by determining if the element is in S_1 and then determining if it is in S_2).

- In the first case, the element is counted once on the left hand side and $1+1-1=1$ time on the right hand side.
- In the second case, the element is counted once on the left hand side and once on the right hand side.
- In the third case, the element is counted once on the left hand side and once on the right hand side.
- In the fourth case, the element is counted zero times on the left hand side and zero times on the right hand side.

This proves the theorem when $n=2$ (since the count is correct for all elements).

Now suppose that the theorem is true for $n-1$ with $n>2$. We will prove the theorem is true for any n sets $S_1, \ldots, S_n$.

$$|S_1 \cup S_2 \cup \ldots \cup S_n| = |S_1 \cup S_2 \cup \ldots \cup S_{n-1}| + |S_n| - |(S_1 \cup S_2 \cup \ldots \cup S_{n-1}) \cap S_n|$$

by applying the theorem for the case $n=2$. Here the first set is $S_1 \cup S_2 \cup \ldots \cup S_{n-1}$ and the second set is S_n.

Now,

$$\begin{aligned}
& |S_1 \cup S_2 \cup \ldots \cup S_{n-1}| + |S_n| - |(S_1 \cup S_2 \cup \ldots \cup S_{n-1}) \cap S_n| \\
= & |S_1 \cup S_2 \cup \ldots \cup S_{n-1}| + |S_n| - |((S_1 \cap S_n) \cup (S_2 \cap S_n) \cup \ldots \cup (S_{n-1} \cap S_n))| \\
= & \sum_{i=1}^{n-1} |S_i| - \sum_{1 \le i < j \le n-1} |S_i \cap S_j| + \sum_{1 \le i < j < k \le n-1} |S_i \cap S_j \cap S_k| - \ldots + (-1)^n |S_1 \cap S_2 \cap \ldots \cap S_{n-1}| \\
+ & |S_n| - |((S_1 \cap S_n) \cup (S_2 \cap S_n) \cup \ldots \cup (S_{n-1} \cap S_n))|
\end{aligned}$$

by applying induction to the first term.

Similarly, we can apply induction to the second term.

$$\begin{array}{lllll}
\sum_{i=1}^{n-1} |S_i| & - \sum_{1 \le i < j \le n-1} |S_i \cap S_j| & + \sum_{1 \le i < j < k \le n-1} |S_i \cap S_j \cap S_k| & - \ldots & +(-1)^n |S_1 \cap S_2 \cap \ldots \cap S_{n-1}| \\
+|S_n| & +|((S_1 \cap S_n) \cup \ldots \cup (S_{n-1} \cap S_n))| & & & \\
= \sum_{i=1}^{n-1} |S_i| & - \sum_{1 \le i < j \le n-1} |S_i \cap S_j| & + \sum_{1 \le i < j < k \le n-1} |S_i \cap S_j \cap S_k| & - \ldots & +(-1)^n |S_1 \cap S_2 \cap \ldots \cap S_{n-1}| \\
+|S_n| & - \sum_{i=1}^{n-1} |S_i \cap S_n| & + \sum_{1 \le i < j \le n-1} |(S_i \cap S_j) \cap S_n| & - \ldots & + \ldots \\
+(-1)^{n+1} & |(S_1 \cap S_n) \cap \ldots \cap (S_{n-1} \cap S_n)| & & & \\
= \sum_{i=1}^{n} |S_i| & - \sum_{1 \le i < j \le n} |S_i \cap S_j| & + \sum_{1 \le i < j < k \le n} |S_i \cap S_j \cap S_k| & - \ldots & +(-1)^{n+1} |S_1 \cap S_2 \cap \ldots \cap S_n|
\end{array}$$

Since $S_1, \ldots, S_n$ was arbitrary, we have proven the theorem by induction. $\square$

We can now apply this to the hats problem

Theorem 3. *The number of derangements from A to A where $|A| = n$ is*

$$\sum_{i=0}^{n} (-1)^i \frac{n!}{i!}$$

Proof. Let $A = \{a_1, \dots, a_n\}$.

Let S_i be the set of all bijections from A to A where a_i is a fixed point.

Let T be the set of all bijections from A to A.

The set of all derangements is then simply $T \setminus (S_1 \cup S_2 \cup \ldots \cup S_n)$.

By the inclusion-exclusion principle,

$$|S_1 \cup S_2 \cup \ldots \cup S_n| = \sum_{i=1}^{n} |S_i| - \sum_{1 \le i < j \le n} |S_i \cap S_j| + \sum_{1 \le i < j < k \le n} |S_i \cap S_j \cap S_k| - \ldots + (-1)^{n+1} |S_1 \cap S_2 \cap \ldots \cap S_n|$$

Now, for any i, $|S_i| = (n-1)!$ (the number of bijections from $A \setminus \{a_i\}$ to itself). Similarly, for any i, $|S_i \cap S_j| = (n-2)!$ (the number of bijections from $A \setminus \{a_i, a_j\}$ to itself) and so on.

$$\begin{aligned} |S_1 \cup S_2 \cup \ldots \cup S_n| &= \sum_{i=1}^{n} (n-1)! - \sum_{1 \le i < j \le n} (n-2)! + \sum_{1 \le i < j < k \le n} (n-3)! - \ldots + (-1)^{n+1} 1! \\ &= \binom{n}{1}(n-1)! - \binom{n}{2}(n-2)! + \binom{n}{3}(n-3)! - \ldots + (-1)^{n+1}\binom{n}{n} 1! \\ &= \sum_{i=1}^{n} (-1)^{i+1} \binom{n}{i} (n-i)! \end{aligned}$$

Note that $|T| = n! = (-1)^2 \binom{n}{0} (n-0)!$, so we obtains

$$\begin{aligned} |T| - |S_1 \cup S_2 \cup \ldots \cup S_n| &= (-1)^2 \binom{n}{0}(n-0)! - \sum_{i=1}^{n} (-1)^{i+1} \binom{n}{i} (n-i)! \\ &= (-1)^2 \binom{n}{0}(n-0)! + \sum_{i=1}^{n} (-1)^{i+2} \binom{n}{i} (n-i)! \\ &= \sum_{i=0}^{n} (-1)^{i+2} \binom{n}{i} (n-i)! \\ &= \sum_{i=0}^{n} (-1)^{i} \binom{n}{i} (n-i)! \\ &= \sum_{i=0}^{n} (-1)^{i} \frac{n!}{i!(n-i)!} (n-i)! \\ &= \sum_{i=0}^{n} (-1)^{i} \frac{n!}{i!} \end{aligned}$$

□