Clifford Algebra in Two and Three Dimensions

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The purpose of Clifford (or "geometrical") algebra is to describe space without using coordinates and in such a way as to describe any number of dimensions without changing the formalism.

The denizens of a d-dimensional space include points, edges, faces, volumes and so on: they themselves are k-dimensional with k ranging from 0 (point) through 1 (edge), 2 (face), 3(volume) and so on up to d (hypervolume). Furthermore, each class of denizen can be described as a linear combination of components: points are their own 0-dimensional components, edges have d 1-dimensional components, faces have $d!2$ ("d choose 2") 2-dimensional components, volumes have $d!3$ 3-dimensional components, etc. This adds up to

$$d!0 + d!1 + d!2 + .. + d!d = 2^d$$

components in all, so a Clifford algebra is a 2^d-dimensional linear algebra to describe d-dimensional space.

The following is a concrete interpretation of the more abstract discussions in books such as [**?**] and [**?**].

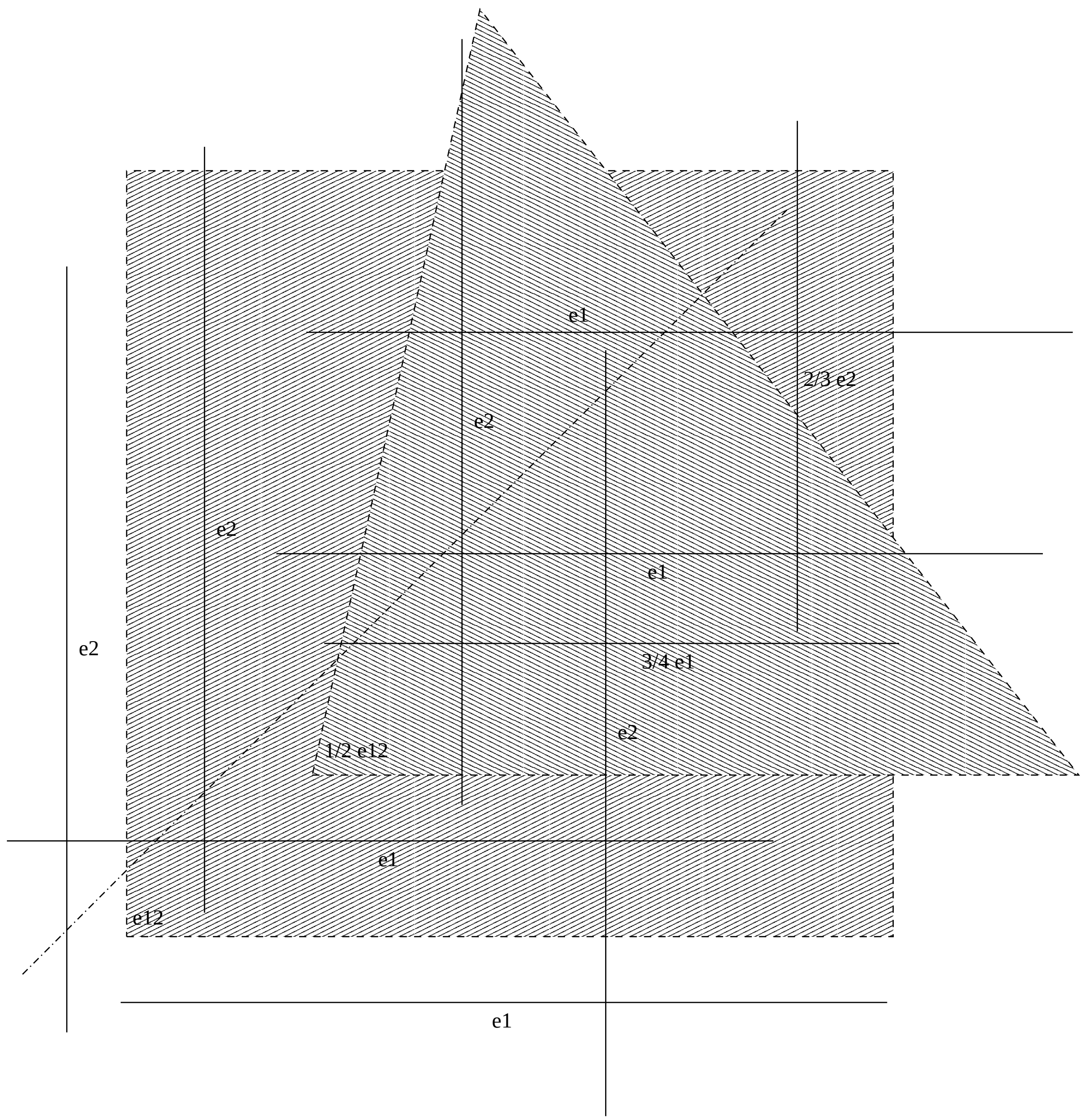
1 Two dimensions

Here are some 1- and 2-dimensional denizens of a 2-dimensional space. Note that they have no absolute position—all the **e1**s are equivalent to each other, as are all the **e2**s—or shape, in the case of faces. They do have magnitudes: the diagram shows a 3/4 **e1** as well as all the other **e1**s (whose lengths are thus 1), and a 2/3 **e2** as well as all the other **e2**s; it also shows a 1/2 **e12** as well as a unit-area **e12**.

The linear combinations appear in the diagonal edge, which is expressed

$$\sqrt{2}\mathbf{e1} + \sqrt{2}\mathbf{e2}$$

(diagrams below will show $\sqrt{2}$ as rt2, and similarly for other roots). This edge also has no absolute position, and could be drawn anywhere in the diagram. (This edge is resolved into two orthonormal components, **e1** and **e2**. Components need be neither orthogonal nor normalized, but orthonormal components are easier to use, and we stick with them in this discussion.)



Thus we see we can add elements of a Clifford algebra and multiply them by scalars. The other Clifford operation is a product of elements. For any edge, **v**, the product **vv** is defined to be its length squared, v^2. Thus **e1e1**=1=**e2e2**, since these components are normalized. We can add to this definition by saying **e1e2**=**e12**. But here a twist comes up: **e2e1**=−**e1e2**, which is to say that **e1** and **e2** anticommute. The reason is that **e12** defines the plane **e1** and **e2** are in (or rather, any area=1 part of the plane, since **e12** is normalized, too).

To see this, think of the plane **e12** in 3-space: it has two sides. If you pretend to push **e1** into **e2** with the fingers of your right hand, your thumb sticks out of that plane in some direction. But if you go around the back of the plane and push **e2** into **e1**, again with the fingers of your right hand, your thumb now points in the other direction. So **e2e1** is the inverse of **e1e2**, hence the sign change.

Although this Clifford product is not commutative, we will define it to be associative. Now we can see that

$$\mathbf{e12e12} = -\mathbf{e12e21} = -\mathbf{e1e2e2e1} = -\mathbf{e1e1} = -1$$

so that **e12** is "the square root of −1". This "the square root of −1" is a red herring. It is more important and much more useful to think of **e12** as a right-angle rotation. Here is why.

$$\mathbf{e1e12} = \mathbf{e2}$$
$$\mathbf{e2e12} = -\mathbf{e1}$$

So postmultiplying by **e12** is a counterclockwise right-angle rotation. (And premultiplying is a clockwise right-angle rotation.)

To work with arbitrary edges but keep the work simple, we will use the general normalized edge, $c\mathbf{e1} + s\mathbf{e2}$, where c is short for $\cos(\theta)$ and s is short for $\sin(\theta)$, for some angle θ (which we will from now on simply define as the *pair* (c, s)). Clearly $c^2 + s^2 = 1$ and we can see, by taking the product, that the edge is normalized:

$$(c\mathbf{e1} + s\mathbf{e2})(c\mathbf{e1} + s\mathbf{e2}) = (c^2 + s^2) + (cs - sc)\mathbf{e12} = 1$$

Let's rotate this by a counterclockwise right angle:

$$(c\mathbf{e1} + s\mathbf{e2})\mathbf{e12} = -s\mathbf{e1} + c\mathbf{e2}$$

1.1 Rotation

Now we can experiment with the product of two edges, not the same, and not orthogonal.

$$(\mathbf{e1})(c\mathbf{e1} + s\mathbf{e2}) = c + s\mathbf{e12}$$

This rather reminds us of the complex number, $c + is$, but, again, it is more profitable to think of it as an operator.

$$(\mathbf{e1})(c + s\mathbf{e12}) = c\mathbf{e1} + s\mathbf{e2}$$

It is the operator that turns **e1** into $c\mathbf{e1} + s\mathbf{e2}$. This is easy to see if we consider any two normalized edges **u** and **v**:

$$\mathbf{uuv} = \mathbf{v}$$

and so the operator **uv**, postmultiplying **u**, turns it into **v**. If $c + s\mathbf{e12}$ has the above effect on every edge in the space, then it is a *rotation* through angle (c, s) (counterclockwise if the angle is positive). Let's try, for any a, b:

$$(a\mathbf{e1} + b\mathbf{e2})(c + s\mathbf{e12}) = a(c\mathbf{e1} + s\mathbf{e2}) + b(-s\mathbf{e1} + c\mathbf{e2})$$

which rotates the $a\mathbf{e1}$ component through (c, s), and likewise the $b\mathbf{e2}$ component: $c + s\mathbf{e12}$ is the rotation operator in 2-dimensional space.

Let's try it explicitly:

$$(c\mathbf{e1} + s\mathbf{e2})(c'\mathbf{e1} + s'\mathbf{e2}) = cc' + ss' + (cs' - sc')\mathbf{e12} = C + S\mathbf{e12}$$

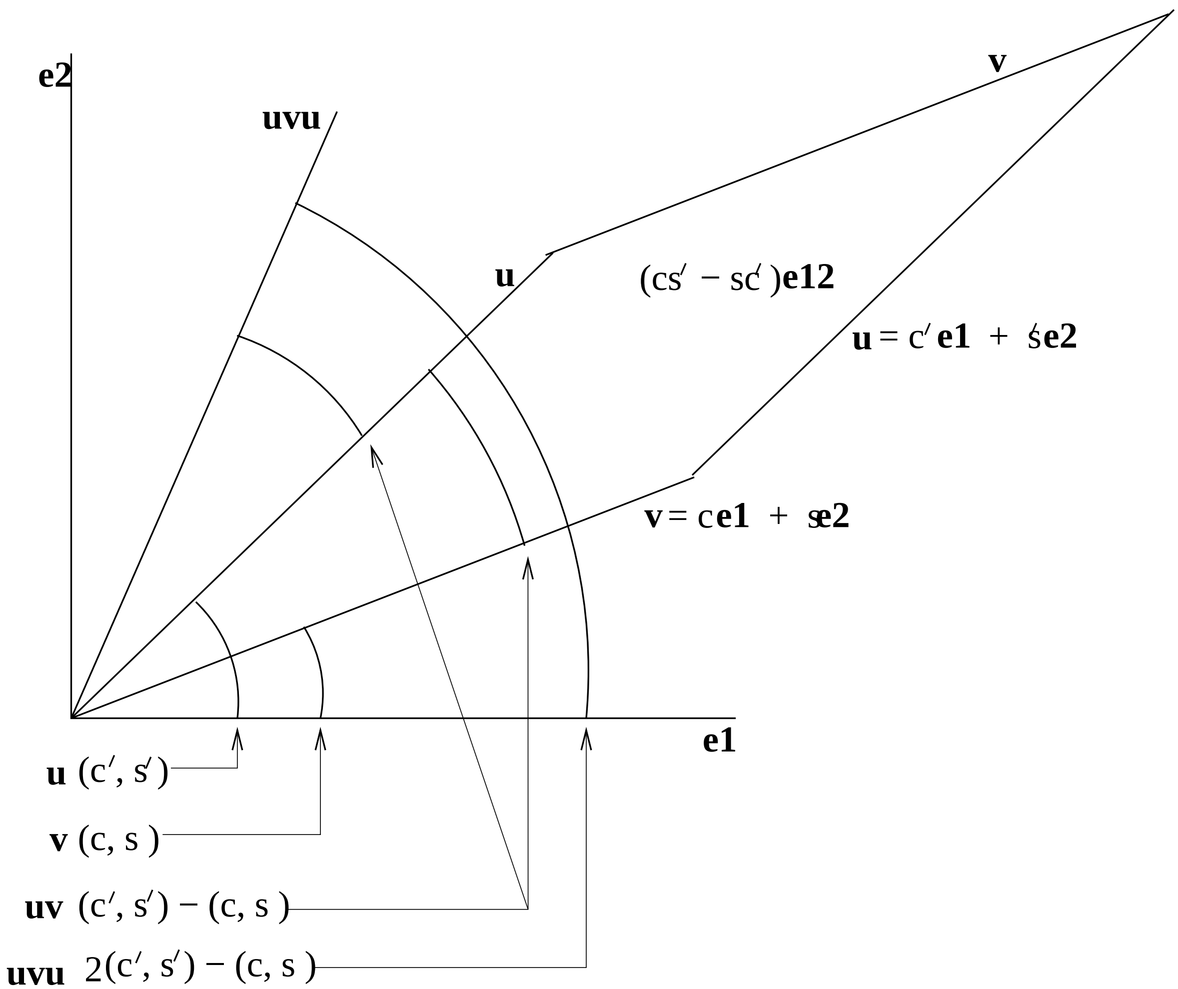
where $C = \cos((c', s') - (c, s))$ and $S = \sin((c', s') - (c, s))$: the rotation here is through the difference between the two angles. The diagram shows the generated face, $S\mathbf{e12}$, with $S = cs' - sc'$ being the area of the parallelopiped made by the two edges and their duplicates. (Note that for clarity the diagram shows edges starting from a common origin: they could in fact be drawn anywhere in the plane.) It is obvious that the area of this face is 1 if the two operands of the product are normalized and orthogonal to each other; in this case, the scalar component, C, is zero. Conversely, if the two operands are normalized and identical, the area, S, is zero and the scalar component is 1, namely the length.

We can also use a matrix to describe this rotation. We show it here for two reasons. First, writing them in matrix form makes complicated calculations, which we will have later, more manageable. Second, matrix multiplication gives us an easy way to remember the trigonometry rules for combining cosines and sines. Suppose we have an edge $x\mathbf{e1} + y\mathbf{e2}$ which we want to rotate through angle (c, s). We can write

$$(x\mathbf{e1} + y\mathbf{e2})(c + s\mathbf{e12}) = (cx - sy)\mathbf{e1} + (sx + cy)\mathbf{e2}$$

as

$$(\mathbf{e1}, \mathbf{e2}) \begin{pmatrix} c & -s \\ s & c \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$



Now consider two rotations: they produce a third rotation through the sum of the angles,

$$\begin{pmatrix} c & -s \\ s & c \end{pmatrix} \begin{pmatrix} c' & -s' \\ s' & c' \end{pmatrix} = \begin{pmatrix} C & -S \\ S & C \end{pmatrix}$$

where the cosine and sine of the sum are $C = cc' - ss'$ and $S = cs' + sc'$, as we can see by taking the matrix product.

(Note that x and y give the length, $l = \sqrt{x^2 + y^2}$, and the direction, $(x/l, y/l)$, of an *edge*. They are not the coordinates of a *point*. If, however, we supplement the Clifford algebra with an origin, then (x, y) can be seen as the coordinates of a point, and the above matrix form is just the familiar rotation of the point relative to the axes.)

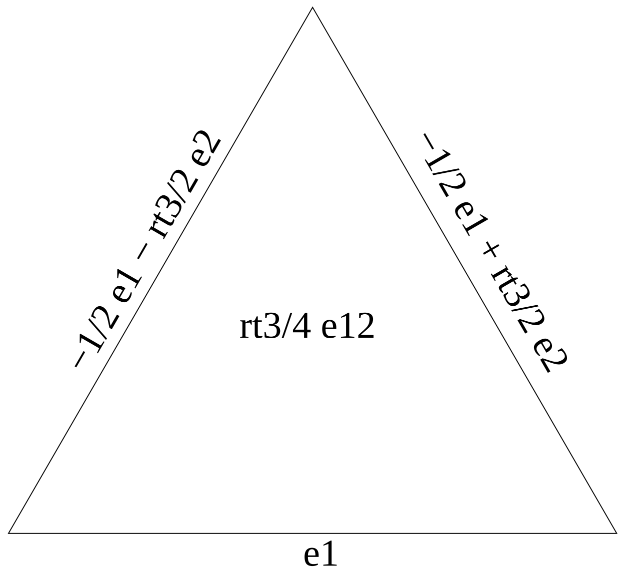
As an exercise, we use rotations to draw an equilateral triangle in the plane **e12** given a side **e1**. For the second side, we rotate **e1** by a third of a revolution

$$\mathbf{e1}(-1/2 + \sqrt{3}/2\ \mathbf{e12}) = -1/2\mathbf{e1} + \sqrt{3}/2\ \mathbf{e2}$$

The same rotation again, or just doubling the rotation, gives the third side.

$$\mathbf{e1}(-1/2 - \sqrt{3}/2\ \mathbf{e12}) = -1/2\mathbf{e1} - \sqrt{3}/2\ \mathbf{e2}$$

Here is the triangle, shown, again for clarity, as a triangle rather than just as three sets of unit-length edges.



1.2 Reflection

For normalized edges **u** and **v** we saw that **uuv** = **v**, and similarly **uvv** = **u**, so premultiplying **v** by **uv** maps (rotates) it into **u**. (Thus, premultiplying **e1** by c + s**e12** rotates it backwards into c**e1** $-s$**e2**.) What does **uvu** mean? This is the *reflection* of **v** in **u**. Try **u**=**e1** and resolve **v** into parallel and perpendicular components, **v** = c**e1** + s**e2**. Then

$$\mathbf{e1}(c\mathbf{e1} + s\mathbf{e2})\mathbf{e1} = c\mathbf{e1} - s\mathbf{e2}$$

The perpendicular component changes sign, so the result is the reflection in **e1**.

We can argue that **e1** could be in any direction, and so the result is generally the reflection, as we said. But it is useful trigonometry practice to spell it out. We set **u**=c'**e1** + s'**e2** and **v**=c**e1** + s**e2**, where (c', s') and (c, s) are two different angles. Then

$$(c'\mathbf{e1} + s'\mathbf{e2})(c\mathbf{e1} + s\mathbf{e2})(c'\mathbf{e1} + s'\mathbf{e2}) = C\mathbf{e1} + S\mathbf{e2}$$

where $C = \cos((c', s') - (c, s) + (c', s'))$ and S=$\sin((c', s') - (c, s) + (c', s'))$. The result is at angle $2(c', s') - (c, s)$, which you can see in the diagram on page **??** is the reflection of the inner edge in the edge written before and after it. These angles, and the reflection, **uvu** of **v** in **u**, are also shown in that diagram.

We can use another argument to persuade ourselves that **uvu** is the reflection of **v** in **u**. Postmultiplied, **vu** rotates any edge by the angle from **v** to **u**. Thus **uvu** is the rotation of **u** by this angle, and that makes it the reflection of **v** in **u**. (There is a similar premultiplication argument.) Finally, since **uuvuu** = **v**, reflecting twice restores the element.

From a reflection, **uvu**, we can get the *projection* of **v** on **u** by averaging (**uvu** + **v**)/2. If we write these as operators, $\mathcal{F}$ and $\mathcal{P}$, then $\mathcal{P} = (\mathcal{F} + \mathcal{I})/2$, where $\mathcal{I}$ is the identity operator. Let's look at the matrix form of reflection, (**e1**,**e2**)$\mathcal{F}\begin{pmatrix} x \\ y \end{pmatrix}$.

$$\mathcal{F} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - 2\begin{pmatrix} s^2 & -cs \\ -cs & c^2 \end{pmatrix} = \mathcal{I} - 2\mathcal{P}$$

The component of **v** orthogonal to its projection on **u** can be found by subtraction,

$$\mathcal{P} - \mathcal{I} = \begin{pmatrix} s^2 & -cs \\ -cs & c^2 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

If there were an origin and **v** were interpreted as a point, then this component is the *perpendicular* from the point to edge **u** and its length is the *distance*.

1.3 Shear

So far we can rotate by postmultiplying an edge by $c + s\mathbf{e12}$, and we can reflect an edge $\mathbf{v}$ in a normalized edge $\mathbf{u}$ to $\mathbf{uvu}$. Reflection is related to projection, which is related, if we add an origin, to the perpendicular and the distance from a point to a line. There are other transformations which Clifford algebra cannot do, short of using tricks. These are the *shear* transformations (and they include scaling differently in different directions).

A special such shear, which preserves the direction (but not the length) of $\mathbf{e1} + \mathbf{e2}$, maps any edge $s\mathbf{e1} + y\mathbf{e2}$ into

$$(\mathbf{e1}, \mathbf{e2}) \begin{pmatrix} e & t \\ t & e \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

where $e^2 - t^2 = 1$. (We can also write this as scaling by factor f along edge $\mathbf{e1} + \mathbf{e2}$ and by $1/f$ along $\mathbf{e1}$ - $\mathbf{e2}$:

$$\begin{pmatrix} e & t \\ t & e \end{pmatrix} = \frac{f}{2} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} + \frac{1}{2f} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$

so $e = (f^2 + 1)/2f$ and $t = (f^2 - 1)/2f$.)

How do we describe this shear in Clifford algebra? The trick is to suppose that $\mathbf{e2e2}{=}{-1}$ instead of 1. Then, $\mathbf{e1e1}{=}1$ as before, but $\mathbf{e12e12}{=}{-}\mathbf{e2e1e1e2}{=}1$. Now let's express the shear transformation as a rotation.

$$(x\mathbf{e1} + y\mathbf{e2})(e + t\mathbf{e12}) = (\mathbf{e1},\mathbf{e2}) \begin{pmatrix} e & t \\ t & e \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

(For physics, where Clifford algebra is mostly used, this describes a Minkowski space, as opposed to Euclidean. The shear transformation we have written is the Lorentz transformation of special relativity.)

2 Three dimensions

When we add a third dimension, our rotation operator no longer works.

$$\mathbf{e3}(c + s\mathbf{e12}) = c\mathbf{e3} + s\mathbf{e123}$$

(Of course it still works for any edge completely in the $\mathbf{e12}$ plane; but it fails for components orthogonal to that plane.) We need to try something else.

Let's think of a rotation as two reflections. To rotate $x\mathbf{e1} + y\mathbf{e2}$ by an angle (c, s) we could reflect it first in $\mathbf{e1}$ and then in the edge $c_J\mathbf{e1} + s_J\mathbf{e2}$ which makes angle (c_J, s_J) with $\mathbf{e1}$, half of (c, s). (Instead of writing $\cos(\theta/2)$, we invent the abbreviation c_J, since J looks a little like 2 upside-down.) Clearly, such reflections will leave any edge component invariant if it is orthogonal to the plane containing $\mathbf{e1}$ and $c_J\mathbf{e1} + s_J\mathbf{e2}$, namely $\mathbf{e12}$, and that is what we want. The diagram shows that, in the plane, the two reflections are indeed the rotation.

The two reflections, applied to $x\mathbf{e1} + y\mathbf{e2}$, are written

$$\begin{aligned}&(c_J\mathbf{e1} + s_J\mathbf{e2})\mathbf{e1}(x\mathbf{e1} + y\mathbf{e2})\mathbf{e1}(c_J\mathbf{e1} + s_J\mathbf{e2})\\ &= (c_J - s_J\mathbf{e12})(x\mathbf{e1} + y\mathbf{e2})(c_J + s_J\mathbf{e12})\\ &= (\mathbf{e1}, \mathbf{e2}) \begin{pmatrix} c & -s \\ s & c \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}\end{aligned}$$

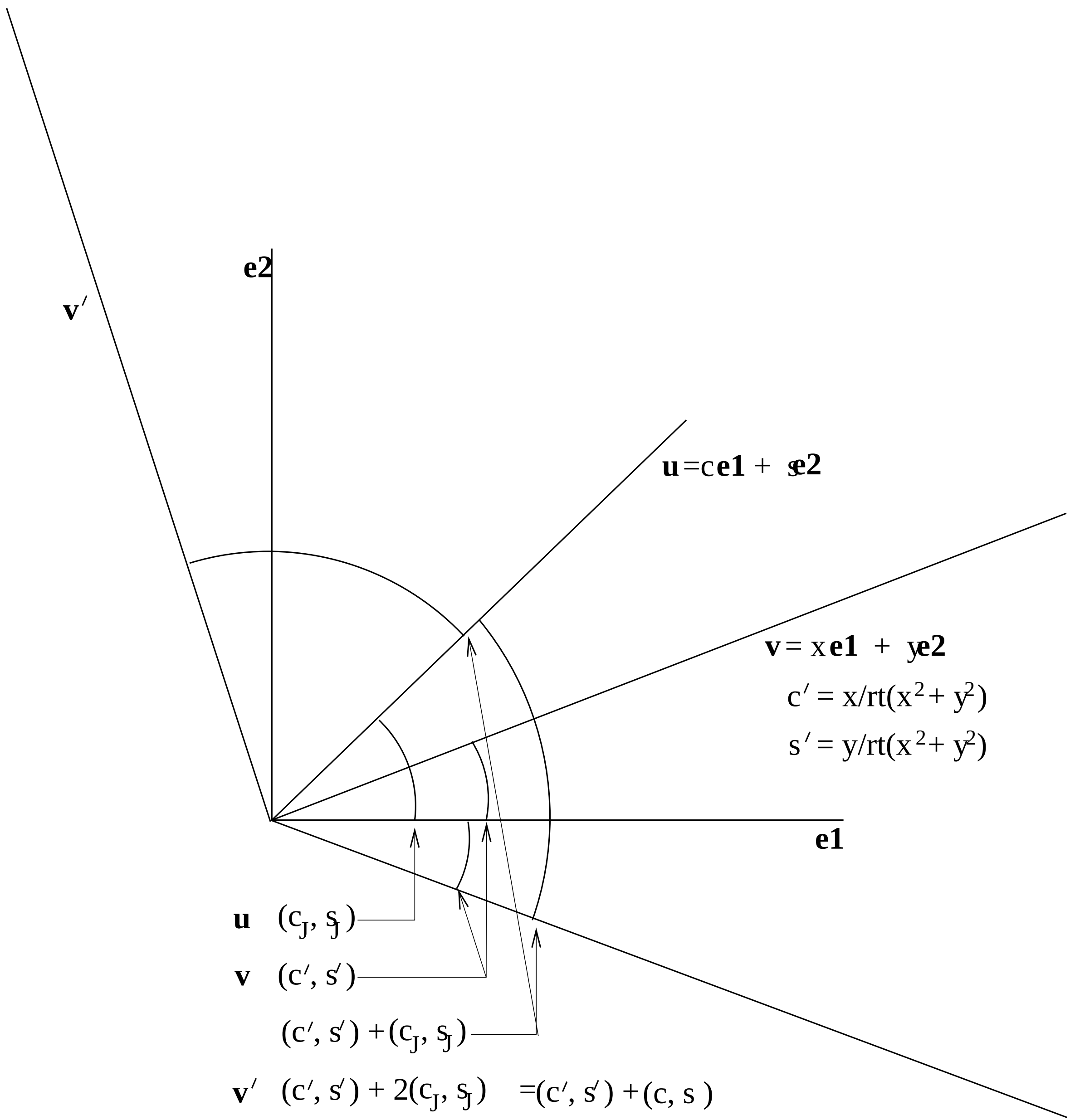
2.1 Rotations

In 3D, this half-angle operator will rotate any edge $x\mathbf{e1} + y\mathbf{e2} + z\mathbf{e3}$ by angle (c, s) in any plane $r\mathbf{e12} + p\mathbf{e23} + q\mathbf{e31}$ (normalized so that $p^2 + q^2 + r^2 = 1$):

$$(c_J - s_J(r\mathbf{e12} + p\mathbf{e23} + q\mathbf{e31}))(x\mathbf{e1} + y\mathbf{e2} + z\mathbf{e3})(c_J + s_J(r\mathbf{e12} + p\mathbf{e23} + q\mathbf{e31}))$$

gives, after some algebra and doubling of half-angles,

$$(\mathbf{e1}, \mathbf{e2}, \mathbf{e3}) \left(\begin{pmatrix} c & -sr & sq \\ sr & c & -sp \\ -sq & sp & c \end{pmatrix} + (1 - c) \begin{pmatrix} p \\ q \\ r \end{pmatrix} (p, q, r) \right) \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$



Furthermore, we can show that this plane, $r\mathbf{e12} + p\mathbf{e23} + q\mathbf{e31}$, is orthogonal to edge $p\mathbf{e1} + q\mathbf{e2} + r\mathbf{e3}$, which is thus the axis of rotation:

$$(p\mathbf{e1} + q\mathbf{e2} + r\mathbf{e3})\mathbf{e123} = r\mathbf{e12} + p\mathbf{e23} + q\mathbf{e31}$$

using the 3D volume, $\mathbf{e123}$, to produce the orthogonal element, as we used the 2D area, $\mathbf{e12}$, earlier. ((p, q, r) is an eigenvector of the above matrix: try it!)

(Well, this argument just extends what we found in 2D to 3D, so to prove the orthogonality of the face to the edge, we must take the product of two different edges which we know are orthogonal to $p\mathbf{e1} + q\mathbf{e2} + r\mathbf{e3}$. These could be $q\mathbf{e1} - p\mathbf{e2}$ and $pr^2\mathbf{e1} + qr^2\mathbf{e2} - r(p^2 + q^2)\mathbf{e3}$ (which are also orthogonal to each other—but not normalized). Their product gives $(1 - r^2)r(r\mathbf{e12} + p\mathbf{e23} + q\mathbf{e31})$, which is the (unnormalized) plane we started with.)

With some more algebra, we can now get the famous formula for two 3D rotations, first about $p\mathbf{e1} + q\mathbf{e2} + r\mathbf{e3}$ then about $p'\mathbf{e1} + q'\mathbf{e2} + r'\mathbf{e3}$. We'll write the half-angles as (c, s) and (c', s'), respectively, dropping the Js for readability: we must remember that all angles are half-angles in the rest of this paragraph. The result is a rotation by (c'', s'') about $p''\mathbf{e1} + q''\mathbf{e2} + r''\mathbf{e3}$:

$$
\begin{aligned}
&(c + s(r\mathbf{e12} + p\mathbf{e23} + q\mathbf{e31}))(c' + s'(r'\mathbf{e12} + p'\mathbf{e23} + q'\mathbf{e31}))\\
&= cc' - ss'(rr' + pp' + qq')\\
&\quad + (sc'r + cs'r' + ss'(qp' - pq'))\mathbf{e12}\\
&\quad + (sc'p + cs'p' + ss'(rq' - qr'))\mathbf{e23}\\
&\quad + (sc'q + cs'q' + ss'(pr' - rp'))\mathbf{e31}
\end{aligned}
$$

Note that two 3D rotations do not commute: (c', s', p', q', r') before (c, s, p, q, r) is different from what we just did, $(c', s'.p', q', r')$ after (c, s, p, q, r). Check this with 90-degree rotations about (1,0,0) and (0,1,0)!

This rotation operation applies to faces as well as edges. Here is a complete description of a rotation (c, s) in **e12** applied to each of the base elements, in matrix form.

$$
\begin{pmatrix}
1 & & & & & & & \\
 & c & s & & & & & \\
 & -s & c & & & & & \\
 & & & 1 & & & & \\
 & & & & 1 & & & \\
 & & & & & c & s & \\
 & & & & & -s & c & \\
 & & & & & & & 1
\end{pmatrix}
\begin{pmatrix}
1\\ \mathbf{e1}\\ \mathbf{e2}\\ \mathbf{e3}\\ \mathbf{e12}\\ \mathbf{e23}\\ \mathbf{e31}\\ \mathbf{e123}
\end{pmatrix}
$$

Clearly, **e3** and **e12** are unaltered by the rotation, and **e23** and **e31** transform together in the same way **e1** and **e2** do.

We can use rotations to extend the equilateral triangle example to a tetrahedron. First, we must find one of the edges above the plane of the **e12** face: call this edge $p\mathbf{e1} + q\mathbf{e2} + r\mathbf{e3}$, with $p^2 + q^2 + r^2 = 1$ since we gave all edges unit length. This edge makes a 60-degree angle with the two edges of the triangle it shares a vertex with, let's say **e1** and $-(-1/2\mathbf{e1} - \sqrt{3}/2)\mathbf{e2})$ (we've changed the direction of the second, so that it starts from the same vertex as **e1**). Since the product of two edges gives the angle between them and the plane they are in, we get two equations.

$$\mathbf{e1}(p\mathbf{e1} + q\mathbf{e2} + r\mathbf{e3}) = 1/2 + \sqrt{3}/2(\text{some face})$$

$$(\mathbf{e1} + \sqrt{3}/2\mathbf{e2})(p\mathbf{e1} + q\mathbf{e2} + r\mathbf{e3})/2 = 1/2 + \sqrt{3}/2(\text{some other face})$$

The first gives $p = 1/2$ and the second gives $(p + \sqrt{3}q)/2 = 1/2$, hence $q = 1/2\sqrt{3}$. Normalization gives two solutions for r, one above and one below the plane of **e1**: $r = \pm\sqrt{2/3}$. We choose the positive solution, above the plane.

This gives both the new edge, $1/2\mathbf{e1} + 1/2\sqrt{3}\ \mathbf{e2} + \sqrt{2/3}\ \mathbf{e3}$, and the two faces it shares: $1/4\sqrt{3}\ \mathbf{e12}\ -1/\sqrt{6}\ \mathbf{e31}$, with **e1**; and $-1/4\sqrt{3}\ \mathbf{e12} + 1/2\sqrt{2}\mathbf{e23}\ -\sqrt{3}/2\sqrt{2}\ \mathbf{e31}$, with $\mathbf{e1} + \sqrt{3}/2\mathbf{e2}$.

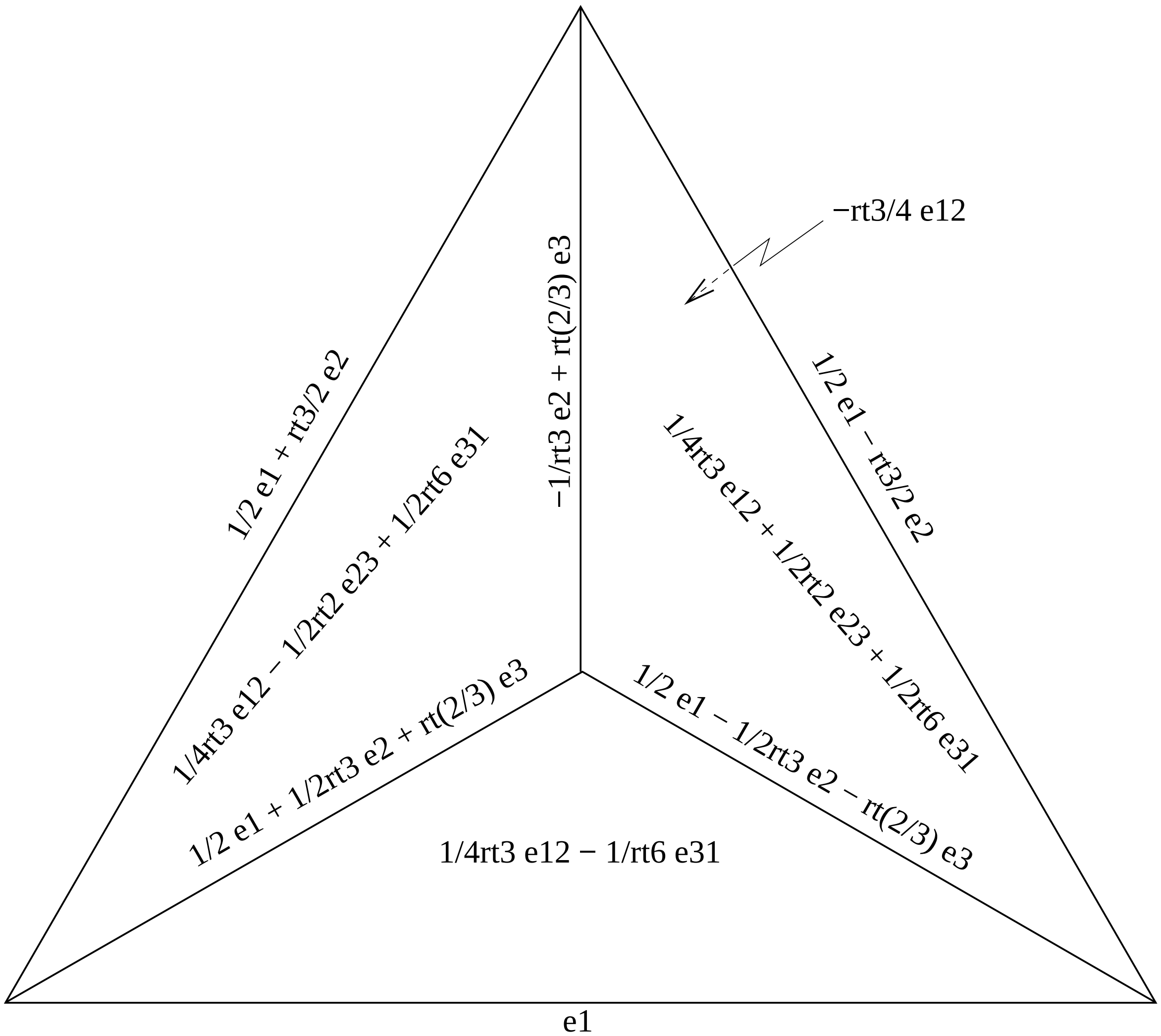
We can rotate these in **e12** by one third and two thirds of a revolution to generate the other edges and faces, just as we rotated **e1** to generate the equilateral triangle. (We must use the half-angle transformation, since none of the edges or faces is in the plane of **e12**.) The results are shown in the diagram. This tetrahedron has been derived entirely without coordinates. (Fixing the origin at some suitable place, such as the lower left vertex, enables us to derive coordinates of all the other vertices directly from the expressions for the edges.)

2.2 Angles

We already know that the cosine of the angle between two edges is the scalar component of the product of the edges, and that this product also gives us the face defined by a parallelopiped of the two edges. In three dimensions, we also have the "dihedral" angle between two faces. Let's try the product of two of the faces of the tetrahedron.

$$-\mathbf{e12}(\mathbf{e12}\ -2\sqrt{2}\ \mathbf{e31})/3 = 1/2 + 2\sqrt{2}/3\ \mathbf{e23}$$

The plane orthogonal to both these faces is **e23**, and in that plane the angle between the two is $(-1/3, -2\sqrt{2}/3)$, which is an internal angle of just over 70 degrees. Taking all six pair-of-face products in the tetrahedron gives the same angle each time but different orthogonal planes.



2.3 Reflection

Three-dimensional reflections in faces are analogous to two-dimensional reflections in edges. For example, we can reflect the edge $a\mathbf{e1} + b\mathbf{e2} + c\mathbf{e3}$ in the face $(\mathbf{e12} - \mathbf{e23})/\sqrt{2}$, which contains **e2** and falls 45 degrees between **e1** (or **e12**) and **e3** (or **e23**). (Note the minus sign!) The result swaps a and c:

$$(\mathbf{e12} - \mathbf{e23})(a\mathbf{e1} + b\mathbf{e2} + c\mathbf{e3})(\mathbf{e12} - \mathbf{e23})/2 = c\mathbf{e1} + b\mathbf{e2} + a\mathbf{e3}.$$

Reflecting an arbitrary edge, $x\mathbf{e1} + y\mathbf{e2} + z\mathbf{e3}$, in the normalized face $r\mathbf{e12} + p\mathbf{e23} + q\mathbf{e31}$ gives

$$(r\mathbf{e12} + p\mathbf{e23} + q\mathbf{e31})(x\mathbf{e1} + y\mathbf{e2} + z\mathbf{e3})(r\mathbf{e12} + p\mathbf{e23} + q\mathbf{e31}) =$$

$$(\mathbf{e1}, \mathbf{e2}, \mathbf{e3})\left(\begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix} - 2\begin{pmatrix} p \\ q \\ r \end{pmatrix}(p, q, r)\right)\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

Reflecting the face orthogonal to that edge, $c\mathbf{e12} + a\mathbf{e23} + b\mathbf{e13}$ in this same face gives

$$(\mathbf{e12} - \mathbf{e23})(c\mathbf{e12} + a\mathbf{e23} + b\mathbf{e31})(\mathbf{e12} - \mathbf{e23})/2 = a\mathbf{e12} + c\mathbf{e23} + b\mathbf{e13}.$$

We could specify this reflection in another way, by building from the base edges, as **e1** $\leftrightarrow$ **e3**, **e2** $\leftrightarrow$ **e2**, and swap pairs (because it is a reflection): **e12** = **e1e2** $\leftrightarrow$ **e3e2** (swap)$\leftrightarrow$ **e2e3** = **e23**, and so on.

We can also ask what it means to "reflect" an edge, $a\mathbf{e1} + b\mathbf{e2} + c\mathbf{e3}$, in an edge, say $\mathbf{e1}$. This preserves the $\mathbf{e1}$ coordinate but inverts the other two:

$$\mathbf{e1}(a\mathbf{e1} + b\mathbf{e2} + c\mathbf{e3})\mathbf{e1} = a\mathbf{e1} + -b\mathbf{e2} + -c\mathbf{e3}.$$

A similar thing happens if we "reflect" the face, $c\mathbf{e12} + a\mathbf{e23} + b\mathbf{e13}$, in edge $\mathbf{e1}$:

$$\mathbf{e1}(c\mathbf{e12} + a\mathbf{e23} + b\mathbf{e13})\mathbf{e1} = -c\mathbf{e12} + a\mathbf{e23} + -b\mathbf{e13},$$

and this checks against an alternative prescription: $\mathbf{e1} \leftrightarrow \mathbf{e1}$, $\mathbf{e2} \leftrightarrow -\ \mathbf{e2}$, $\mathbf{e3} \leftrightarrow -\ \mathbf{e3}$, and don't swap pairs (because it's an inversion, not a reflection).

Here are transformations of each base component in the base faces and edges. In these special cases, each transformation agrees with our prescriptions for reflection (in planes) or inversion (in lines).

	element, e	$\mathbf{e1}$	$\mathbf{e2}$	$\mathbf{e3}$	$\mathbf{e12}$	$\mathbf{e23}$	$\mathbf{e31}$
reflections	$\mathbf{e12}\ e\ \mathbf{e12}$	$\mathbf{e1}$	$\mathbf{e2}$	$-\mathbf{e3}$	$-\mathbf{e12}$	$\mathbf{e23}$	$\mathbf{e31}$
	$\mathbf{e23}\ e\ \mathbf{e23}$	$-\mathbf{e1}$	$\mathbf{e2}$	$\mathbf{e3}$	$\mathbf{e12}$	$-\mathbf{e23}$	$\mathbf{e31}$
	$\mathbf{e31}\ e\ \mathbf{e31}$	$\mathbf{e1}$	$-\mathbf{e2}$	$\mathbf{e3}$	$\mathbf{e12}$	$\mathbf{e23}$	$-\mathbf{e31}$
inversions	$\mathbf{e1}\ e\ \mathbf{e1}$	$\mathbf{e1}$	$-\mathbf{e2}$	$-\mathbf{e3}$	$-\mathbf{e12}$	$\mathbf{e23}$	$-\mathbf{e31}$
	$\mathbf{e2}\ e\ \mathbf{e2}$	$-\mathbf{e1}$	$\mathbf{e2}$	$-\mathbf{e3}$	$-\mathbf{e12}$	$-\mathbf{e23}$	$\mathbf{e31}$
	$\mathbf{e3}\ e\ \mathbf{e3}$	$-\mathbf{e1}$	$-\mathbf{e2}$	$\mathbf{e3}$	$\mathbf{e12}$	$-\mathbf{e23}$	$-\mathbf{e31}$

2.4 Projection

Using $\mathcal{P} = (\mathcal{F} + \mathcal{I})/2$, we can find the projection of a edge in a face. For $a\mathbf{e1} + b\mathbf{e2} + c\mathbf{e3}$ in $(\mathbf{e12} - \mathbf{e23})/\sqrt{2}$, average this with $c\mathbf{e1} + b\mathbf{e2} + a\mathbf{e3}$ to get $(a+c)/2(\mathbf{e1} + \mathbf{e3}) + b\mathbf{e2}$. Since $\mathbf{e1}$ and $\mathbf{e3}$ have the same coefficient, this projected edge clearly lies in the $\mathbf{e12} - \mathbf{e23}$ plane. We can see it more clearly if we rotate this face 45 degrees in $\mathbf{e31}$ to get the two-dimensional edge $(a + c)/\sqrt{2}$ $\mathbf{e1} + b\ \mathbf{e2}$.

Projecting an arbitrary edge, $x\mathbf{e1} + y\mathbf{e2} + z\mathbf{e3}$, in the normalized face $r\mathbf{e12} + p\mathbf{e23} + q\mathbf{e31}$ gives

$$(\mathbf{e1}, \mathbf{e2}, \mathbf{e3}) \left(\begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix} - \begin{pmatrix} p \\ q \\ r \end{pmatrix} (p, q, r) \right) \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

To project a face onto another face, we must subtract instead of add: $\mathcal{P} = (\mathcal{F} - \mathcal{I})/2$ (remember the minus sign in $\mathbf{e12} - \mathbf{e23}$). Thus, projecting the face $c\mathbf{e12} + a\mathbf{e23} + b\mathbf{e31}$ in $\mathbf{e12} - \mathbf{e23}$ gives $(c - a)/2(\mathbf{e12} - \mathbf{e23})$, and this clearly lies in face $\mathbf{e12} - \mathbf{e23}$. Rotating it gives $(c - a)/\sqrt{2}\mathbf{e12}$.

Finally, as an example, let's reflect the tetrahedron in the plane formed by edge $1/2\ \mathbf{e1} + 1/2\sqrt{3}$ $\mathbf{e2} + \sqrt{2/3}\ \mathbf{e3}$ and its projection on plane $\mathbf{e12}$. This plane is $\sqrt{3}/2\ \mathbf{e31} - 1/2\ \mathbf{e23}$ (and the angle between the two edges in this plane is $(1/\sqrt{3}, \sqrt{2}/\sqrt{3})$). It reflects $\mathbf{e1} \leftrightarrow 1/2\ \mathbf{e1} + \sqrt{3}/2\ \mathbf{e2}$, $\mathbf{e2} \leftrightarrow \sqrt{3}/2\ \mathbf{e1} - 1/2\ \mathbf{e2}$, and $\mathbf{e3} \leftrightarrow \mathbf{e3}$. As a result, this reflection maps the edges and the faces of the tetrahedron into the appropriate other edges, or into themselves, with sign reversals as appropriate. (Reflecting faces give $\mathbf{e12} \leftrightarrow \mathbf{e12}$, $\mathbf{e23} \leftrightarrow 1/2\ \mathbf{e23} + \sqrt{3}/2\ \mathbf{e31}$, and $\mathbf{e31} \leftrightarrow \sqrt{3}/2\ \mathbf{e23} + 1/2\ \mathbf{e31}$, and map, say, $1/4\sqrt{3}\ \mathbf{e12} - 1/\sqrt{6}\ \mathbf{e31}$ correctly into $1/4\sqrt{3}\ \mathbf{e12} - 1/2\sqrt{2}\ \mathbf{e23} + 1/2\sqrt{6}\ \mathbf{e31}$.)