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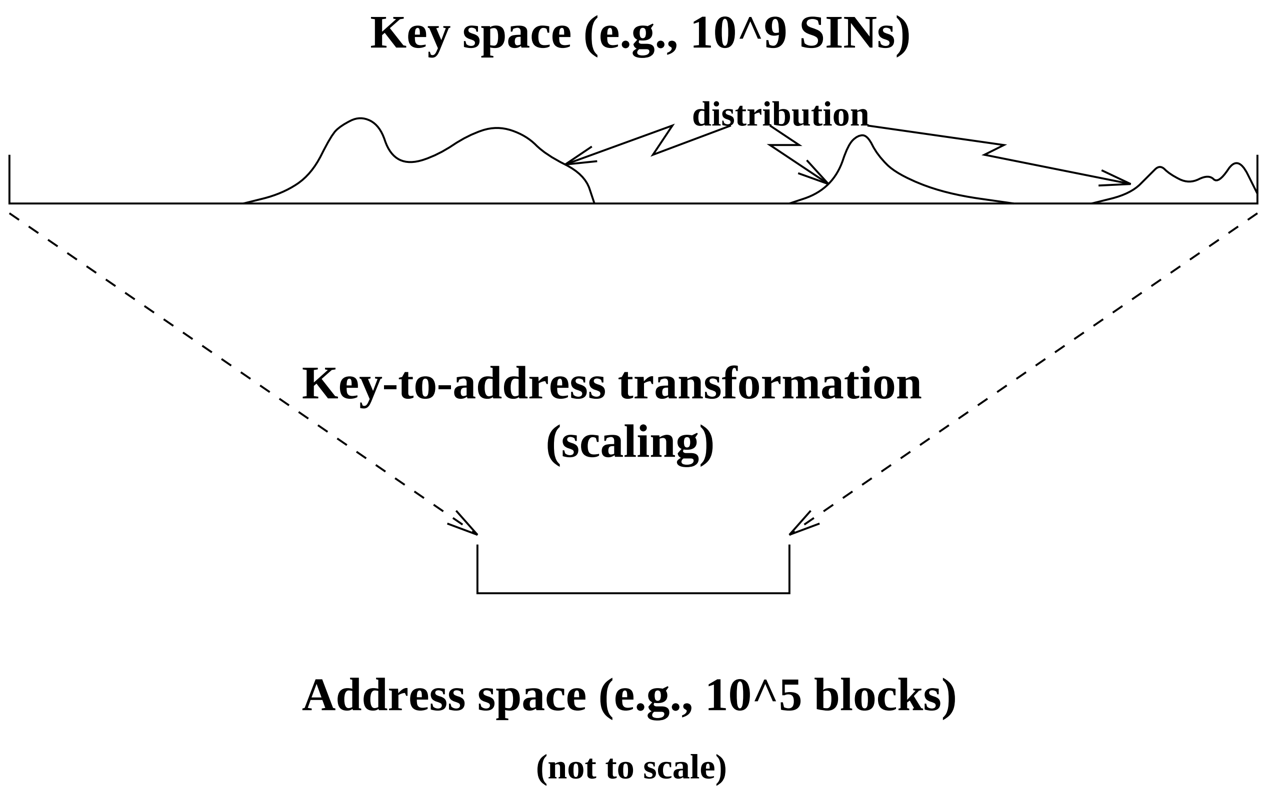
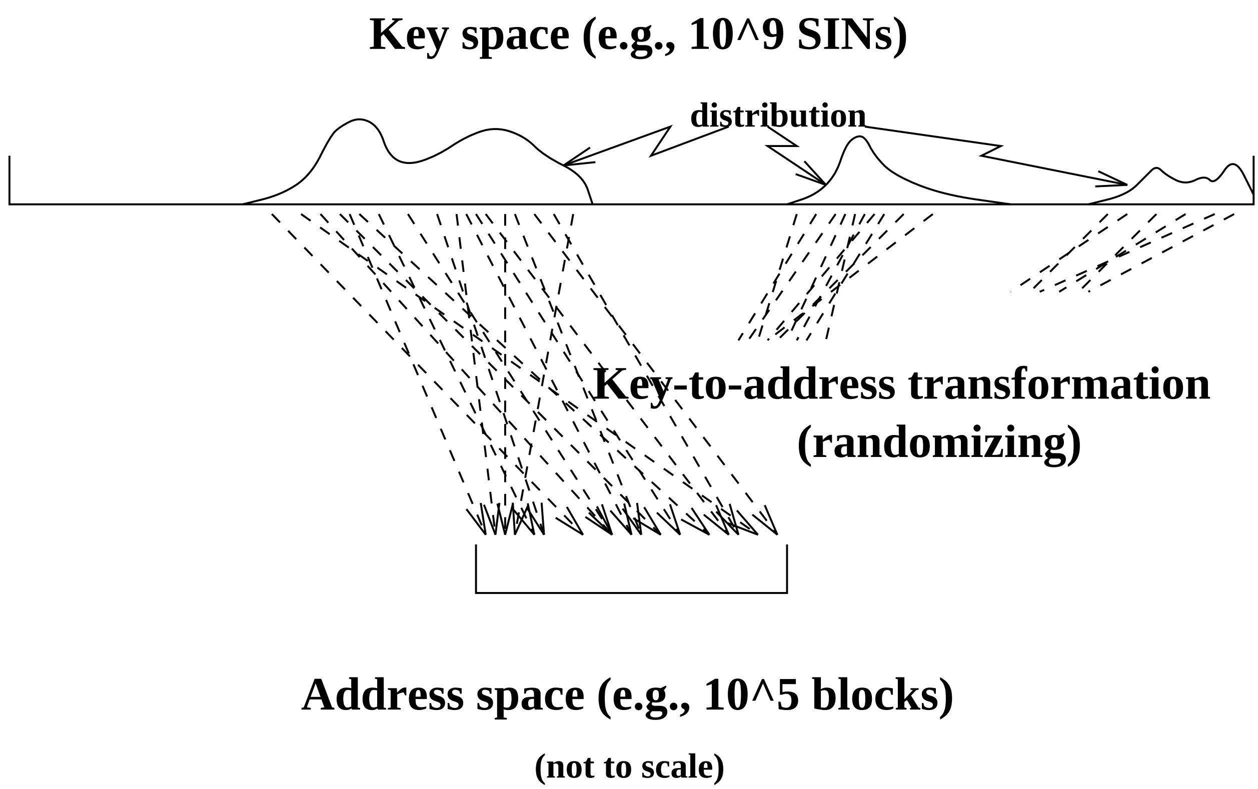
Key-to-Address Transformation Scaling



Key-to-Address Transformation Randomizing



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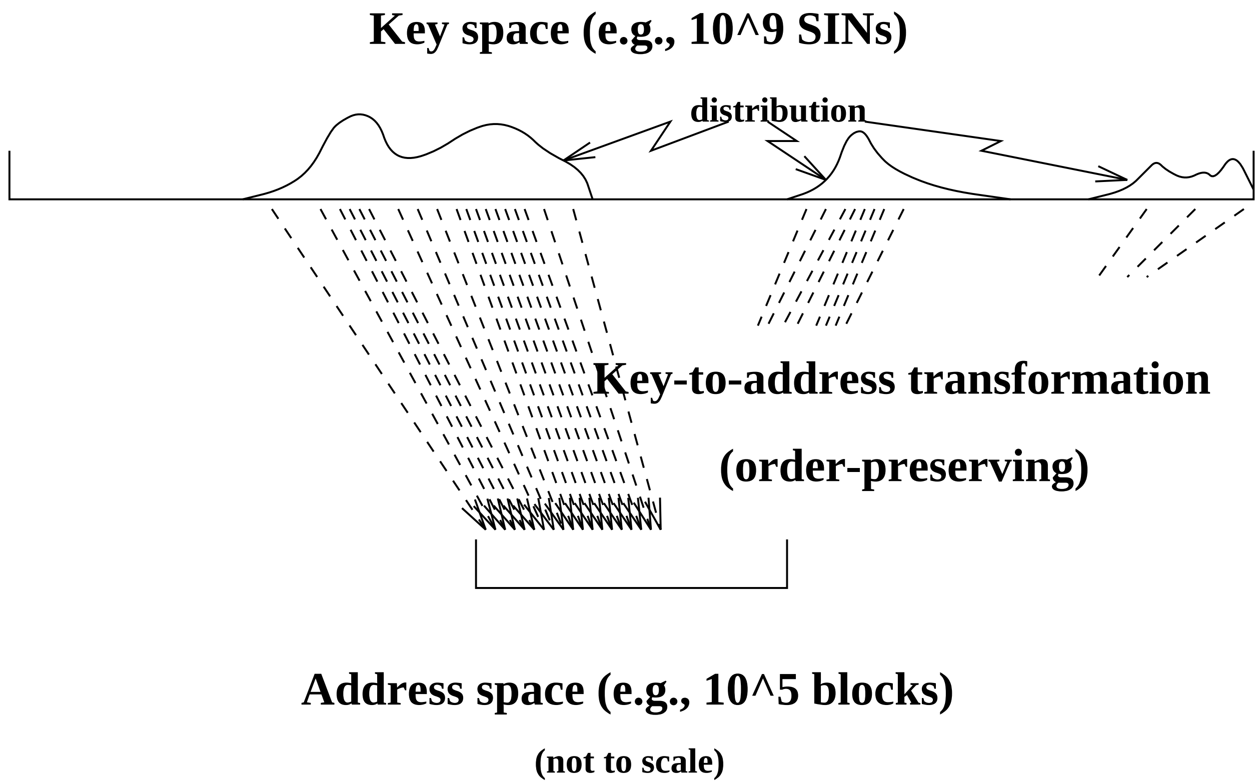
Key-to-Address Transformation
Order-preserving



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Direct Access

(Compute block address from keys,
instead of comparing them.)

Hash Functions

(Randomizing key-to-address transformations)

- Hash functions
  - Division-remainder, $h(k) = k \bmod n, n$ prime
  - Multiplicative, $h(k) =$ middle m bits of Ak, $n = 2^m$
- Collision resolution
  - Linear probing (cyclically downward, no pointers)
  - Separate chaining (pointer chain from each block)
  - $\mathcal{O}(N)$, worst case when all keys hash to one block or (linear probing) when file full and all blocks in an overflow chain.

Hash Algorithms

- Algorithm SS (Hash search with separate chaining)

- Algorithm SI (Hash insert with separate chaining)

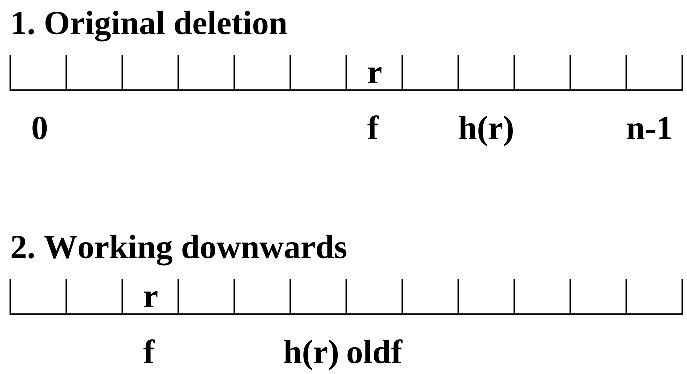
- Algorithm SD (Hash delete with separate chaining)

- Algorithm LS (Hash search with linear probing)

- Algorithm LI (Hash insert with linear probing)

- Algorithm LD (Hash delete with linear probing)

Algorithm LD: Delete Record r Found on Block f



LD1 (Remove record)
ℓ <− loc(r); *oldf* <− f; mark ℓ on *oldf* empty

LD2 (Move later records up)
if block f not full (apart from deletion, if any,
    just made by LD1), stop.
f <− if $f = 0$ then $n - 1$ else $f - 1$
if f = original home block, stop.
    /* else full file can thrash */
for each record, r, on block f
  if ncycle($f, h(r)$, *oldf*)
  /*$h(r)$ not cyclically between f, *oldf*
  or = *oldf*; i.e., $h(r)$ at or beyond *oldf**/
    then copy r to ℓ on *oldf*; goto LD1
goto LD2

Analysing Hash Functions

Parameters

- $\alpha = \frac{N}{nb}$, load factor
- $\pi = \frac{\text{total no. probes}}{N}$, probe factor
- $pi_{\text{op}} = 1 + \frac{\text{total overflows}}{N}$, optimistic probe factor

For ideal $h(k)$, given one of n blocks,

prob(1 record hashes to it) $= \frac{1}{n}$

prob(k records hash to it) $= (\frac{1}{n})^k$

prob(exactly k preselected records of N hash to it)

$= \left(\frac{1}{n}\right)^k \left(1 - \frac{1}{n}\right)^{N-k}$

prob(any k of N records hash to it)

$= \left(\begin{array}{c} N \\ k \end{array} \right) \left(\frac{1}{n}\right)^k \left(1 - \frac{1}{n}\right)^{N-k}$

This *binomial distribution*, $B(k, N, n) = P(k, \frac{N}{n}) + \mathcal{O}(\frac{1}{n})$
where the *Poisson distribution*, $P(k, \frac{N}{n}) = P(k, \alpha b) = \mathrm{e}^{-\alpha b} \frac{(\alpha b)^k}{k!}$

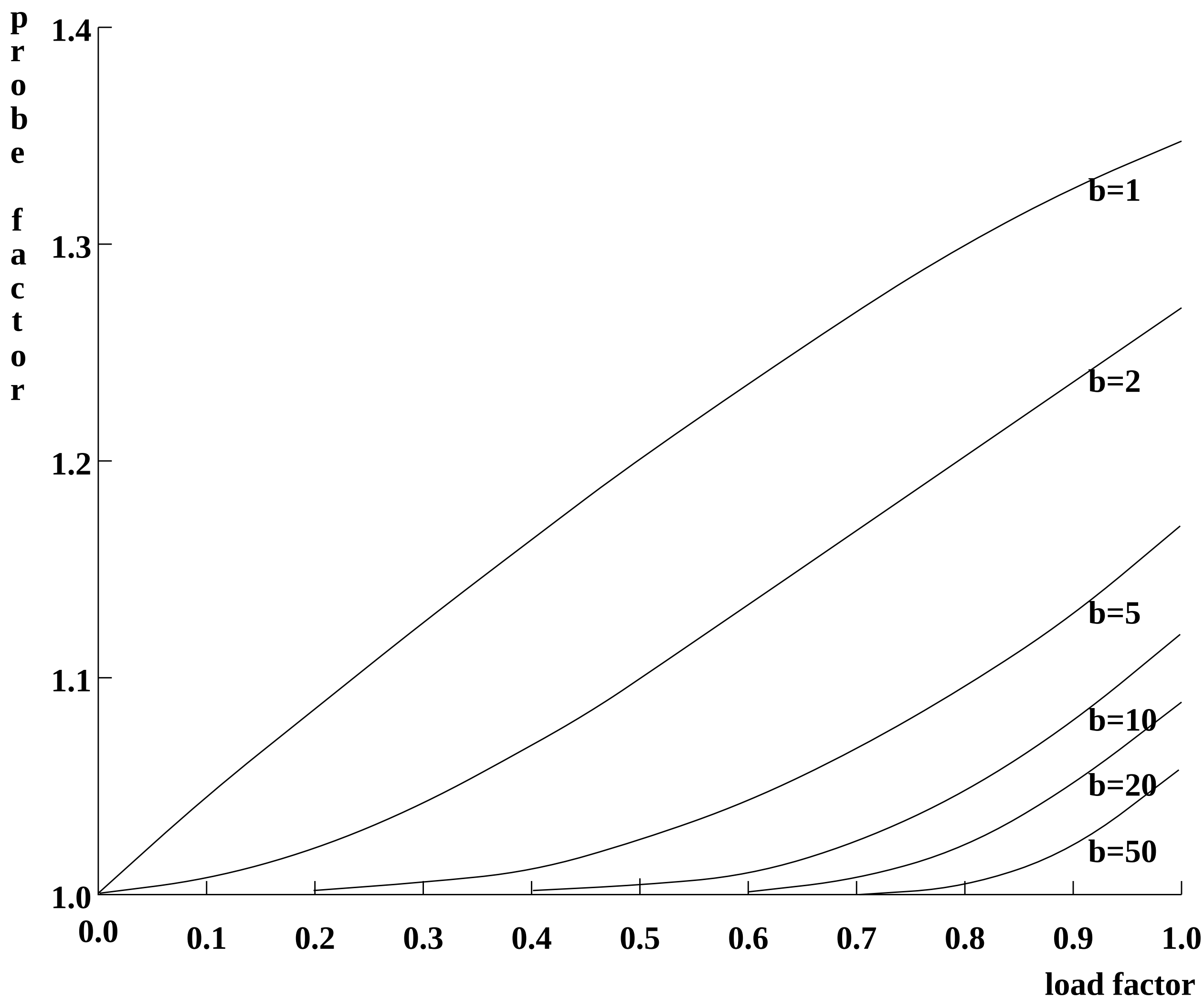
So the number of overflows per block,

$$
\begin{aligned}
\Omega(\alpha, b) &= \Sigma_{k>b}(k-b)B(k, N, n) \\
&\approx \Sigma_{k>b}(k-b)P(k, \alpha b) \\
&= \Sigma_{k=0}^{b}(b-k)P(k, \alpha b) - (1-\alpha)b
\end{aligned}
$$

And the number of overflows per record, $\omega(\alpha, b) = \frac{\Omega(\alpha,b)}{\alpha b}$

Finally, $pi_{\text{op}} = 1 + \omega(\alpha, b)$, and this can be plotted.

$1 + w(\alpha, b)$
$= 1 +$ Number of Overflows per Record



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Direct vs. Sequential Breakeven Activity

Usage Distributions I: Direct Access

$u(L)\mathrm{d}L =$ probability(access record in $L ... L+\mathrm{d}L$)

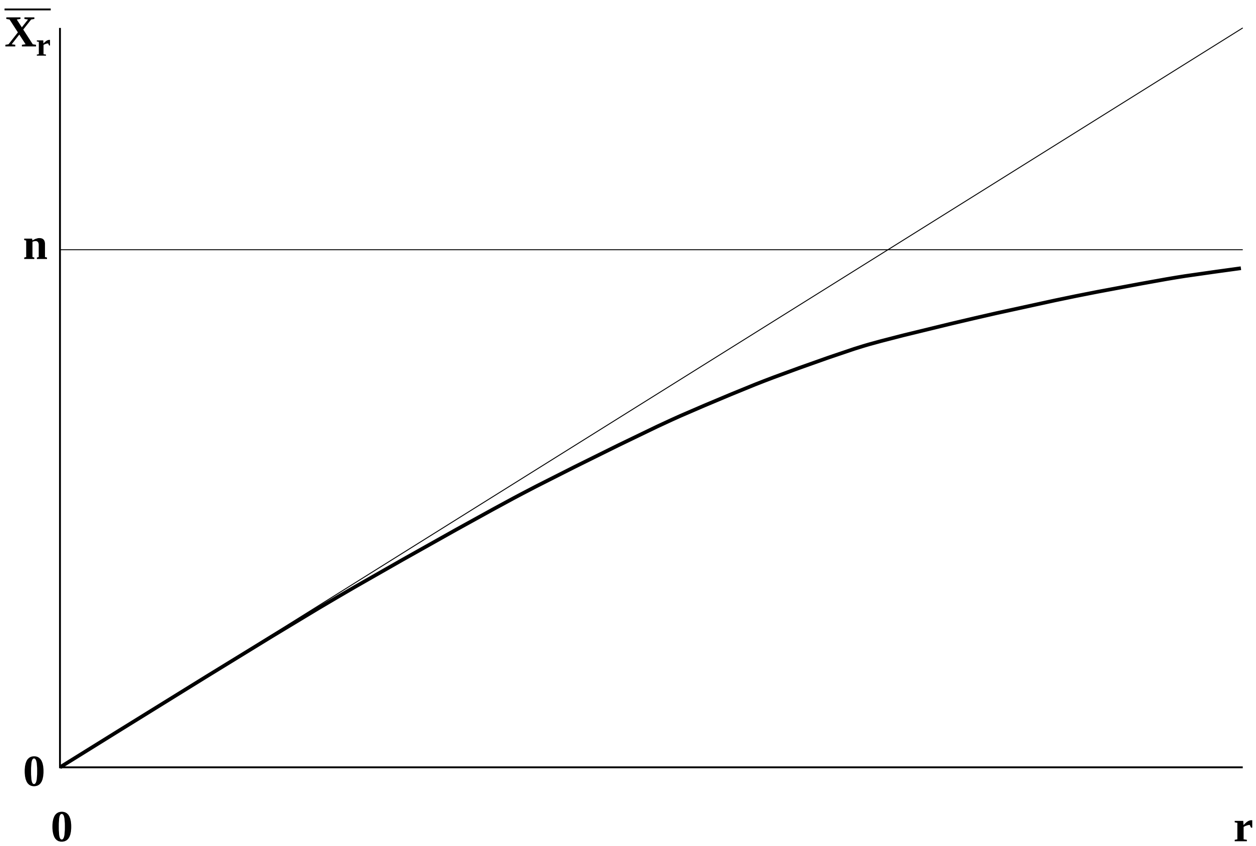
$u_\ell = \int_{(\ell-1)b}^{\ell b} u(L)\mathrm{d}L =$ probability(access block ℓ)

$$X_\ell = \begin{cases} 0 & \text{if block } \ell \text{ is not accessed} \quad \text{prob. } (1-u_\ell)^r \\ 1 & \text{if block } \ell \text{ is accessed} \quad \text{prob. } 1-(1-u_\ell)^r \end{cases}$$

Expected no. blocks accessed ("hit rate"):

$$\begin{aligned} \overline{X_r} &= \sum_{\ell=1}^{n} 0 \times (1-u_\ell)^r + 1 \times (1-(1-u_\ell)^r) \\ &= \sum_{\ell=1}^{n} 1-(1-u_\ell)^r \\ &= n(1-(1-\frac{1}{n})^r) \quad \text{uniform } u_\ell = \frac{1}{n} \\ &\approx r \quad \text{small } r \\ &\approx n \quad \text{large } r \end{aligned}$$

Direct Access Hit Rates for Uniform Usage



The “80-20” Distribution

$\frac{\sum_{\ell=1}^{0.2m} u_\ell}{\sum_{\ell=1}^{m} u_\ell} = 0.8$ $\frac{\int_0^{0.2m} u(L)\mathrm{d}L}{\int_0^{m} u(L)\mathrm{d}L} = 0.8$

try $\sum_{\ell=1}^{m} u_\ell = cm^\theta$ try $\int_0^{m} u(L)\mathrm{d}L = cm^\theta$

try $u_\ell = c(\ell^\theta - (\ell - 1)^\theta)$ try $u(L) = c'L^{\theta-1}$

When $\theta =$log 0.8/log 0.2 = 0.1386, 80-20 distribution.

When $\theta =$log 0.5/log 0.5 = 1, uniform (50-50) distribution.

When $\theta =$0 (continuous case), “Zipf” distribution.

For normalization, $u(L) = \theta L^{\theta-1}/(N/\alpha)^\theta$.

Direct vs. Sequential Breakeven Activity

Usage Distributions II: Sequential Access

Derive "distribution of depths"

$u_r(L)\mathrm{d}L =$ prob.($\mathbf{max}(L_1, ..., L_r)$ is in $L..L+\mathrm{d}L$)

via cumulative distributions:

$$\begin{aligned}
u_r(L) &= \frac{\mathrm{d}}{\mathrm{d}L}U_r(L) \\
&= \frac{\mathrm{d}}{\mathrm{d}L}\text{prob.}(\max(L_1, ..., L_r) \leq L) \\
&= \frac{\mathrm{d}}{\mathrm{d}L}\text{prob.}(L_1 \leq L \textbf{ and...and } L_1 \leq L) \\
&= \frac{\mathrm{d}}{\mathrm{d}L}(\text{prob.}(L_1 \leq L))^r \\
&= \frac{\mathrm{d}}{\mathrm{d}L}(U(L))^r \\
&= \frac{\mathrm{d}}{\mathrm{d}L}(\int_0^L u(L)\mathrm{d}L)^r \\
&= ru(L)(U(L))^{r-1}
\end{aligned}$$

For the 80-20 family, $u(L) = \theta L^{\theta-1}/N^{\theta}$,

$u_r(L) = r\theta L^{r\theta-1}/(N/\alpha)^{r\theta}$.

Discretize,

$$
\begin{aligned}
d_{\delta}^{(r)} &= \int_{\delta-1)b}^{\delta b} u_r(L)\mathrm{d}L \\
&\approx (\delta^{r\theta} - (\delta-1)^{r\theta})/n^{r\theta}
\end{aligned}
$$

Expected depth,

$$
\begin{aligned}
\overline{\delta} &= \sum_{\delta=1}^{n} \delta d_{\delta}^{(r)} \\
&\approx (n^{r\theta+1} - \sum_{\delta=1}^{n-1} \delta^{r\theta})/n^{r\theta} \\
&\approx \frac{nr\theta}{r\theta+1}
\end{aligned}
$$

Direct vs. Sequential Breakeven Activity

For uniform usage,
compare $n(1-(1-\frac{1}{n})^r)$ with $\frac{nr}{r+1}$

Assume

- direct needs 1 probe, ρ, per hit, but only reads the *record*, R bytes;
- sequential reads block after block without probing, and probe to find first block is negligible.

"Effective passes" identified for breakeven r:

$$\left[rR+\rho n(1-(1-\frac{1}{n})^r)\right]/NR = \frac{r}{r+1}$$

$$a_{\text{be}} = \frac{r}{N}$$

$$= \frac{R}{R+\rho}+\mathcal{O}(\frac{1}{N})$$

For $\rho = 10^6$

record size, R	10	100	1000
breakeven activity	0.001%	0.01%	0.1

Hashing and Volatility

- Virtual hashing

- Linear hashing

- Splitting criteria

  - (Greedy): split unless this makes $\alpha < \alpha_0$.

  - (Lazy): split if $\pi > \pi_0$

- Algorithm LHI (Linear hash insert)

Algorithm LHI, Linear Hash Insert

Insert Record r with Key k

- Initially, $j = 0, p = 0, n = \nu$ and, for any k, $h_{-1}(k) = -1$.

LHI1 (Hash.) $a \leftarrow$ if $p = 0$ or $h_{j-1}(k) < p$ then $h_j(k)$ else $h_{j-1}(k)$. If not already there, store r in block a or as an overflow to block a. $N{+}{+}$.

LHI2 (Split disallowed.) If $N/(n+1)b < \alpha_0$ then terminate.

LHI3 (Allocate.) If $p = 0$ then $j{+}{+}$. Allocate block $p + 2^{j-1}\nu$. $n{+}{+}$.

LHI4 (Split.) Rehash block p, including overflows, using h_j.

LHI5 (Increment pointer.) $p \leftarrow$ if $p \geq 2^{j-1}\nu$ then 0 else $p + 1$.

Algorithm LHI, Example

n	j	p	k	$h_j(k)$	$h_{j-1}(k)$	a		
1	0	0	3	0		0	$_0$ 3	1/4
			7	0		0	$_0$ 3,7	2/4
			2	0		0	$_0$ 3,7 2	3/4
			5	0		0	$_0$ 3,7 2,5	4/4
2	1	0					$_0$ 2 $_1$ 3,7 5	
			6	0		0	$_0$ 2,6 $_1$ 3,7 5	5/6
3	2	1					$_0$ $_1$ 3,7 5 $_2$ 2,6	
			11	3	1	1	$_0$ $_1$ 3,7 5,11 $_1$ 2,6	6/8
			4	0	0	0	$_0$ 4 $_1$ 3,7 5,11 $_1$ 2,6	7/8
4		0					$_0$ 4 $_1$ 5 $_2$ 2,6 $_3$ 3,7 11	
			1	1	1	1	$_0$ 4 $_1$ 5,1 $_2$ 2,6 $_3$ 3,7 11	8/1
5	3	1					$_0$ $_1$ 5,1 $_2$ 2,6 $_3$ 3,7 11 $_4$ 4	
			9	1	1	1	$_0$ $_1$ 5,1 9 $_2$ 2,6 $_3$ 3,7 11 $_4$ 4	9/1

Algorithm LHD, Linear Hash Delete

- Initially, j, p, and n are as they were left by the last call to LHI or LHD.

LHD1 (Hash.) $a \leftarrow$ if $p = 0$ or $h_{j-1}(k) < p$ then $h_j(k)$ else $h_{j-1}(k)$. If found, remove r from block a or from the overflows to block a. $N - -$.

LHD2 (Merge disallowed.) If $N/nb \geq \alpha_0$ then terminate.

LHD3 (Decrement pointer.) $p \leftarrow$ if $p = 0$ then $2^{j-1}\nu - 1$ else $p - 1$.

LHD4 (Merge.) Rehash blocks p and $p + 2^{j-1}\nu$ using h_{j-1}.

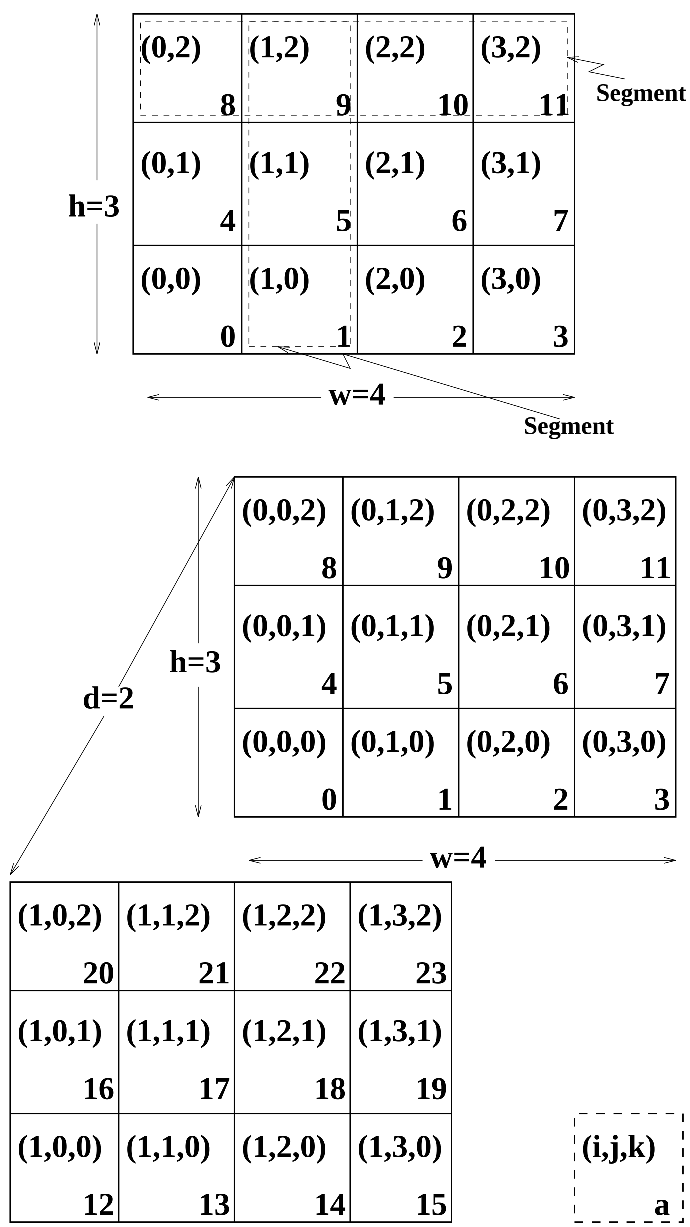
LHD5 (Deallocate.) Deallocate block $p + 2^{j-1}\nu$. $n - -$. If $p = 0$ then $j - -$.

Algorithm LHD, Example

n	j	p	k	$h_j(k)$	$h_{j-1}(k)$	a		
5	3	1	7	7	3	1	0 [ ] 1 [5,1 9] 2 [2,6] 3 [3,11] 4 [4]	8/10 $\geq$
			6	6	2	2	0 [ ] 1 [5,1 9] 2 [2] 3 [3,11] 4 [4]	7/10 $<$
		0					0 [4] 1 [5,1 9] 2 [2] 3 [3,11]	
4	2							
			2	2	0	2	0 [4] 1 [5,1 9] 2 [ ] 3 [3,11]	6/8 $<$
		1					0 [4] 1 [5,1 9, 3,11] 2 [ ]	
3								
			1	1	1	1	0 [4] 1 [5,9 3,11] 2 [ ]	5/6 $\geq$
			11	3	1	1	0 [4] 1 [5,9 3] 2 [ ]	4/6 $<$
		0					0 [4] 1 [5,9 3]	
2	1							

Hashing and Symmetry

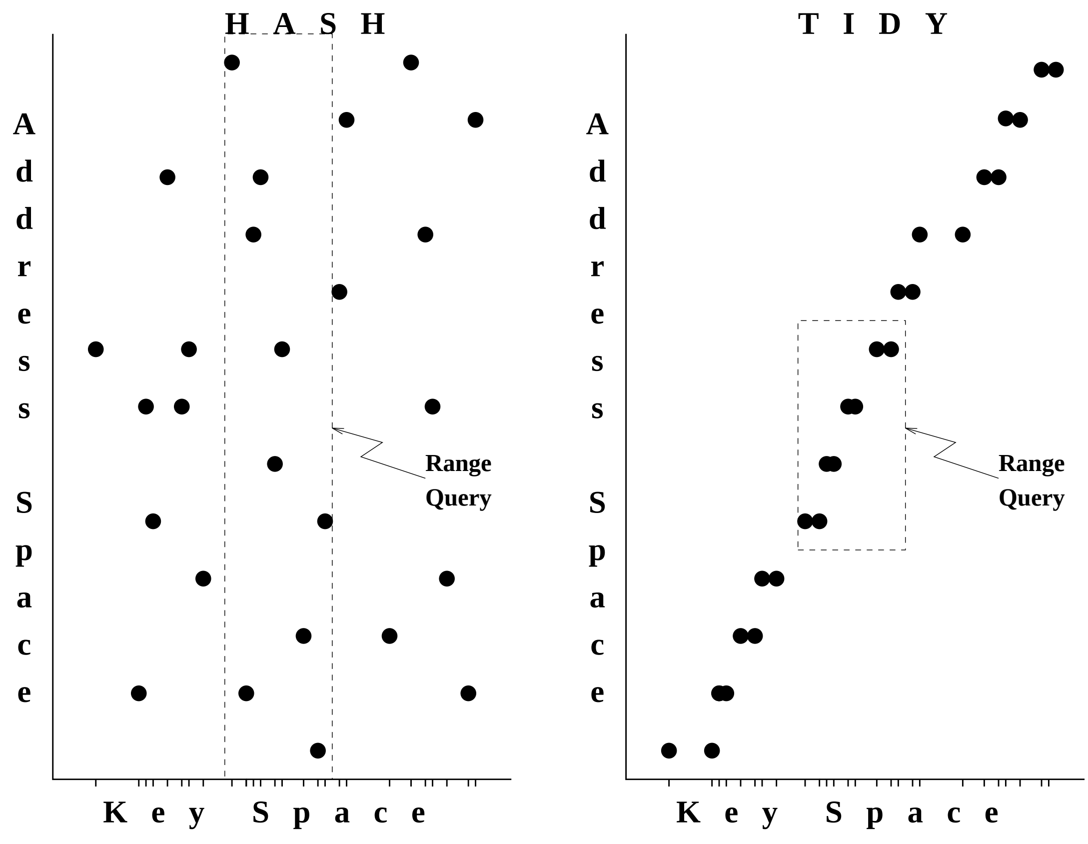
- Maintain separate hash functions on each key field.
- Use an array-addressing expression to combine the segment coordinates into an address, e.g.: $a(i,j) = wj + i$ (2D); $a(i,j,k) = hwk + wj + i$ (3D).



Hashing and Activity

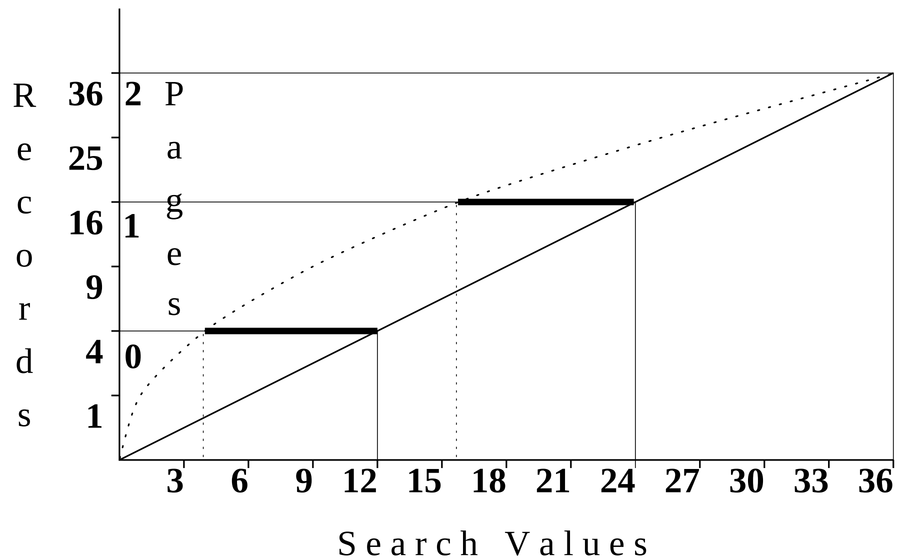
- Sort requests (keys k) by $h(k)$, then merge.
- What about *range* queries?
  Must *preserve order*

Order-Preserving Key-to-Address Transformations (*OPK2AX*): *Tidy Functions*



Tidy Functions

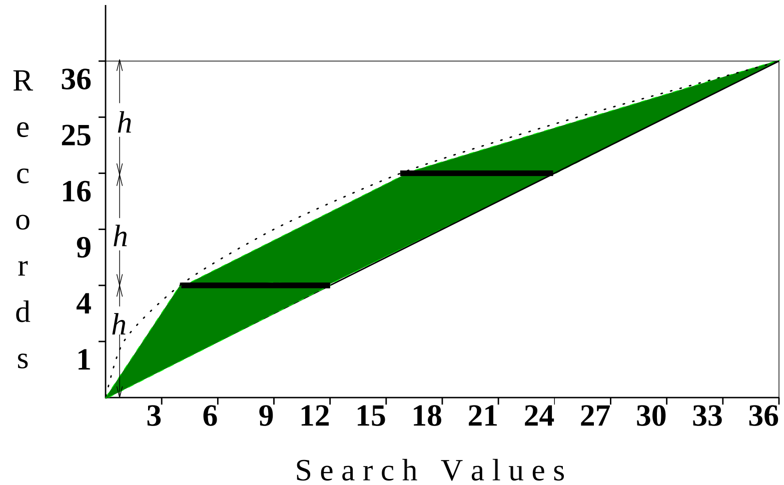
(Order-preserving key-to-address transformations)



1. Records 1,4,9,16,25,36 loaded into file, 2 per page.
2. Ideal index would show pages end at 4,16,36. (But this would need 1 index entry per page.) ....
3. Approximate index is linear interpolation; needs only store 36, 0, 3(pages). ________________________
4. 0–4, 13–16, and 25–36 will be correctly directed to their home pages (0, 1, 2, resp.), and need only 1 access.
5. 5–12 and 17–24 will be incorrectly directed, and an upwards linear probe will be needed: total, 2 accesses.
6. The number of these "overflows" $\propto$ length of horizontal lines between the true curve and the approximate curve.

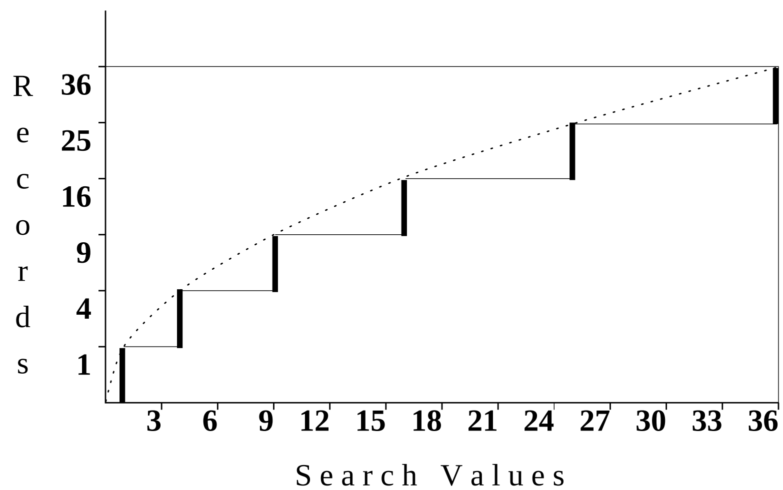
Tidy Functions

The sum of these lengths $\propto$ area between curves:



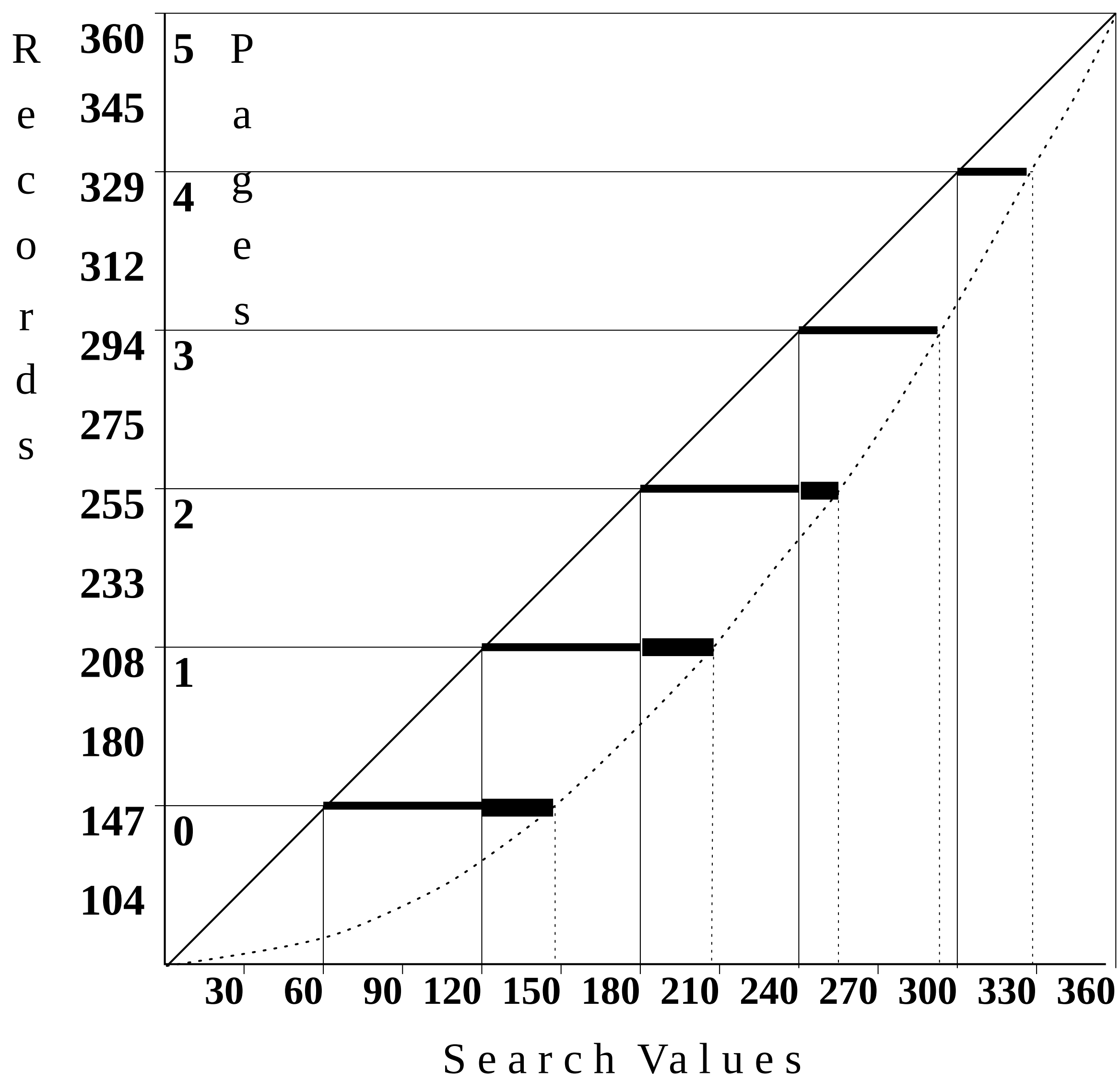
Area $= (l_1/2 + (l_1 + l_2)/2 + l_2/2)h = h\Sigma l_i$

The ideal curve may be seen as the cumulative distribution:



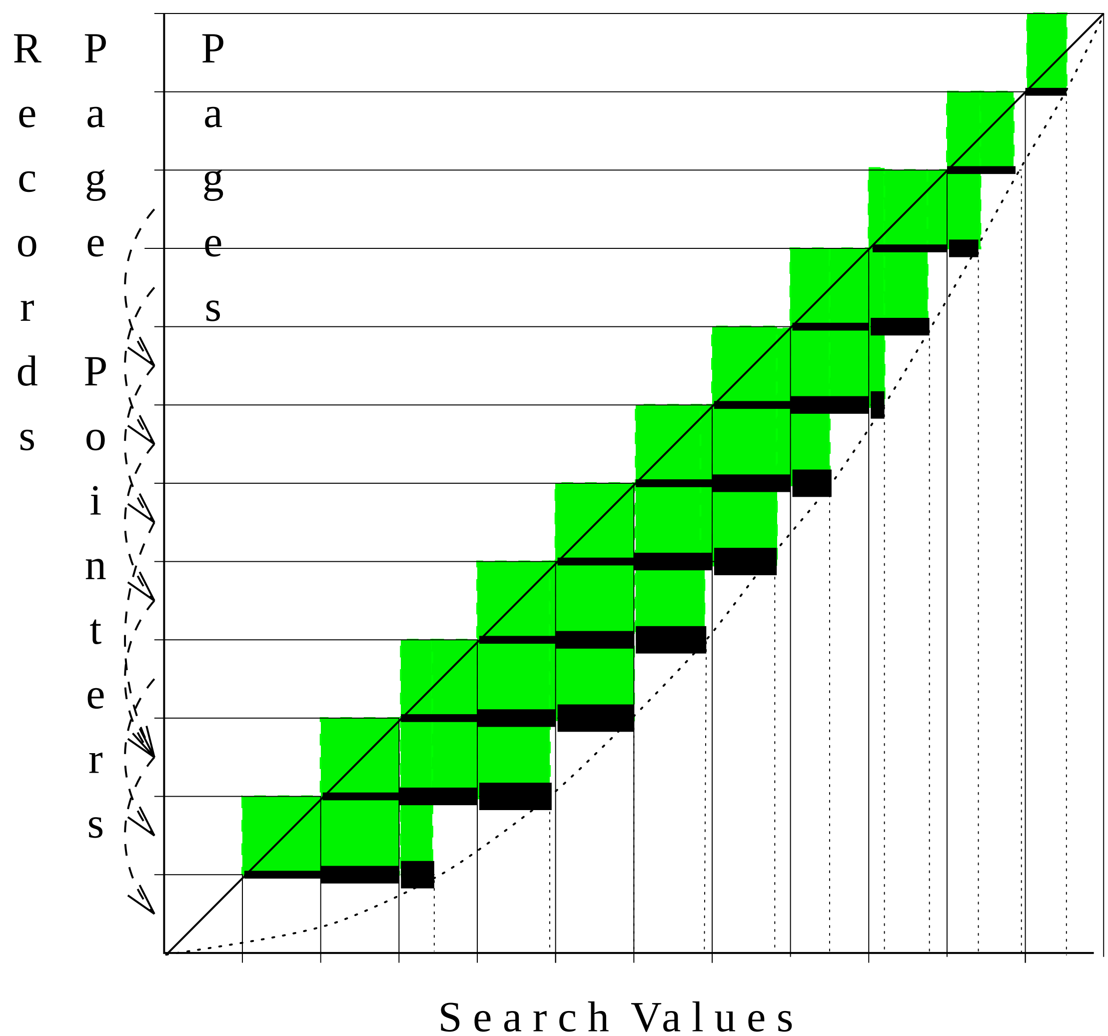
Tidy Functions

Here is a case where a) the linear probing must go downwards and b) there may be more than one extra probe (values 121–147, 181–208, 241–255).



Tidy Functions "Overflows" = Area

In more extreme cases, > 1 probe may be needed before we find *any* of the records that the approximation says are on the page. (Shaded areas = $h\times$ number of extra probes $\approx$ area between curves.)



Pointers on each page tell where to start, reducing total probes.

Tidy Functions Optimal partitioning

Can we improve the approximation by making a new one of two (or more) linear pieces (*piece-wise linear*)?

Since the overflows are proportional to the area between the curves, find a partitioning which minimizes the area.

For the previous example, the cumulative distribution ("true curve") is $y = x^2$ (up to a scale factor which can be removed).

Let's make two linear pieces, which meet each other and the true curve at (ξ, ξ^2). The equations of these two pieces are $y = \xi x$ and $y = (1 + \xi)x - \xi$. The area between them and the true curve is

$$\int_0^\xi (\xi x - x^2)\mathrm{d}x + \int_\xi^1 ((1+\xi)x - \xi - x^2)\mathrm{d}x = \frac{1}{2}(\xi^2 - \xi + \frac{1}{3})$$

Minimizing this,

$$0 = \frac{\mathrm{d}}{\mathrm{d}\xi}\frac{1}{2}(\xi^2 - \xi + \frac{1}{3}) = \xi - \frac{1}{2}$$

So the optimal approximation by two linear pieces partitions the range of search values in the middle.

Tidy Functions Optimal partitioning

We can't get far with calculus:

- Systems of equations, e.g., $p = 3$ partitions gives two simultaneous quadratic equations to solve.
- The true curve is not analytic.

So we use *dynamic programming* (X.Y.Zhao, 1995)

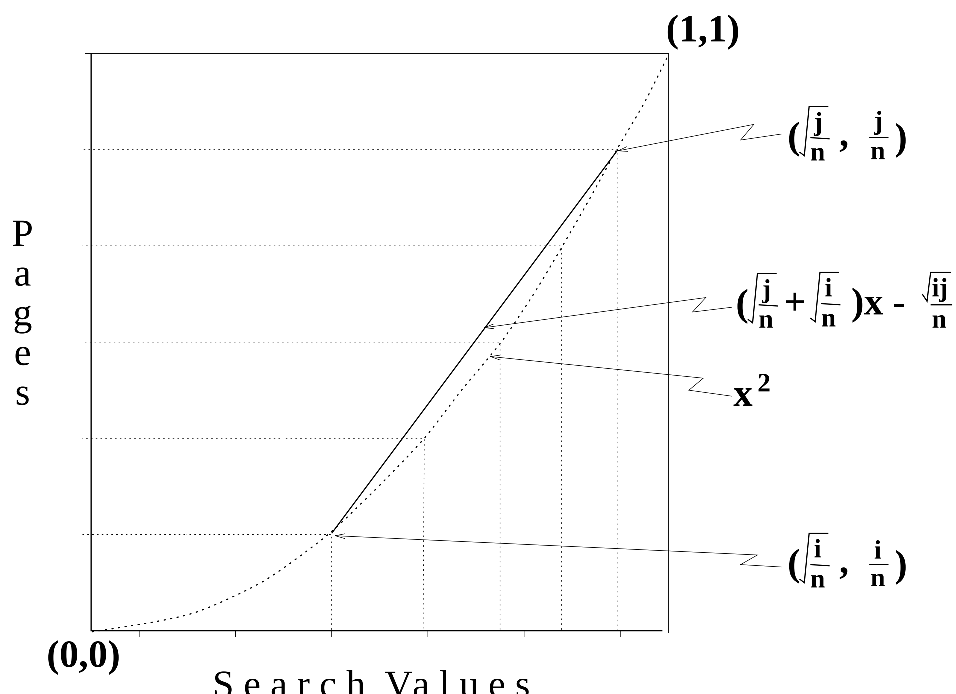
e.g., for $p = 2$ (again), choose k which gives the minimum

$$a(0,k) + a(k,n)$$

where, for the parabolic true curve (again)

$$a(i,j) = \int_{\sqrt{\frac{i}{n}}}^{\sqrt{\frac{j}{n}}} \left(\left(\sqrt{\frac{j}{n}} + \sqrt{\frac{i}{n}} \right) x - \frac{\sqrt{ij}}{n} - x^2 \right) \mathrm{d}x = \frac{1}{6} \left(\sqrt{\frac{j}{n}} - \sqrt{\frac{i}{n}} \right)^3$$

This works out to be $k = n/4$, or, for $n = 6$, $k = 2$.



Tidy Functions Dynamic programming

Find $m(p,n)$ (minumum area) given $a(i,j)$ (and $m(1,j) = a(0,j)$):

There are
$$\left(\begin{array}{c} n-1 \\ p-1 \end{array}\right) = \sum_{k=p-2}^{n-2} \left(\begin{array}{c} k \\ p-2 \end{array}\right)$$
ways of placing $p-1$ partition boundaries on the $n-1$ page boundaries. Problem is not exponential ($\mathcal{O}(n^p)$) but cubic, if we *memoize*, because need at most $\frac{n(n+1)}{2} - \frac{(n-p)(n-p+1)}{2}$ areas.

For $p = 3$ partitions and $n = 6$ pages

$m(3,6)$ (10 subproblems) = min:

- $m(2,5)$ $+a(5,6)$
  - (4 subproblems) = min:
    - $m(1,4) + a(4,5)$
    - $m(1,3) + a(3,5)$
    - $m(1,2) + a(2,5)$
    - $m(1,1) + a(1,5)$
- $m(2,4)$ $+a(4,6)$
  - (3 subproblems) = min:
    - $m(1,3) + a(3,4)$
    - $m(1,2) + a(2,4)$
    - $m(1,1) + a(1,4)$
- $m(2,3)$ $+a(3,6)$
  - (2 subproblems) = min:
    - $m(1,2) + a(2,3)$
    - $m(1,1) + a(1,3)$
- $m(2,2)$ $+a(2,6)$
  - (1 subproblem) = min:
    - $m(1,1) + a(1,2)$

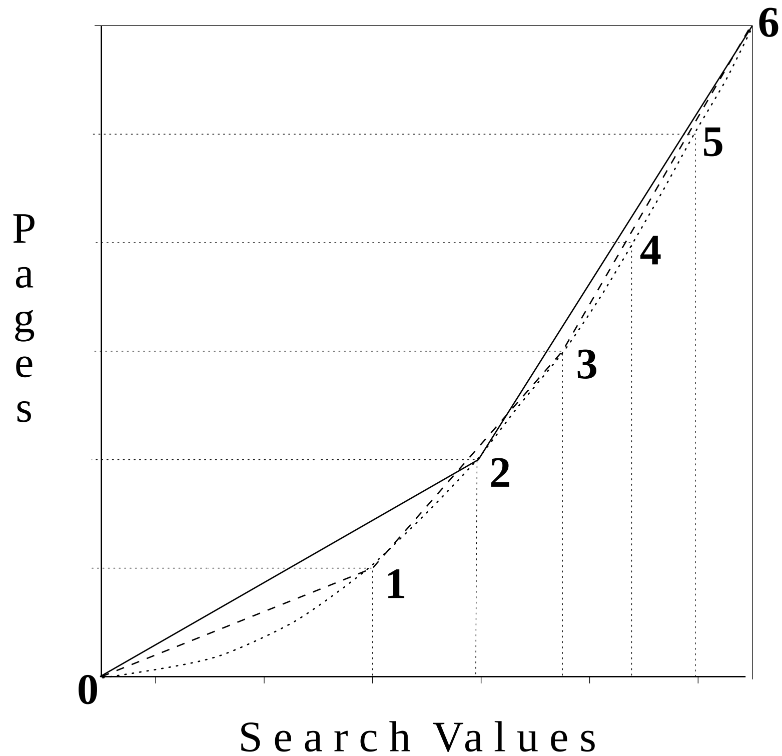
Tidy Functions Dynamic programming

Here are all the areas, $(\sqrt{j}-\sqrt{i})^3$, (parabolic curve),

$i\backslash j$	0	1	2	3	4	5	6
0		1	2.828	5.196	8	11.18	14.7
1			0.071	0.39	1	1.89	3.04
2				0.032	0.2	0.56	1.1
3					0.019	0.13	0.37
4						0.013	0.09
5							0.009
6							

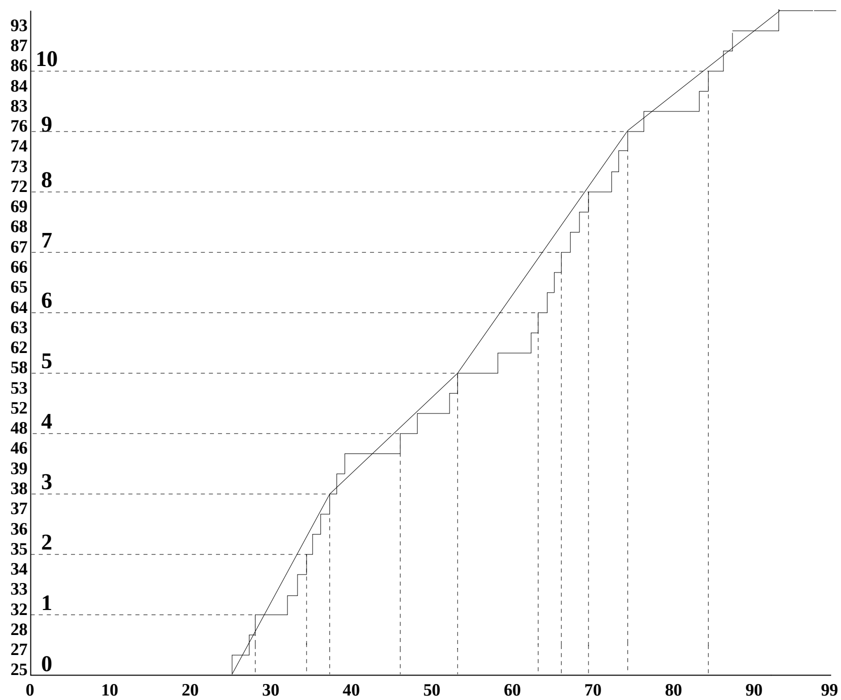
From this we can see

- $p = 2$: partition is at page 2 ________________
- $p = 3$: partitions are at pages 1,3 ....



Tidy Functions Dynamic programming

First two digits of Tbook phone numbers.



Areas for 11 pages (2× | *step* − *triangle* |)

28	34	37	46	53	63	66	69	74	84	93
1	23	56	221	380	662	767	890	1121	1649	2188
	12	3	114	231	453	540	645	846	1314	1799
13_{28}		3	54	129	291	360	447	618	1026	1457
4_{28}	16_{34}		3	36	138	189	258	399	747	1124
59_{37}	7_{37}	19_{37}		9	33	66	117	228	516	839
92_{37}	40_{37}	16_{46}	28_{46}		18	3	30	111	339	608
194_{37}	92_{46}	40_{46}	34_{53}	46_{53}		3	12	63	231	446
245_{37}	95_{53}	43_{53}	19_{53}	31_{53}	49_{63}		3	18	126	287
314_{37}	122_{53}	70_{53}	46_{66}	22_{66}	34_{66}	52_{66}		9	39	146
449_{46}	203_{53}	113_{66}	61_{66}	37_{66}	31_{69}	43_{69}	61_{69}		12	41
719_{53}	353_{69}	161_{69}	109_{69}	73_{74}	49_{74}	43_{74}	55_{74}	73_{74}		1
988_{53}	460_{69}	244_{74}	154_{74}	102_{74}	74_{84}	50_{84}	44_{84}	56_{84}	74_{94}	

Minimum areas (dynamic programming)

The cubic dynamic programming problem is still too big, since $n =$ the number of pages.

We might divide the whole problem into a number of equal-sized smaller problems, then run the optimizing algorithm on each of these.

For example, a 1 Gbyte file of 1 Kbyte pages would have $n = 10^6$ pages.

Dividing this into 10,000 subproblems of 100 pages each would require 10,000 $\mathcal{O}(100^3)$-sized optimizations.

Tidy Functions Dynamic programming

All the above has been concerned with minimizing *areas*.
Interpretation: minimize probes for successful or unsuccessful searches.

Could also minimize probes for successful searches: overflows are given by number of page boundaries crossed by *vertical* lines at search values corresponding to the records actually present:

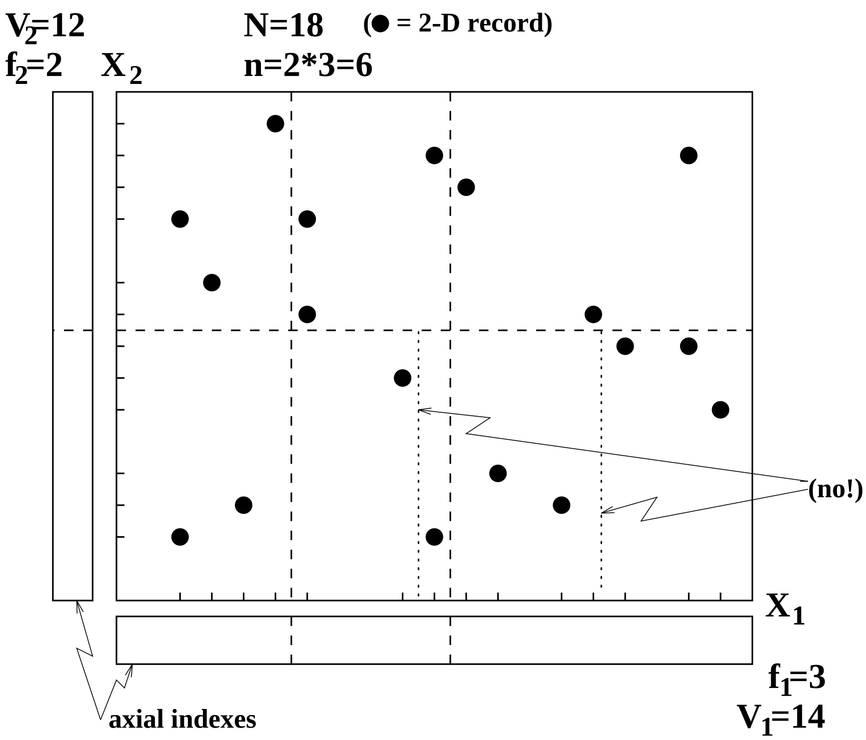
$$\begin{aligned}\pi &= 1 + (\sum_i \lfloor t(r_i) - L(r_i) \rfloor)/n \\ &= 1 + (\sum_i \lfloor i - L(t^{-1}(i)) \rfloor)/n\end{aligned}$$

where $t()$ is the true curve, $L()$ is the (linear) approximation, $r_i = t^{-1}(i)$ is the search value position for the i^{th} record, $i = 1..N$.

These vertical distances could be used in the dynamic programming instead of the areas.

Tidying and Symmetry

- Addressing must be via *axes* of address space. (These are 1-D tidy functions.)
- So key and address spaces must be partitioned *rectilinearly*.



In d dimensions,
$f_i = n^{\frac{1}{d}}$.
The d axial indexes,
each of size $n^{\frac{1}{d}}$,
might fit in RAM.

Axial distributions:

X_1 2 1 1 1 2 1 2 1 1 1 1 1 2 1

X_2 2 2 1 1 1 2 2 1 2 1 2 1

Multipage search

1. Use axial indices to find coordinates for page(s) that can hold the data requested.

2. For each page needed (coordinates $i, j, k, ..$), use an array addressing formula to give the page address. (See “hashing and symmetry”.)

Multidimensional **paging**

Algorithm MP (N records)

MP1 For each axis, $i = 1..d$, find axial distributions and V_i. (d sorts: $\mathcal{O}(dN \log N)$)

MP2 Given approximate values for b (blocksize) and α (load factor), choose partitioning factors, $f_i, i = 1..d$. ($\mathcal{O}(1)$)

$$n = \Pi_1^d f_i; \text{heuristic} : \frac{V_i}{f_i} = \text{const}$$

MP3 For each axis
find candidate(s) for axial partition (scan forward then back, cost $2V_i$)

MP4 Build histograms for all combinations of axial partitions by 1 pass of the data. Do π-α comparisons to find the best. ($\mathcal{O}(N)$)

MP1 Finding Axial Distributions

Maker\Toy	Caboose	Calico Cat	Car	Locomotive	Toy Train	Tractor	Tricycle	Truck	Ukulele	
Amloco Toys				1						1
Canloco Ltd.				1						1
Dink Inc.			1	1						2
Extrafun		1	1	1						3
Fischerman	1		1	1						4
General Toy Corp.	1	1	1	1	1	1	1	1	1	9
Mettal Toys	1		1	1		1	1	1		6
Noisy Toys		1	1	1	1	1				5
Playloco				1						1
	3	3	6	9	4	2	2	2	1	

MP2 Finding f_i:

Given $N, b, \alpha,\ n = \lceil N/b\alpha \rceil$.

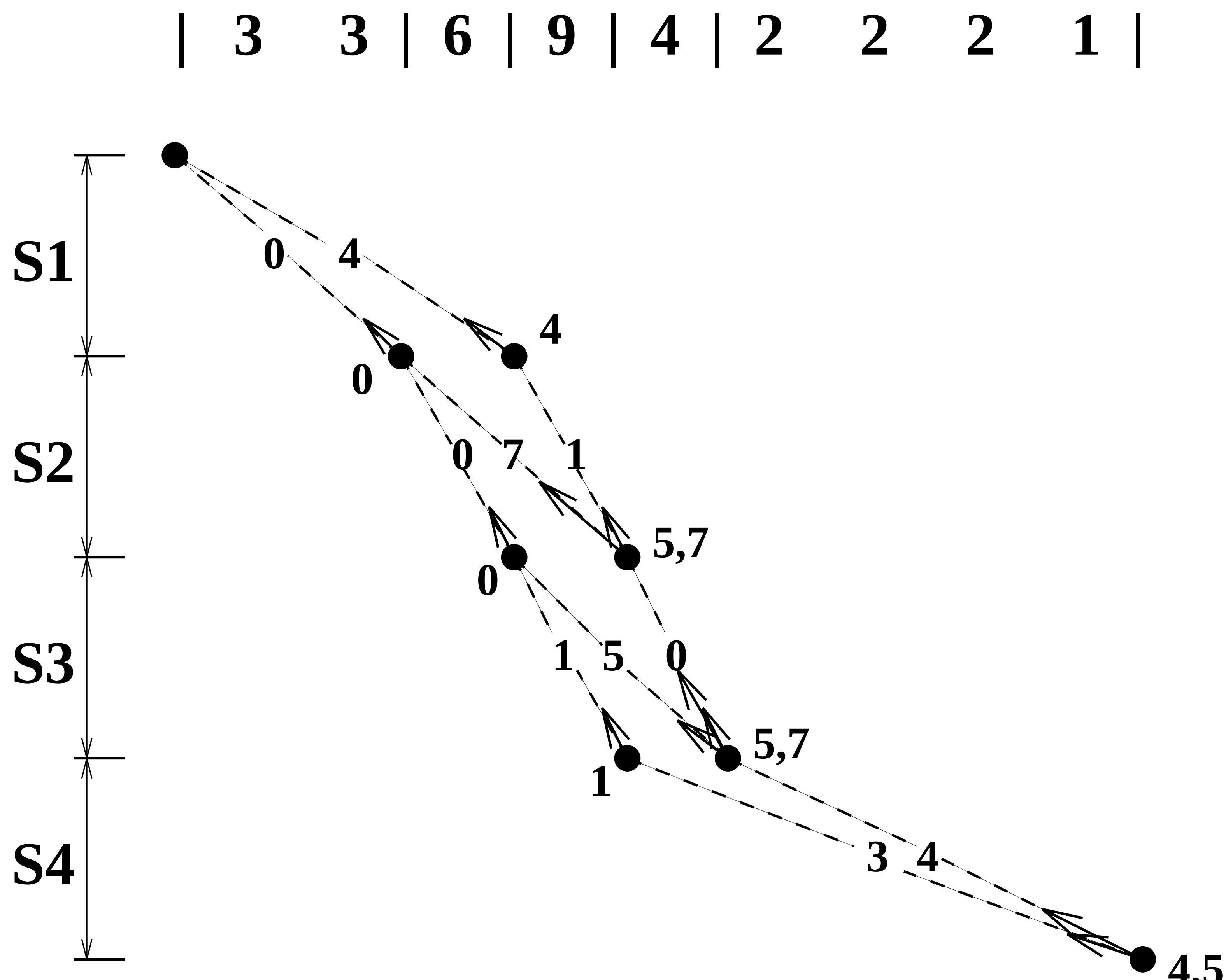
So $f_i = cV_i, n = c^d \Pi V_i, c = (\Pi V_i/n)^{1/d}$.

But this does not give integer values for f_i.

Rounding, flooring, or ceiling may not give factors of n (n may be prime).

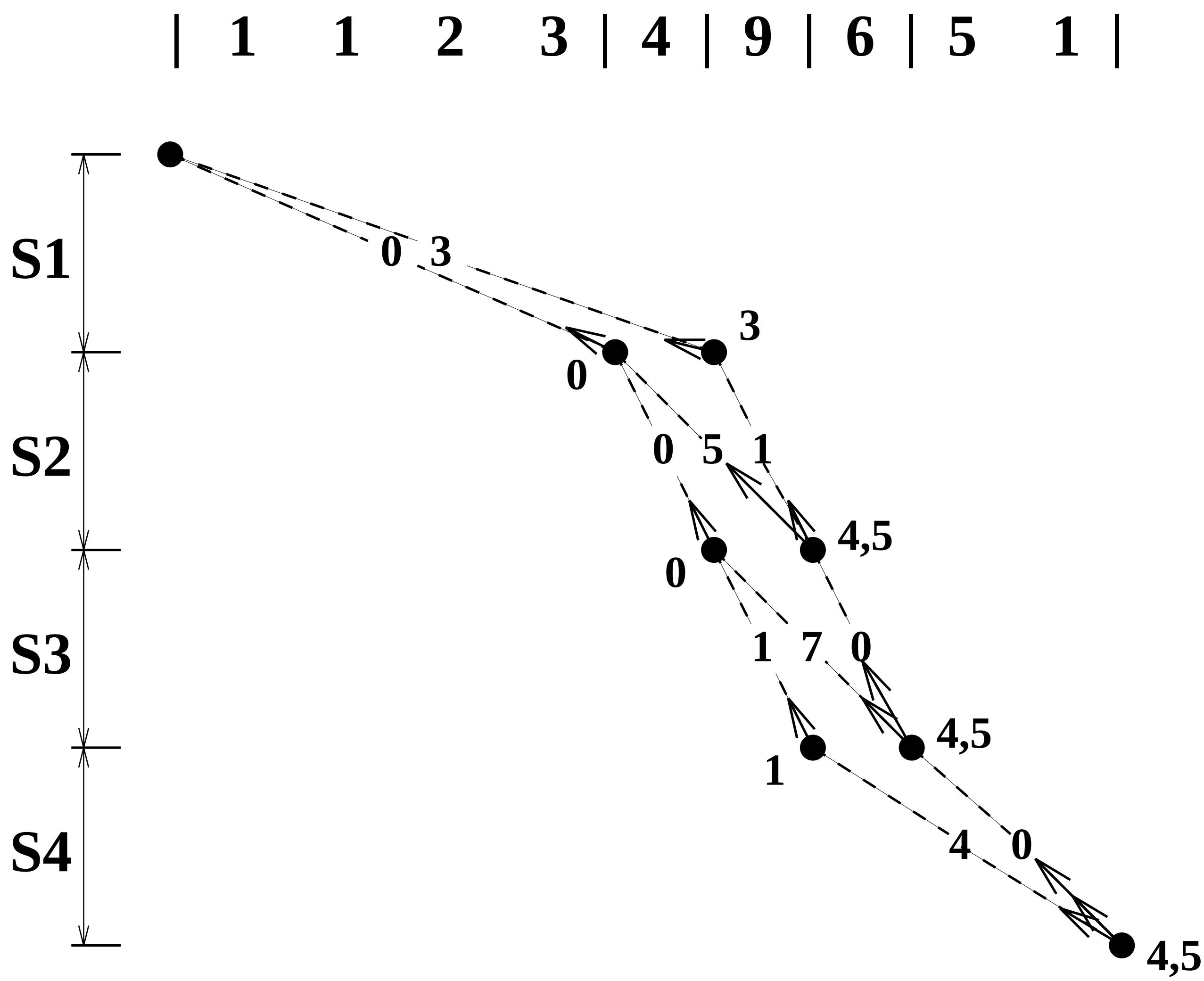
So we must remain flexible, by allowing changes to b, α.

MP3 Optimal Partition of *Toy* Axis:



| 3 3 | 6 | 9 | 4 | 2 2 2 1 |
S1
S2
S3
S4
0
4
4
0
0
7
1
5,7
0
1
5
0
5,7
1
3
4
4,5

MP3 Optimal Partition of *Maker* Axis:



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Histograms for Candidate Partitionings

aA

2	3	5	1
2	1	1	5
1	1	1	3
1	1	2	2

α	0.67	1.0
π_{opt}	1.13	1.25

aB

2	3	6	0
2	1	2	4
1	1	2	2
1	1	3	1

α	0.67	1.0
π_{opt}	1.13	1.25

aC

5	5	1	0
3	1	1	4
2	1	1	2
2	2	1	1

α	0.67	1.0
π_{opt}	1.16	1.28

bA

1	2	4	0
3	2	2	6
1	1	1	3
1	1	2	2

α	0.67	1.0
π_{opt}	1.13	1.25

bB

1	2	4	0
3	2	4	4
1	1	2	2
1	1	3	1

α	0.67	1.0
π_{opt}	1.09	1.25

bC

3	4	0	0
5	2	2	4
2	1	1	2
2	2	1	1

α	0.67	1.0
π_{opt}	1.13	1.25

cA

1	2	4	0
1	1	1	1
2	1	1	5
2	2	3	5

α	0.67	1.0
π_{opt}	1.16	1.28

cB

1	2	4	0
1	1	2	0
2	1	2	4
2	2	5	3

α	0.67	1.0
π_{opt}	1.13	1.25

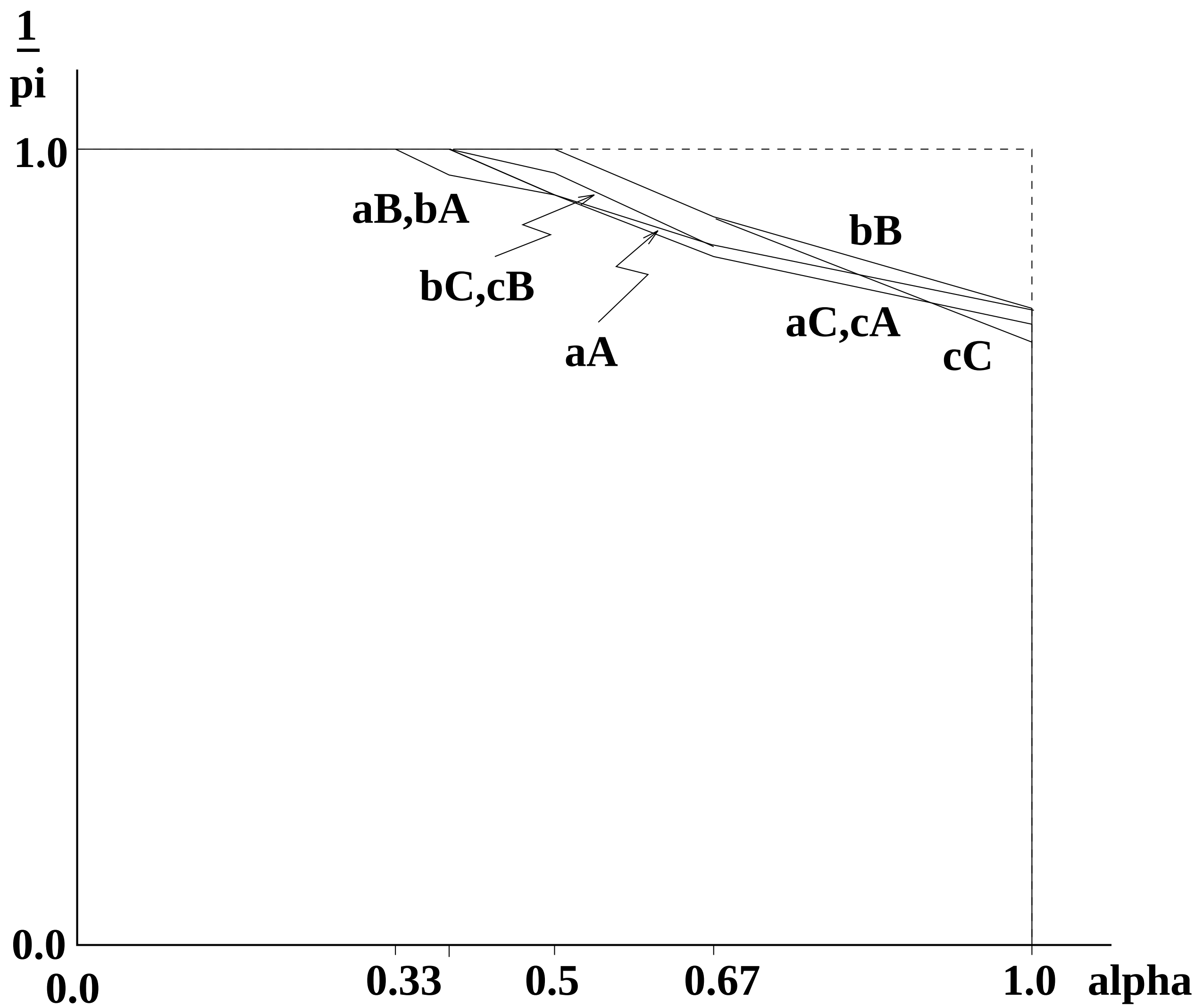
cC

3	4	0	0
2	1	1	0
3	1	1	4
4	3	2	3

α	0.67	1.0
π_{opt}	1.09	1.31

MP4 π-α analysis:

The nine cases from the example:



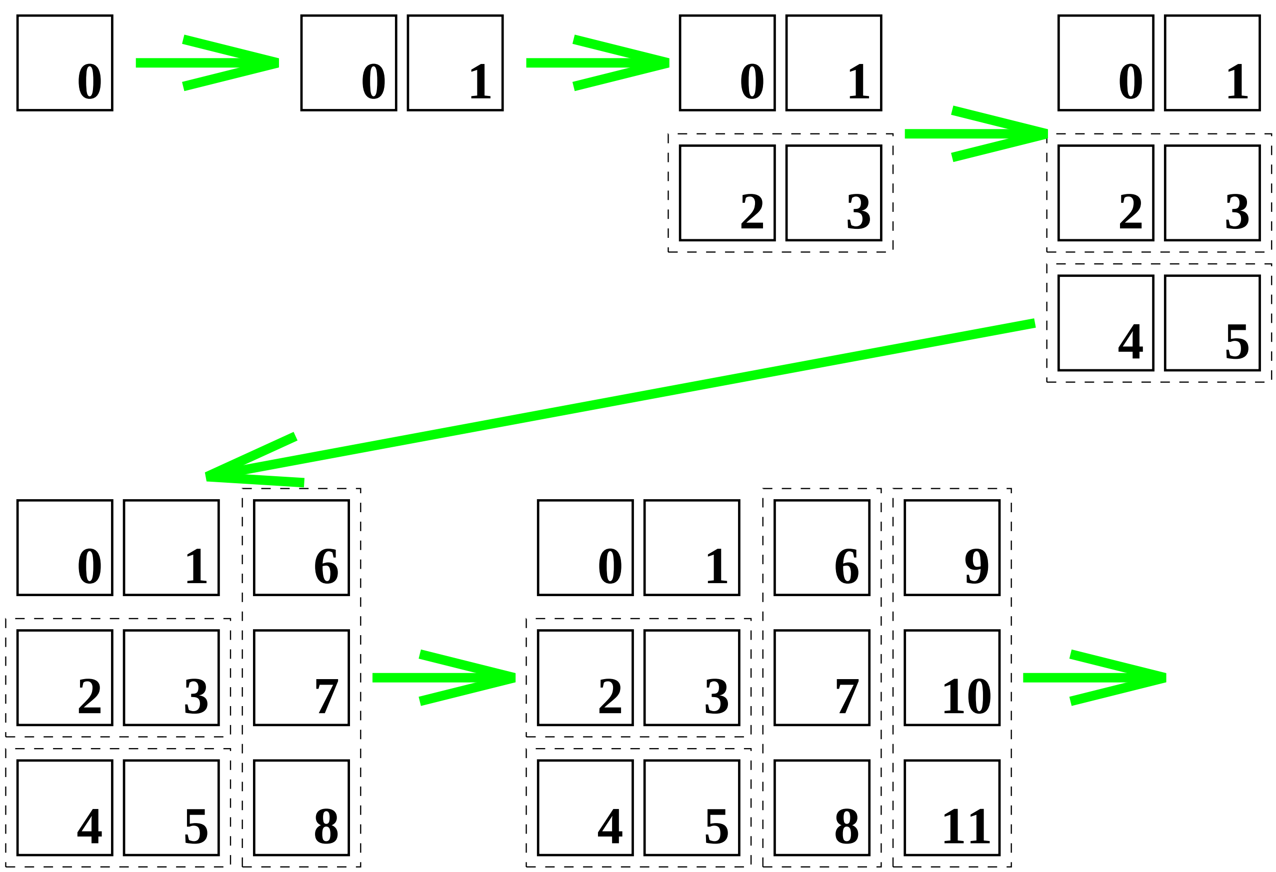
Pick curve with largest area: bB wins!

So far	lo	hi
Symmetry	√	√
Activity	√	√
Volatility	√	×

Dynamic Multipaging

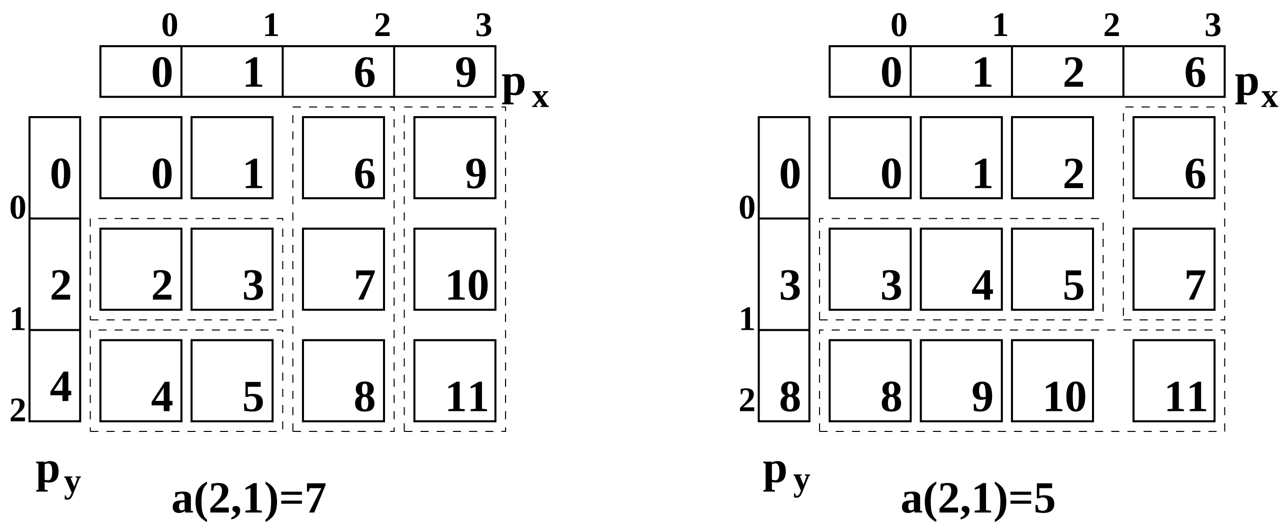
Strategy: *preserve the access method!*

Tactic: *split the pages!*:



Addressing a dynamic multidimensional array

The axial indices are already there: use them also to keep track of the history of splitting.



$$
\begin{aligned}
a(i,j) &= \max(p_x(i), p_y(j)) + \text{ the other one}\\
a(2,1) &= \max(p_x(2), p_y(1)) + \text{ the other one}
\end{aligned}
$$

the left-hand example

$$
\begin{aligned}
&= \max(6,2) + \text{ the other one}\\
&= 6 + j = 6 + 1 = 7
\end{aligned}
$$

the right-hand example

$$
\begin{aligned}
&= \max(2,3) + \text{ the other one}\\
&= 3 + i = 3 + 2 = 5
\end{aligned}
$$

- Similarly for higher dimensions:
  - Use the maximum of the d page entries in the axial indexes to determine the first coordinate,
  - then the remaining coordinates address the page within the $(d-1)$-dimensional slab in the conventional way.

- What about splits in the *middle* of the file?
  - Add the new slab of pages to the outer face, as shown above,
  - and use the axial index to point to it out of order. (Consider swapping 3 and 8 in p_y in the right-hand example, above, for a result of splitting pages 0, 1, 2, 6.)

Multipage Search and Insert

Split Criteria

1. (π) If $\pi > \pi_0$ then split.

2. (α) Split unless $\alpha < \alpha_0$.

Direction criteria

3. (Shape) Increase f_i for the axis, i, for which V_i/f_i is largest (in order to equalize all V_i/f_i, as far as possible).

4. (π) Split in the direction so that π is minimized. (N.B. Keep a log of overflows in the axial index for each row, column, ..)

5. (α) Increase f_i for the axis, i, for which f_i is largest (to create least number, n/f_i, of new pages and so decrease α the least).

Shift criterion

6. (π) If $\pi > \pi_0$ and shifting a boundary will make $\pi \leq \pi_0$, shift in the direction so that π is minimized.

Note that shape and α criteria are easy to calculate. The π criterion in (1) must be tested by doing or simulating the shifts or splits, and so is much more expensive. However, the π criteria deal directly with what is usually the important consideration, namely keeping the probe factor down.

Algorithm MPI

(A collection of alternative algorithms.)

Try to shift first:

A. 6, 1, 3, 4, 5	emphasizes π
B. 6, 1, 3, 5, 4	π, then α
C. 6, 2, 3, 5, 4	split emphasizes α
D. 6, 2, 3, 4, 5	split using α, then π

No shift:

A', B', C', D'	as above, but without 6