# On the Reconfiguration of Chains

(Extended Abstract)

Sue Whitesides* and Naixun Pei**

McGill University***

**Abstract.** A *chain* is a sequence of rigid rods or links consecutively connected at their endjoints, about which they may rotate freely. A *planar chain* is a chain whose rods lie in the plane, with rods allowed to pass over one another as they move. A *convex obtuse polygon* $P$ is a convex polygon with each interior angle not less than $\pi/2$. We consider the following reconfiguration problem.
*Given:* an $n$-link planar chain $\Gamma$ confined inside a convex obtuse polygon $P$ whose sides are all longer that the longest link of $\Gamma$; a point $p \in P$; and an endjoint of $\Gamma$. *Question:* Can $\Gamma$ be moved within $P$ so that the specified endjoint of $\Gamma$ reaches $p$?
We give a necessary and sufficient condition for a "yes" answer, and in this case we further give an algorithm for reaching $p$. The necessary and sufficient condition is independent of the initial configuration of $\Gamma$ and is checkable in time proportional to the number of links in the real RAM model of computation.

## 1 Introduction

A *linkage* is a collection of rigid rods or links, with links connected together at their endjoints. A *planar* linkage has its links confined to the plane; links may cross over one another and the locations of certain joints may be required to remain fixed to the plane. A *chain* is a linkage consisting of a sequence of links consecutively connected at their endjoints.

The *reachability problem* for a linkage $\Gamma$ constrained to lie inside a region $R$ is to determine, given a point $p \in R$ and a joint $A_j$ of $\Gamma$, whether $\Gamma$ can be moved within $R$ so that $A_j$ reaches $p$.

This paper solves the reachability problem for $n$-link planar chains confined within convex obtuse polygons. We define a *convex obtuse polygon* to be a convex polygon whose internal angles each measure $\pi/2$ or more. In particular, our paper gives an algorithm that decides whether a given endjoint of a chain confined within a convex obtuse polygon $P$ can reach a given point $p \in P$ and that produces a sequence of moves that bring the endjoint to $p$ when $p$ is reachable.

* sue@cs.mcgill.ca Supported by FCAR and NSERC.

** pei@cs.mcgill.ca

*** School of Computer Science, 3480 University St. #318, Montreal, Quebec H3A 2A7 CANADA

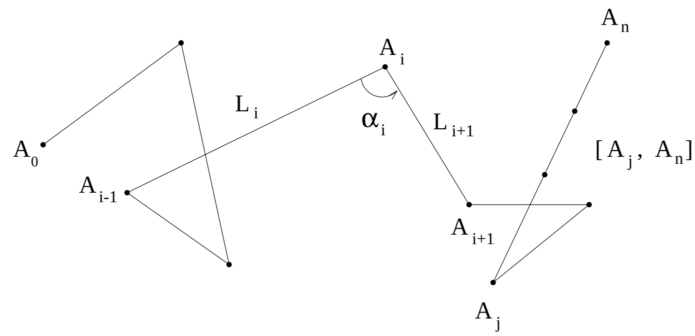


**Fig. 1.** Notation for chains.

The decision phase of the algorithm runs in time proportional to the number of links in the real RAM model of computation. Our results represent a significant improvement to the known results for this type of problem, as we explain further below.

Reachability and reconfiguration problems for linkages have been investigated by several researchers [3, 2, 5, 12, 1, 11, 6, 7, 4, 8, 9]. See Whitesides [16] for a survey.

Reconfiguration problems are often at least NP-hard when the number of degrees of freedom is not bounded [13, 4, 15]. To find fast reconfiguration algorithms, it is essential to understand what relationships between moving objects and their environments give rise to problems that are quickly solvable in spite of having many degrees of freedom.

At present, algorithms for fast reconfiguration of $n$-link chains have been given for very simple confining regions: circles, squares, equilateral triangles, or no confining region at all.

In this paper, we consider chains confined by arbitrary convex obtuse polygons. We require that the minimum side length of the confining polygon be greater than the length of the longest link in the chain. The results of [6] and [7] also require that the longest chain link be no longer than the side of the confining square.

For references on algorithmic, geometric motion planning in general, see for example Latombe's book [10] and books edited by Schwartz and Yap [13], and by Schwartz, Sharir and Hopcroft [14].

Before proceeding further, we introduce some terminology and notation, illustrated in Fig. 1. In an $n$-link chain $\Gamma$ with consecutive joints $A_0, \ldots, A_n$, the initial and final joints $A_0$ and $A_n$ are called *endjoints*; the others are called *intermediate joints.* The link between $A_{i-1}$ and $A_i$ $(0 < i \leq n)$ is denoted by $L_i$, and the length of $L_i$ is denoted by $l_i$. The angle at intermediate joint $A_i$ determined by rotating $L_i$ about $A_i$ counterclockwise to bring $L_i$ to $L_{i+1}$ is denoted by $\alpha_i$.

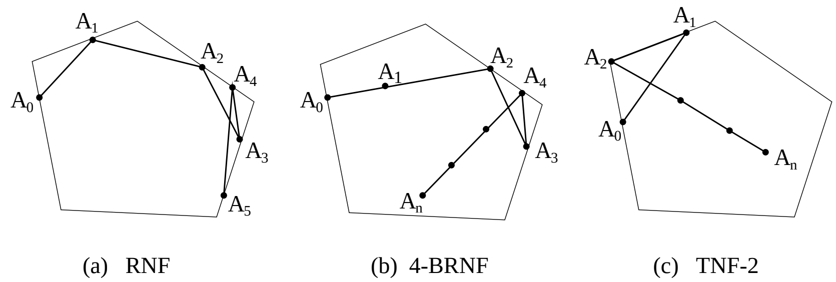


**Fig. 2.** Examples of normal forms.

An intermediate joint $A_i$ is called a *straight joint* if $\alpha_i = \pi$; otherwise, $A_i$ is called a *bending joint*. The subchain of $\Gamma$ with joints $A_i, A_{i+1}, \ldots, A_j (i < j)$, is denoted by $\Gamma(i,j)$. Subchain $\Gamma(i,j)$ is said to be *straight*, denoted by $[A_i, \ldots, A_j]$, if its links form a straight line segment with all interior joints straight.

We denote the distance between two points $x, y$ by $d(x,y)$; the line they determine by $l(x,y)$; the line segment they determine by $xy$; and the length of this segment by $|xy| = d(x,y)$.

We regard a polygon $P$ as a 2-dimensional closed set; we denote the boundary of $P$ by $\partial P$ and the length of the shortest side of $P$ by $s_{\min}$. For a planar closed region $Q$, we use $v_{\max}(p)$ to denote a point of $Q$ farthest from $p$, and $d_{\max}(p)$ to denote $d(p, v_{\max}(p))$. Obviously, if $Q$ is a polygon, $v_{\max}(p)$ is a vertex of $Q$ farthest from $p$.

Let $\Gamma$ be an $n$-link chain confined in an arbitary planar closed region $Q$, not necessarily polygonal or even convex, and let $p \in Q$. It is easy to verify that if $p$ is reachable by the endjoint $A_n$, then the following condition $(*)$ must hold:

**Condition** $(*)$ For all $i \in \{1, \ldots, n\}$, $l_i - \sum_{j=i+1}^{n} l_j \;\leq\; d_{\max}(p)$ . $\qquad (*)$

Note that $(*)$ is independent of the initial configuration of $\Gamma$. Also $(*)$ can be tested in time proportional to the number of links in the real RAM model of computation whenever $d_{\max}(p)$ is given.

In [7], Kantabutra proved that if $\Gamma$ is confined to a *square* whose side is longer than the longest link, $(*)$ is also *sufficient* for $A_n$ to reach $p$. Our paper generalizes this result from squares to arbitrary convex obtuse polygons.

**Assumptions:** From now on, we assume that $\Gamma$ denotes an $n$-link chain confined inside a convex obtuse polygon $P$ whose shortest side $s_{\min}$ is at least as long as the longest link of $\Gamma$. Joints of $\Gamma$ may lie on $\partial P$.

The rest of this paper is organized as follows. Section 2 shows that any chain can be brought to certain normal form configurations to be defined. Section 3 shows how to determine the points reachable by the endjoints of a chain and presents an algorithm to bring an endjoint to any given reachable point. The algorithm moves the chain in turn through the various normal forms given in Sect. 2. Section 4 concludes.

## 2 Normal Forms

We define three special configurations for $\Gamma$ as follows (refer to Fig. 2).

**Normal Forms:** $\Gamma$ is in *Rim Normal Form*, denoted RNF, if all its joints lie on $\partial P$. $\Gamma$ is in *k-Bending Rim Normal Form*, denoted $k$-BRNF, if there exist $k$ joints $A_{i_1}, \dots, A_{i_k}$ such that $A_{i_1}, \dots, A_{i_k}$ lie on $\partial P$, while any intermediate joints not among these $k$ are straight. Finally, $\Gamma$ is in *Tail Normal Form with index* $i_o$, denoted TNF-$i_o$, if there exists $i_0$, $i_0 \neq n$, such that $A_o, \dots, A_{i_o}$ lie on $\partial P$, $A_{i_o}$ lies at a vertex of $P$, and subchain $\Gamma(i_0, n)$ is straight.

Note that by the above definitions, $\Gamma$ is in $k$-BRNF for some $k$ if and only if all intermediate joints of $\Gamma$ are either straight or on $\partial P$; $\Gamma$ is in $n$-BRNF if and only if $\Gamma$ is in RNF; and $\Gamma$ is in 0-BRNF if and only if $\Gamma$ is a straight chain with no joints on $\partial P$.

Bringing a chain to these normal forms plays a crucial role in our reachability algorithm, described in Sect. 3. The rest of this section elaborates on moving a chain to normal forms. Section 2.1 shows that any chain can be brought to RNF, and Sect. 2.2 shows how to bring a chain already in RNF to TNF-$i_0$ for some $i_0$.

Before we proceed to the subsections, we state two essential lemmas. The first gives a key property of convex obtuse polygons and the lemma that follows it demonstrates the utility of RNF.

**Lemma 1.** *Let $x, y$ be points on nonadjacent sides of convex obtuse polygon $P$. Then $\mid xy \mid \geq s_{\min}$.*

One consequence of the above lemma is the property for chains in RNF given by the next lemma.

**Lemma 2.** *For an $n$-link chain $\Gamma$ in RNF, any joint of $\Gamma$ can be moved along any path on $\partial P$ while keeping $\Gamma$ in RNF.*

Here, notice that the assumptions that $P$ is convex obtuse and that $\Gamma$ has no long links are crucial, as illustrated in Fig. 3.

### 2.1 Bringing a Chain to RNF

The key idea of the algorithm for bringing a chain to RNF is to use $k$-BRNF as a bridge. More specifically, we will show that if $\Gamma$ takes the form $[A_0, \dots, A_n]$ with $A_0$ and $A_n$ on $\partial P$ (a special 2-BRNF), then $\Gamma$ can be brought to 3-BRNF while keeping one of its endjoints fixed and while keeping the other on $\partial P$. By applying this manoeuvre to various subchains of $\Gamma$, it is possible to bring $\Gamma$ to 4-BRNF, to 5-BRNF, ..., and finally, to $n$-BRNF, which is just RNF.

The algorithm consists of three main phases, which we describe in the next three lemmas. Recall that the **Assumptions** of Sect. 1 hold throughout.

**Figure Conventions:** In some multi-part figures, the parts are intended to show possibilities for configurations, but the chain depicted may not be the same in

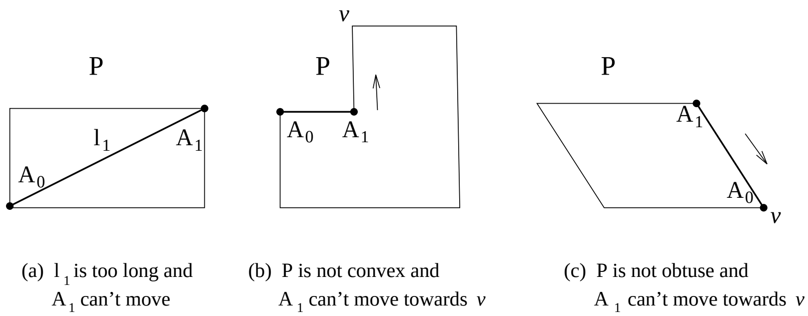


**Fig. 3.** The **Assumptions** are essential.

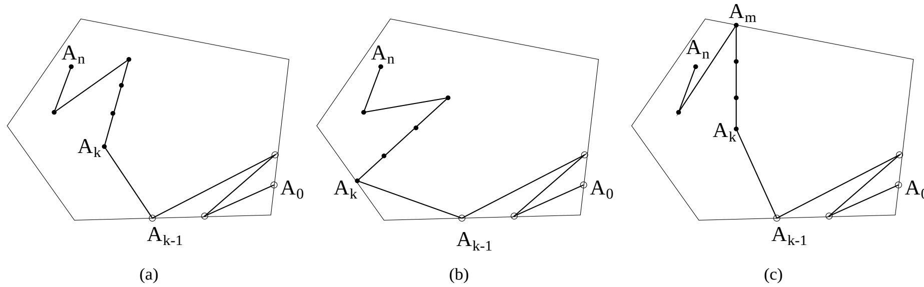


**Fig. 4.** An initial configuration (a) and two possible final configurations (b) and (c) for Lemma 2.3

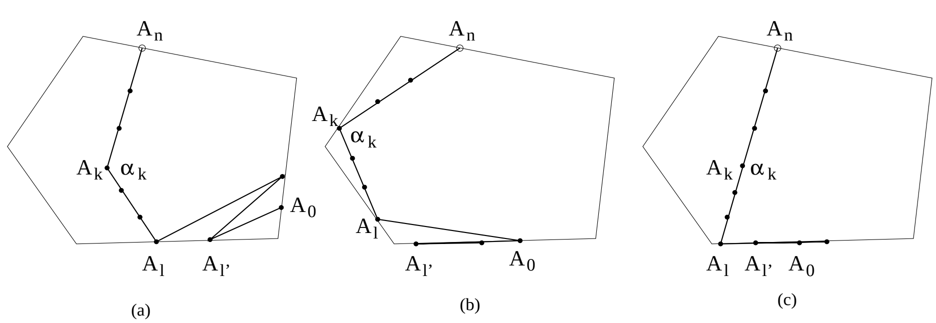


**Fig. 5.** An initial configuration (a) and two possible final configurations (b) and (c) for Lemma 2.4. In (b) and (c), some joints are folded and some links overlap.

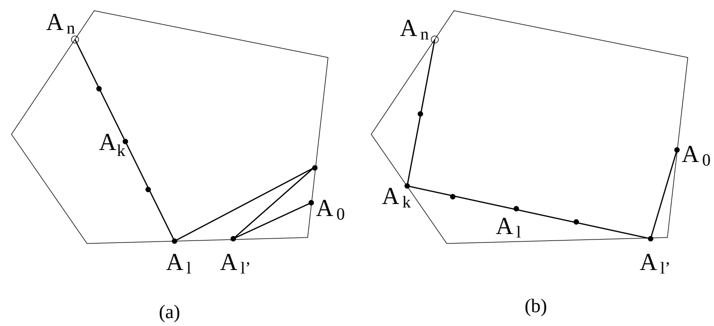


**Fig. 6.** The initial (a) and final (b) configurations for Lemma 2.5

all parts of the figure. Also, an unfilled circle $o$ at a joint in a figure indicates that the joint is to be kept fixed during some motion of $\Gamma$.

**Lemma 3.** *Suppose joints $A_0, \ldots, A_{k-1}$ of $\Gamma$ lie on $\partial P$. Then with $A_0, \ldots, A_{k-1}$ kept fixed, $\Gamma$ can be moved to a configuration in which either $A_k$ lies on $\partial P$ or some $A_m$ lies on $\partial P$, where $m > k$ and $\Gamma(k, m)$ is straight. See Fig. 4.*

**Lemma 4.** *Suppose that $A_0, \ldots, A_l$ and $A_n$ lie on $\partial P$ and that $[A_l, \ldots, A_k]$ and $[A_k, \ldots, A_n]$ are straight, for some $l < k < n$. Then while keeping $A_n$ fixed and $\Gamma(0, l)$ in RNF, and while keeping $[A_l, \ldots, A_k]$ and $[A_k, \ldots, A_n]$ straight, $\Gamma$ can be brought to a configuration in which $A_k$ either lies on $\partial P$ or has $\alpha_k = \pi$. See Fig. 5.*

**Lemma 5.** *Suppose that $A_0, \ldots, A_l$ and $A_n$ lie on $\partial P$ and that $\Gamma(l, n)$ is straight, and suppose that $k$ satisfies $l < k < n$. Then there exists an $l' \leq l$ such that $\Gamma$ can be brought to a configuration in which $A_0, \ldots, A'_l$ and $A_k$ lie on $\partial P$ and $\Gamma(l', k)$ is straight. Furthermore, during this reconfiguration, $A_n$ can be kept fixed, $[A_k, \ldots, A_n]$ can be kept straight, and $\Gamma(0, l')$ can be kept in RNF. See Fig. 6.*

**Corollary 6.** *Suppose, as in Lemma 5, that $A_0, \ldots, A_l$ and $A_n$ lie on $\partial P$ and that $\Gamma(l, n)$ is straight. Then $\Gamma$ can be brought to RNF while $A_n$ is kept fixed.*

We are now ready to show that any chain can be brought to RNF.

**Theorem 7.** *Any $n$-link chain $\Gamma$ confined inside a convex obtuse polygon whose shortest side is at least as long as the longest chain link can be brought to RNF.*

*Proof.* We give an algorithmic proof. The algorithm consists of an initial step, in which $A_0$ is brought to $\partial P$, followed by a main step, in which the lowest indexed joint not on $\partial P$ is brought to $\partial P$. This main step is repeated until all joints have been moved to $\partial P$.

*initial step*: To bring $A_0$ to $\partial P$, proceed as follows. For $k = 1, 2, \ldots$, fix $A_k$ and rotate $[A_0, \ldots, A_k]$ about $A_k$. Repeat this process until either $A_0$ hits $\partial P$ or the whole chain $\Gamma$ becomes straight. If $\Gamma$ straightens before $A_0$ hits $\partial P$, complete this initial step by sliding the straightened $\Gamma$ along the line it determines towards $\partial P$ until $A_0$ hits $\partial P$.

*main step*: For an $A_k$ not on $\partial P$, with $k > 0$ and $\Gamma(0, k-1)$ in RNF, bring $A_k$ to $\partial P$ as follows. In accordance with Lemma 3, keep $A_0, \ldots, A_{k-1}$ fixed and move $\Gamma$ to a configuration in which either $A_k$ lies on $\partial P$ or $A_m$ for some $m > k$ lies on $\partial P$, where $[A_k, \ldots, A_m]$ is straight.

In the latter case, where some $A_m$ moves to $\partial P$ and $\Gamma(k, m)$ is straight, continue the step by fixing $A_m, \ldots, A_n$ and moving $\Gamma(0, m)$ in accordance with Lemma 4 so that either $A_k$ hits $\partial P$ or $\alpha_k$ straightens to $\pi$. If $\alpha_k$ straightens to $\pi$, again continue the step by fixing $A_m \ldots, A_n$ and moving $\Gamma(0, m)$ to RNF in accordance with Corollary 6. This puts $A_k$ on $\partial P$.

*iteration steps*: Once $A_0, \ldots, A_{k-1}, A_k$ lie on $\partial P$, repeat the main step to bring the $A_{k+1}, \ldots, A_n$ in turn to $\partial P$. □

### 2.2 Bringing a Chain to TNF-$i_0$

**Theorem 8.** *Let $\Gamma$ be an $n$-link chain confined within a convex obtuse polygon $P$. Suppose there exists a point $p \in P$, a vertex $v$ of $P$, and an index $i_0 < n$ such that $d(p, v) \geq$ the sum of the lengths of the links in $\Gamma(i_0, n)$. Then $\Gamma$ can be brought to TNF-$i_0$ with $A_{i_0}$ at $v$.*

**Sketch of the Proof:** In accordance with Theorem 7, bring $\Gamma$ to RNF. Then, keeping $\Gamma$ in RNF, move $A_{i_0}$ around $\partial P$ to $v$ in accordance with Lemma 2.

Let $p_{i_0}, p_{i_o+1}, \ldots, p_n$ be the points that $A_{i_0}, A_{i_0+1}, \ldots, A_n$ now occupy, respectively, as shown in Fig. 7. Note that by the triangle inequality, the property of $i_0$ specified in the statement of the lemma implies that for any $k$ with $i_0 \leq k < n$, $d(p, p_k) > l_{k+1} + \cdots + l_n$. This inequality ensures the validity of certain operations described below.

To begin, bring $[A_{n-1}, A_n]$ onto the line segment $p_{n-1}p$. To do this, fix $A_0, \ldots, A_{n-1}$ and rotate $L_n$ about $p_{n-1}$ until $L_n$ and $p$ become collinear. See Fig. 7.

Next, straighten $\Gamma(n-2, n)$ and move it onto $p_{n-2}p$ as follows. Fix $A_0, \ldots, A_{n-2}$ and, while keeping $L_n$ and $p$ collinear, rotate $L_{n-1}$ about its endjoint $A_{n-2}$ at $p_{n-2}$ until $[A_{n-2}, A_{n-1}, A_n]$ and $p$ become collinear. See Fig. 8.

Repeat this process until $\Gamma(i_0, n)$ is straight and lies on $vp$ with $A_{i_o}$ at $v$. □

## 3 Reachable Points

Now we give our main result.

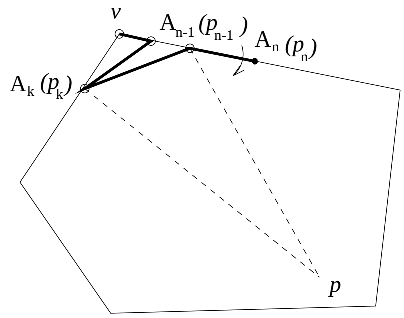


**Fig. 7.** Bringing $L_n$ onto $p_{n-1}p$. Point $p_{i_0}$ and joint $A_{i_0}$ are at $v$.

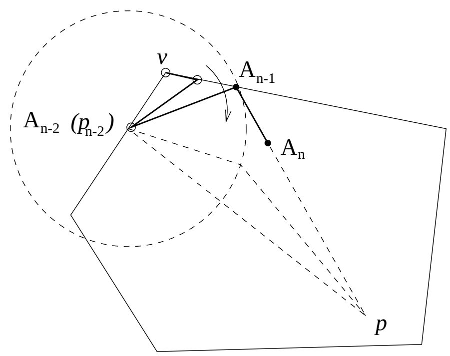


**Fig. 8.** Straightening $\Gamma(n-2, n)$ onto $p_{n-2}p$.

**Theorem 9.** *Let $\Gamma$ be an $n$-link chain confined within a convex obtuse polygon $P$ whose shortest side is at least as long as the longest link of the chain. Then $(*)$ is a necessary and sufficient condition for $p$ to be reachable by $A_n$. Furthermore, if $p$ is reachable by $A_n$, then it is possible to compute a sequence of motions bringing $A_n$ to $p$.*

**Sketch of the Proof:** The necessity of condition $(*)$ is clear. Now we show the sufficiency by giving an algorithm to produce a sequence of motions to bring $A_n$ to $p$ when $(*)$ is satisfied. This will complete the proof of the theorem.

Let $i_0$ be the least index such that

$$\sum_{j=i_0+1}^{n} l_j \;\leq\; d_{\max}(p) \;\; .$$

Note that taking $i = n$ in $(*)$ gives $l_n \leq d_{\max}(p)$, so $i_0 < n$. In accordance with Theorem 8, bring $\Gamma$ to TNF-$i_0$ with $A_{i_0}$ at $v_{\max}(p) = v$.

If $i_0 = 0$, then $\Gamma$ is now straight. Slide $\Gamma$ along $l(v, p)$ towards $p$ until $A_n$ reaches $p$. If $i_0 > 0$, we consider two cases, $l_{i_0} \leq d_{\max}(p)$ and $l_{i_0} > d_{\max}(p)$.

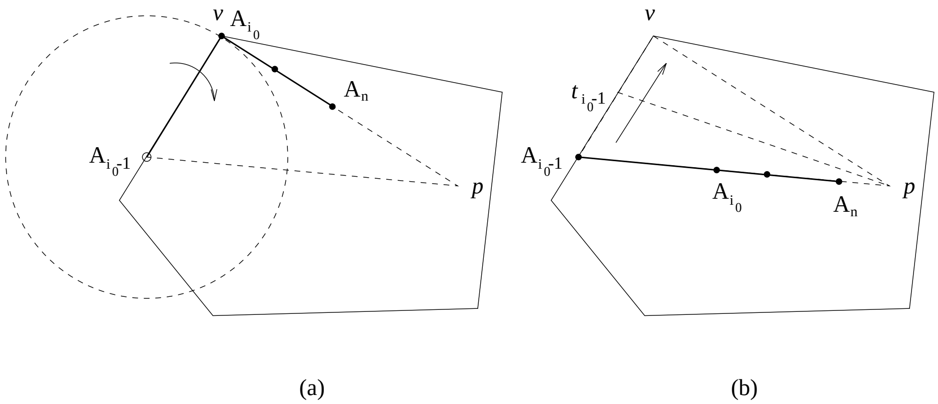


**Fig. 9.** Case $l_{i_0} \leq d_{\max}(p)$.

*case* $l_{i_0} \leq d_{\max}(p)$: Keeping $A_0, \ldots, A_{i_0-1}$ fixed and keeping $[A_{i_0}, \ldots, A_n]$ straight and collinear with $p$, rotate $L_{i_0}$ about $A_{i_0-1}$ until $A_n$ reaches $p$ or $\alpha_{i_0}$ straightens to $\pi$. See Fig. 9(a).

If $A_n$ reaches $p$ first, then we are done. If $\alpha_{i_0}$ straightens to $\pi$ as shown in Fig. 9(b), then clearly $l_{i_0} + \cdots + l_n < d(p_{i_o-1}, p)$. While keeping $\Gamma(0, i_0 - 1)$ in RNF and while keeping $[A_{i_0-1}, A_{i_0}, \ldots, A_n]$ straight and collinear with $p$, move $A_{i_0-1}$ along $\partial P$ towards $v$. We claim that $A_n$ will reach $p$ before $A_{i_0-1}$ reaches $v$ during this process.

Suppose otherwise, i.e., that $A_{i_0-1}$ reaches $v$ first. By definition of $i_0$, we have that $l_{i_0} + \cdots + l_n > d_{\max}(p) = d(v, p)$. Since we know from the previous paragraph that the left side of this inequality is less than $d(p_{i_0-1}, p)$, there exists some intermediate configuration having $A_{i_0-1}$ at some $t_{i_0-1} \in \partial P$ and having $[A_{i_0-1}, A_{i_0}, \ldots, A_n]$ straight and collinear with $p$, such that $l_{i_0} + \cdots + l_n = d(t_{i_0-1}, p)$. This implies that $A_n$ lies at $p$ in this intermediate configuration, a contradiction.

*case* $l_{i_0} > d_{\max}(p)$: Fixing $A_0, \ldots, A_{i_0-1}$ and keeping $[A_{i_0}, \ldots, A_n]$ straight and collinear with $p$, rotate $L_{i_0}$ about $A_{i_0-1}$ until $A_n$ reaches $p$ or $\alpha_{i_0} = 0$ or $A_{i_0}$ hits $\partial P$. The remaining details are similar to those of the previous case. □

## 4 Conclusions

We have studied the reachability problem of *planar chains* confined within *convex obtuse polygons*, a notion of our invention. For a planar chain $\Gamma$ within a convex obtuse polygon $P$ whose shortest side is longer than the longest link of $\Gamma$, we have characterized the reachable points of the endjoints of $\Gamma$ and have presented an algorithm for reconfiguring $\Gamma$ within $P$ so that a specified endjoint reaches a given reachable point. This significantly extends the best known results for this type of problem and contributes to the goal of understanding the geometry of chains and their constraining environments; in particular, it contributes to finding relationships that ensure that typically hard reconfiguration problems become easy.

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