COMP 648 Cell Decomposition Problems

due Tuesday, October 25

You may work together, but your write-ups should be your own.

1) Sketch the curve

$$f(x) = x^3 - 2x^2 + 1.$$

Treat this as a simple calculus problem – i.e., plot $f(x)$ for large magnitudes of x, for values of x where the tangent line is horizontal, etc. How many real roots does $f(x)$ have? Assuming that $f(x)$ has at least one real root, enumerate these roots according to increasing value $k_1, k_2, \ldots$ (There are at most three of these.) Now, put this information "on hold" – it will be useful in part 4) below.

2) Consider the polynomial

$$P^1(x, y) = 5(y - x^2).$$

Find the simplest (smallest number of cells) *c.a.d.* of the real line R^1 so that as x varies over any given cell, the number of distinct real roots of $P^1(x, y)$ remains constant throughout the cell.

3) Repeat part 2) for the polynomial

$$P^2(x, y) = (x - 2)y + 1.$$

4) Now consider the *product polynomial*

$$P(x, y) = (P^1(x, y))(P^2(x, y)).$$

Repeat part 2) for this polynomial. Is it good enough to "merge" the *c.a.d.*s from 2) and 3)? Careful – part 1) is relevant here.

5) Give a *c.a.d.* for R^2 (2-dimensional Euclidean space) such that in each cell, P^1 and P^2 maintain constant sign (+, -, 0) as the point (x, y) ranges over the cell. Do this (anyway you can) by using the *c.a.d.* for R^1 in part 4), together with the product polynomial $P_x(y)$. If you can "see" the answer, you can just write it down.

6) Suppose the free positions in C-space are described by the Tarski set

$$\{(x,y) \mid [5(y - x^2) \leq 0] \; OR \; [(x-2)y+1<0]\}.$$

Describe this set as a union of cells in the *c.a.d.* of part 5).

7) Computing a *c.a.d.*

a)Write the product polynomial $P(x,y)$ in the form
$a_2(x)y^2 + a_1(x)y^1 + a_0(x)$.

b) Suppose that x is fixed, and compute the derivative of the product polynomial with respect to y. This will give another polynomial $Q(x,y)$ in y whose coefficients are polynomials in x, namely:

$$Q(x,y) = 2a_2(x)y + a_1(x).$$

c) Depending on the value of x, $Q(x,y)$ is either a constant function, or a linear function of y. Hence for any fixed value of x, $Q(x,y)$ has either no factors or one factor in common with $P(x,y)$.
For what value(s) of x does $Q_x(y)$ have a factor in common with $P_x(y)$?

Note that for this (these) value(s), the degree of the greatest common divisor polynomial of P and Q is 1, and for all other values of x, the degree of the gcd polynomial is 0. (The gcd of two polynomials is just the product of their common factors, raised to the appropriate exponents).

d) Give a *c.a.d.* for R^1 such that in each cell, $P(x,y)$ has constant degree n and such that the degree of the gcd polynomial of $P(x,y)$ and $Q(x,y)$ is some constant m.

e) Compute the value of $n-m$ in each cell of the c.a.d. in d). This should give the number of distinct roots (real and complex) of $P(x,y)$. As it turns out, $P(x,y)$ has no non-real roots, so $n-m$ should give the number of distinct real roots in each cell. Does it?

f) Compare the *c.a.d.* from part d) with the *c.a.d.* from part 5).