Stone Duality for Markov Processes

Prakash Panangaden[1]

[1]School of Computer Science
McGill University

Radboud University, Nijmegen, May 2013

Collaborators

- This paper (IEEE Symposium On Logic In Computer Science 2013) is joint work with Dexter Kozen, Kim G. Larsen and Radu Mardare.
- Previous work was with Josée Desharnais, Vincent Danos, François Laviolette (Info. Comp 2006) and Philippe Chaput, Vincent Danos and Gordon Plotkin (ICALP 2009).

Outline

Outline

Outline

Outline

Outline

Outline

Outline

Outline

Outline

Recap of Stone Duality

$$\begin{tikzcd} \mathcal{C} \arrow[rr, bend left, "F"] & & \mathcal{D}^{op} \arrow[ll, bend left, "G"] \end{tikzcd}$$

Stone Duality

We have a (contravariant) adjunction between categories $\mathcal{C}$ and $\mathcal{D}$, which is an *equivalence* of categories.

Examples: Finite sets and finite Boolean algebras, Boolean algebras and Stone spaces, Finite-dimensional vector spaces and itself, commutative unital C^*-algebras and compact Hausdorff spaces, ...

Duality for Stone-Markov Processes

1. We extend completeness theorems for probabilistic logics to a Stone-type duality.

Duality for Stone-Markov Processes

1. We extend completeness theorems for probabilistic logics to a Stone-type duality.
2. It builds on classical Stone-type duality: we define Markov processes on top of Stone spaces. **Stone-Markov** processes.

Duality for Stone-Markov Processes

1. We extend completeness theorems for probabilistic logics to a Stone-type duality.
2. It builds on classical Stone-type duality: we define Markov processes on top of Stone spaces. **Stone-Markov** processes.
3. The algebraic counterpart is called *Aumann algebra*: an algebraic form of Aumann's probabilistic logic.

Duality for Stone-Markov Processes

1. We extend completeness theorems for probabilistic logics to a Stone-type duality.
2. It builds on classical Stone-type duality: we define Markov processes on top of Stone spaces. **Stone-Markov** processes.
3. The algebraic counterpart is called *Aumann algebra*: an algebraic form of Aumann's probabilistic logic.
4. Previous completeness proofs are *conditional* on a logic satisfying Lindenbaum's Lemma.

Technical Issue and Tools

- Some subtle topological issues arise that we need to confront.

Technical Issue and Tools

- Some subtle topological issues arise that we need to confront.
- We are going to have to use infinitary operations. No hope of completeness without it.

Technical Issue and Tools

- Some subtle topological issues arise that we need to confront.
- We are going to have to use infinitary operations. No hope of completeness without it.
- We will use the Rasiowa-Sikorski Lemma, which is based on the Baire category theorem.

Technical Issue and Tools

- Some subtle topological issues arise that we need to confront.
- We are going to have to use infinitary operations. No hope of completeness without it.
- We will use the Rasiowa-Sikorski Lemma, which is based on the Baire category theorem.
- We will RST to *prove* that every consistent set of formulas can be extended to a maximal consistent set (Lindenbaum's Lemma).

What are labelled Markov processes?

- Labelled Markov processes are probabilistic versions of labelled transition systems. Labelled transition systems where the final state is governed by a probability distribution - no other indeterminacy.

What are labelled Markov processes?

- Labelled Markov processes are probabilistic versions of labelled transition systems. Labelled transition systems where the final state is governed by a probability distribution - no other indeterminacy.
- All probabilistic data is *internal* - no probabilities associated with environment behaviour.

What are labelled Markov processes?

- Labelled Markov processes are probabilistic versions of labelled transition systems. Labelled transition systems where the final state is governed by a probability distribution - no other indeterminacy.
- All probabilistic data is *internal* - no probabilities associated with environment behaviour.
- We observe the interactions - not the internal states.

What are labelled Markov processes?

- Labelled Markov processes are probabilistic versions of labelled transition systems. Labelled transition systems where the final state is governed by a probability distribution - no other indeterminacy.
- All probabilistic data is *internal* - no probabilities associated with environment behaviour.
- We observe the interactions - not the internal states.
- **In general, the state space of a labelled Markov process may be a *continuum*.**

Motivation

Model and reason about systems with *continuous* state spaces or continuous time evolution or both.

- hybrid control systems; e.g. flight management systems.
- telecommunication systems with spatial variation; e.g. cell phones
- performance modelling,
- continuous time systems,
- probabilistic process algebra with recursion.

Formal Definition of LMPs

- An LMP is a tuple $(S, \Sigma, \mathsf{L}, \forall\alpha \in \mathsf{L}.\tau_\alpha)$ where
- S is the space of states, *assumed to be an analytic space*,
- Σ is the Borel σ-algebra of S, and
- where $\tau_\alpha : S \times \Sigma \longrightarrow [0, 1]$ is a *transition probability* function such that
- $\forall s : S.\lambda A : \Sigma.\tau_\alpha(s, A)$ is a subprobability measure
  and
  $\forall A : \Sigma.\lambda s : S.\tau_\alpha(s, A)$ is a measurable function: the reals are equipped with the Borel σ-algebra.

Logical Characterization

- $$\mathcal{L}_0 ::== \mathsf{T} | \phi_1 \wedge \phi_2 | \langle a \rangle_q \phi$$
- We say $s \models \langle a \rangle_q \phi$ iff
$$\exists A \in \Sigma . (\forall s' \in A . s' \models \phi) \wedge (\tau_a(s, A) > q).$$

Logical Characterization

- $$\mathcal{L}_0 ::== \mathsf{T}|\phi_1 \wedge \phi_2|\langle a\rangle_q\phi$$
- We say $s \models \langle a\rangle_q\phi$ iff
$$\exists A \in \Sigma.(\forall s' \in A.s' \models \phi) \wedge (\tau_a(s, A) > q).$$
- Two systems are bisimilar iff they obey the same formulas of $\mathcal{L}$. [DEP 1998 LICS, I and C 2002]

Logical Characterization

- $$\mathcal{L}_0 ::== \mathsf{T} | \phi_1 \wedge \phi_2 | \langle a \rangle_q \phi$$
- We say $s \models \langle a \rangle_q \phi$ iff
$$\exists A \in \Sigma . (\forall s' \in A . s' \models \phi) \wedge (\tau_a(s, A) > q).$$
- Two systems are bisimilar iff they obey the same formulas of $\mathcal{L}$. [DEP 1998 LICS, I and C 2002]
- Analyticity is crucial for proving this.

Logical Characterization

- $$\mathcal{L}_0 ::== \mathsf{T} | \phi_1 \wedge \phi_2 | \langle a \rangle_q \phi$$
- We say $s \models \langle a \rangle_q \phi$ iff
$$\exists A \in \Sigma . (\forall s' \in A . s' \models \phi) \wedge (\tau_a(s, A) > q).$$
- Two systems are bisimilar iff they obey the same formulas of $\mathcal{L}$. [DEP 1998 LICS, I and C 2002]
- Analyticity is crucial for proving this.
- Later [DDLP 2006], we introduced a co-bisimulation (cocongruence) and proved that the above logic characterizes co-bisimulation for *all* Markov processes, not just ones defined on analytic spaces.

Polish and Analytic Spaces

- Why are we talking about topology at all? It turns out that many aspects of measure theory work much more nicely if the spaces are metric spaces. A *Polish* space is the topological space underlying a complete, separable metric space.

Polish and Analytic Spaces

- Why are we talking about topology at all? It turns out that many aspects of measure theory work much more nicely if the spaces are metric spaces. A *Polish* space is the topological space underlying a complete, separable metric space.
- Separable means that there is a *countable* base of open sets; for metric spaces this is the same as saying that there is a countable dense subset.

Polish and Analytic Spaces

- Why are we talking about topology at all? It turns out that many aspects of measure theory work much more nicely if the spaces are metric spaces. A *Polish* space is the topological space underlying a complete, separable metric space.
- Separable means that there is a *countable* base of open sets; for metric spaces this is the same as saying that there is a countable dense subset.
- If X, Y are Polish and $f : X \longrightarrow Y$ is continuous then $f(X) \subset Y$ is an analytic space.

Polish and Analytic Spaces

- Why are we talking about topology at all? It turns out that many aspects of measure theory work much more nicely if the spaces are metric spaces. A *Polish* space is the topological space underlying a complete, separable metric space.
- Separable means that there is a *countable* base of open sets; for metric spaces this is the same as saying that there is a countable dense subset.
- If X, Y are Polish and $f : X \longrightarrow Y$ is continuous then $f(X) \subset Y$ is an analytic space.
- It does not matter if f is continuous or measurable (!), or if we take a Borel subset of X.

Polish and Analytic Spaces

- Why are we talking about topology at all? It turns out that many aspects of measure theory work much more nicely if the spaces are metric spaces. A *Polish* space is the topological space underlying a complete, separable metric space.
- Separable means that there is a *countable* base of open sets; for metric spaces this is the same as saying that there is a countable dense subset.
- If X, Y are Polish and $f : X \longrightarrow Y$ is continuous then $f(X) \subset Y$ is an analytic space.
- It does not matter if f is continuous or measurable (!), or if we take a Borel subset of X.
- Analytic spaces have some remarkable properties that were crucial in the proof of the logical characterization theorem.

Stone spaces

- A Stone space is a compact Hausdorff space with a base of *clopen* sets: zero-dimensional space.
- A space is said to be *totally disconnected* if the only connected sets are singletons. For (locally) compact Hausdorff spaces zero dimensional is equivalent to totally disconnected. Ultrametric spaces are zero-dimensional.
- Many, but not all, Stone spaces are Polish.

Boolean Algebras

A Boolean algebra is a set A equipped with two constants, $0, 1$, a unary operation $(\cdot)'$ and two binary operations $\vee, \wedge$ which obey the following axioms, p, q, r are arbitrary members of A:

$$\begin{aligned} 0' &= 1 & 1' &= 0 \\ p \wedge 0 &= 0 & p \vee 1 &= 1 \\ p \wedge 1 &= p & p \vee 0 &= p \\ p \wedge p' &= 0 & p \vee p' &= 1 \\ p \wedge p &= p & p \vee p &= p \end{aligned}$$

Boolean Algebras II

$$
\begin{aligned}
p'' &= p \\
(p \wedge q)' &= p' \vee q' \\
(p \vee q)' &= p' \wedge q' \\
p \wedge q &= q \wedge p \\
p \vee q &= q \vee p \\
p \wedge (q \wedge r) &= (p \wedge q) \wedge r \\
p \vee (q \vee r) &= (p \vee q) \vee r \\
p \wedge (q \vee r) &= (p \wedge q) \vee (p \wedge r) \\
p \vee (q \wedge r) &= (p \vee q) \wedge (p \vee r)
\end{aligned}
$$

The operation $\vee$ is called *join*, $\wedge$ is called *meet* and $(\cdot)'$ is called *complement*.

Classical Stone Duality

- Given a Stone space $\mathcal{S}$ the clopens form a Boolean algebra: $\mathcal{C}l(\mathcal{S})$.

Classical Stone Duality

- Given a Stone space $\mathcal{S}$ the clopens form a Boolean algebra: $\mathcal{C}l(\mathcal{S})$.
- Given a Boolean algebra $\mathcal{B}$, a Stone space $\mathcal{U}(\mathcal{B})$ is constructed from the space $\mathcal{U}$ of maximal filters (ultrafilters).

Classical Stone Duality

- Given a Stone space $\mathcal{S}$ the clopens form a Boolean algebra: $\mathcal{C}l(\mathcal{S})$.
- Given a Boolean algebra $\mathcal{B}$, a Stone space $\mathcal{U}(\mathcal{B})$ is constructed from the space $\mathcal{U}$ of maximal filters (ultrafilters).
- Given an element b of the Boolean algebra, a *basic open* is defined by $U_b := \{u \in \mathcal{U} \mid b \in u\}$.

Classical Stone Duality

- Given a Stone space $\mathcal{S}$ the clopens form a Boolean algebra: $\mathcal{Cl}(\mathcal{S})$.
- Given a Boolean algebra $\mathcal{B}$, a Stone space $\mathcal{U}(\mathcal{B})$ is constructed from the space $\mathcal{U}$ of maximal filters (ultrafilters).
- Given an element b of the Boolean algebra, a *basic open* is defined by $U_b := \{u \in \mathcal{U} \mid b \in u\}$.
- Ultrafilters correspond precisely to maps from $\mathcal{B}$ to $\mathbf{2}$:

Classical Stone Duality

- Given a Stone space $\mathcal{S}$ the clopens form a Boolean algebra: $\mathcal{C}l(\mathcal{S})$.
- Given a Boolean algebra $\mathcal{B}$, a Stone space $\mathcal{U}(\mathcal{B})$ is constructed from the space $\mathcal{U}$ of maximal filters (ultrafilters).
- Given an element b of the Boolean algebra, a *basic open* is defined by $U_b := \{u \in \mathcal{U} \mid b \in u\}$.
- Ultrafilters correspond precisely to maps from $\mathcal{B}$ to $\mathbf{2}$:
- $\{b \in \mathcal{B} \mid f(b) = 1\}$ is always an ultrafilter.

Classical Stone Duality

- Given a Stone space $\mathcal{S}$ the clopens form a Boolean algebra: $\mathcal{C}l(\mathcal{S})$.
- Given a Boolean algebra $\mathcal{B}$, a Stone space $\mathcal{U}(\mathcal{B})$ is constructed from the space $\mathcal{U}$ of maximal filters (ultrafilters).
- Given an element b of the Boolean algebra, a *basic open* is defined by $U_b := \{u \in \mathcal{U} \mid b \in u\}$.
- Ultrafilters correspond precisely to maps from $\mathcal{B}$ to $\mathbf{2}$:
- $\{b \in \mathcal{B} \mid f(b) = 1\}$ is always an ultrafilter.
- Continuous maps $f : \mathcal{S}_1 \longrightarrow \mathcal{S}_2$ between Stone spaces give Boolean algebra homomorphisms $f^{-1} : \mathcal{C}l(\mathcal{S}_2) \longrightarrow \mathcal{C}l(\mathcal{S}_1)$.

Classical Stone Duality

- Given a Stone space $\mathcal{S}$ the clopens form a Boolean algebra: $\mathcal{C}l(\mathcal{S})$.
- Given a Boolean algebra $\mathcal{B}$, a Stone space $\mathcal{U}(\mathcal{B})$ is constructed from the space $\mathcal{U}$ of maximal filters (ultrafilters).
- Given an element b of the Boolean algebra, a *basic open* is defined by $U_b := \{u \in \mathcal{U} \mid b \in u\}$.
- Ultrafilters correspond precisely to maps from $\mathcal{B}$ to $\mathbf{2}$:
- $\{b \in \mathcal{B} \mid f(b) = 1\}$ is always an ultrafilter.
- Continuous maps $f : \mathcal{S}_1 \longrightarrow \mathcal{S}_2$ between Stone spaces give Boolean algebra homomorphisms $f^{-1} : \mathcal{C}l(\mathcal{S}_2) \longrightarrow \mathcal{C}l(\mathcal{S}_1)$.
- Boolean algebra homomorphisms $h : \mathcal{B}_1 \longrightarrow \mathcal{B}_2$ give rise to continuous functions $(\cdot) \circ f : \mathcal{U}(\mathcal{B}_2) \longrightarrow \mathcal{U}(\mathcal{B}_1)$ between the Stone spaces.

Classical Stone Duality

- Given a Stone space $\mathcal{S}$ the clopens form a Boolean algebra: $\mathcal{C}l(\mathcal{S})$.
- Given a Boolean algebra $\mathcal{B}$, a Stone space $\mathcal{U}(\mathcal{B})$ is constructed from the space $\mathcal{U}$ of maximal filters (ultrafilters).
- Given an element b of the Boolean algebra, a *basic open* is defined by $U_b := \{u \in \mathcal{U} \mid b \in u\}$.
- Ultrafilters correspond precisely to maps from $\mathcal{B}$ to $\mathbf{2}$:
- $\{b \in \mathcal{B} \mid f(b) = 1\}$ is always an ultrafilter.
- Continuous maps $f : \mathcal{S}_1 \longrightarrow \mathcal{S}_2$ between Stone spaces give Boolean algebra homomorphisms $f^{-1} : \mathcal{C}l(\mathcal{S}_2) \longrightarrow \mathcal{C}l(\mathcal{S}_1)$.
- Boolean algebra homomorphisms $h : \mathcal{B}_1 \longrightarrow \mathcal{B}_2$ give rise to continuous functions $(\cdot) \circ f : \mathcal{U}(\mathcal{B}_2) \longrightarrow \mathcal{U}(\mathcal{B}_1)$ between the Stone spaces.
- Everything that can and should be an isomorphism is an isomorphism.

Baire Category Theorem

1. A set $N \subset X$ is said to be *nowhere dense* if every open set U contains an open set $V \subset U$ with V, A disjoint. A set is said to be of the *first category* or *meager* if it is the countable union of nowhere dense sets.

Baire Category Theorem

1. A set $N \subset X$ is said to be *nowhere dense* if every open set U contains an open set $V \subset U$ with V, A disjoint. A set is said to be of the *first category* or *meager* if it is the countable union of nowhere dense sets.
2. Meagre sets need not be closed.

Baire Category Theorem

1. A set $N \subset X$ is said to be *nowhere dense* if every open set U contains an open set $V \subset U$ with V, A disjoint. A set is said to be of the *first category* or *meager* if it is the countable union of nowhere dense sets.
2. Meagre sets need not be closed.
3. Baire proved: The complement of any meager subset of $\mathbb{R}$ is dense. This is equivalent to: no interval is meager and also to the intersection of countably many dense open sets is dense.

Baire Category Theorem

1. A set $N \subset X$ is said to be *nowhere dense* if every open set U contains an open set $V \subset U$ with V, A disjoint. A set is said to be of the *first category* or *meager* if it is the countable union of nowhere dense sets.
2. Meagre sets need not be closed.
3. Baire proved: The complement of any meager subset of $\mathbb{R}$ is dense. This is equivalent to: no interval is meager and also to the intersection of countably many dense open sets is dense.
4. A Baire space is one in which no non-empty open set is meagre.

Baire Category Theorem

1. A set $N \subset X$ is said to be *nowhere dense* if every open set U contains an open set $V \subset U$ with V, A disjoint. A set is said to be of the *first category* or *meager* if it is the countable union of nowhere dense sets.
2. Meagre sets need not be closed.
3. Baire proved: The complement of any meager subset of $\mathbb{R}$ is dense. This is equivalent to: no interval is meager and also to the intersection of countably many dense open sets is dense.
4. A Baire space is one in which no non-empty open set is meagre.
5. Modern BCT: (1) Every complete metric space is a Baire space. (2) Every (locally) compact Hausdorff space is a Baire space.

Baire Category Theorem

1. A set $N \subset X$ is said to be *nowhere dense* if every open set U contains an open set $V \subset U$ with V, A disjoint. A set is said to be of the *first category* or *meager* if it is the countable union of nowhere dense sets.
2. Meagre sets need not be closed.
3. Baire proved: The complement of any meager subset of $\mathbb{R}$ is dense. This is equivalent to: no interval is meager and also to the intersection of countably many dense open sets is dense.
4. A Baire space is one in which no non-empty open set is meagre.
5. Modern BCT: (1) Every complete metric space is a Baire space. (2) Every (locally) compact Hausdorff space is a Baire space.
6. The version we use: In a compact Hausdorff space the intersection of a family of dense open sets is dense.

Baire Category Theorem

1. A set $N \subset X$ is said to be *nowhere dense* if every open set U contains an open set $V \subset U$ with V, A disjoint. A set is said to be of the *first category* or *meager* if it is the countable union of nowhere dense sets.
2. Meagre sets need not be closed.
3. Baire proved: The complement of any meager subset of $\mathbb{R}$ is dense. This is equivalent to: no interval is meager and also to the intersection of countably many dense open sets is dense.
4. A Baire space is one in which no non-empty open set is meagre.
5. Modern BCT: (1) Every complete metric space is a Baire space. (2) Every (locally) compact Hausdorff space is a Baire space.
6. The version we use: In a compact Hausdorff space the intersection of a family of dense open sets is dense.
7. The boundary of any open set is closed and nowhere dense.

The Rasiowa-Sikorski Lemma

Let $\mathcal{B}$ be a Boolean algebra and $T \subset \mathcal{B}$ be a set with $\bigvee T$ defined. An ultrafilter U is said to *respect* T if

$$\bigvee T \in U \Rightarrow T \cap U \neq \emptyset.$$

The Rasiowa-Sikorski Lemma

Let $\mathcal{T}$ be a countable family of subsets of $\mathcal{B}$ each member of which has a join and let $x \neq 0$ in $\mathcal{B}$. Then there is an ultrafilter which respects each member of $\mathcal{T}$ and which contains x.

Dualizing the lemma

We can define $\mathcal{U}$ *dually respects* S by saying that $\bigwedge S \in \mathcal{U} \iff S \subset \mathcal{U}$.
Then we have that $\mathcal{U}$ respects T iff $\mathcal{U}$ dually respects $S := \{\neg t \mid t \in T\}$.
We can use the RS lemma for respects or dually respects: there is no point making a terminological distinction between the two cases.

Definition of a Markov Process

Labels are not important for the present work so we forget about them for now.

Markov Process

Given an analytic space (M, Σ), a *Markov process* is a measurable mapping $\tau \in [\![M \rightarrow \Delta(M, \Sigma)]\!]$.

We have curried the definition of LMPs and eliminated the labels.

Markovian Logic

The formulas of $\mathcal{L}$ are defined, for a set $\mathcal{P}$ of atomic propositions, by the grammar

$$\phi \ ::= \ p \mid \perp \mid \phi \Rightarrow \phi \mid L_r\phi$$

where p can be any element of $\mathcal{P}$ and r any element of $\mathbb{Q}_0$.
The other Boolean operators are defined in the usual way.

Notation: $L_{r_1 \dots r_k}\phi := L_{r_1} \dots L_{r_k}\phi$.

Semantics

Given a Markov Process $\mathcal{M} = (M, \Sigma, \tau)$, $m \in M$ and $i : M \longrightarrow 2^{\mathcal{P}}$ we have

the satisfaction relation:

- $\mathcal{M}, m, i \models p$ if $p \in i(m)$,
- $\mathcal{M}, m, i \models \bot$ never,
- $\mathcal{M}, m, i \models \phi \Rightarrow \psi$ if $\mathcal{M}, m, i \models \psi$ whenever $\mathcal{M}, m, i \models \phi$,
- $\mathcal{M}, m, i \models L_r\phi$ if $\tau(m)([\![\phi]\!]) \geq r$,
  where $[\![\phi]\!] = \{m \in M \mid \mathcal{M}, m, i \models \phi\}$.

It follows that:

- $\mathcal{M}, m, i \models \top$ always,
- $\mathcal{M}, m, i \models \phi \wedge \psi$ iff $\mathcal{M}, m, i \models \phi$ and $\mathcal{M}, m, i \models \psi$,
- $\mathcal{M}, m, i \models \phi \vee \psi$ iff $\mathcal{M}, m, i \models \phi$ or $\mathcal{M}, m, i \models \psi$,
- $\mathcal{M}, m, i \models \neg\phi$ iff not $\mathcal{M}, m, i \models \phi$.

Axioms for Markovian Logic

The axioms of $\mathcal{L}$

(A1): $\vdash L_0\phi$

(A2): $\vdash L_r T$

(A3): $\vdash L_r\phi \to \neg L_s\neg\phi, \quad r+s>1$

(A4): $\vdash L_r(\phi \wedge \psi) \wedge L_s(\phi \wedge \neg\psi) \to L_{r+s}\phi, \quad r+s \leq 1$

(A5): $\vdash \neg L_r(\phi \wedge \psi) \wedge \neg L_s(\phi \wedge \neg\psi) \to \neg L_{r+s}\phi, \quad r+s \leq 1$

(R1): $$\frac{\vdash \phi \to \psi}{\vdash L_r\phi \to L_r\psi}$$

(R2): $\{L_{r_1 \cdots r_n r}\psi \mid r<s\} \vdash L_{r_1 \cdots r_n s}\psi$

Completeness

- Zhou proved strong completeness using a strengthened version of (R1) and assuming Lindenbaum's Lemma.

Completeness

- Zhou proved strong completeness using a strengthened version of (R1) and assuming Lindenbaum's Lemma.
- Goldblatt proved strong completeness by assuming Lindenbaum's Lemma and other infinitary axioms.

Completeness

- Zhou proved strong completeness using a strengthened version of (R1) and assuming Lindenbaum's Lemma.
- Goldblatt proved strong completeness by assuming Lindenbaum's Lemma and other infinitary axioms.
- Our duality theorem implies strong completeness for the above axioms without assuming Lindenbaum's Lemma.

Completeness

- Zhou proved strong completeness using a strengthened version of (R1) and assuming Lindenbaum's Lemma.
- Goldblatt proved strong completeness by assuming Lindenbaum's Lemma and other infinitary axioms.
- Our duality theorem implies strong completeness for the above axioms without assuming Lindenbaum's Lemma.
- Instead we use the RSL to establish Lindenbaum's Lemma.

The Definition of Aumann Algebras

An *Aumann algebra (AA)* is a structure $\mathcal{A} = (A, \Rightarrow, \bot, \{F_r\}_{r\in\mathbb{Q}_0}, \sqsubseteq)$ where

- $(A, \Rightarrow, \bot, \sqsubseteq)$ is a Boolean algebra;
- for each $r \in \mathbb{Q}_0$, $F_r : A \to A$ is an unary operator; and
- the axioms below hold for all $a, b \in A$, $r, s, r_1, \ldots, r_n \in \mathbb{Q}_0$.

Axioms

(AA1) $\top \sqsubseteq F_0 a$

(AA2) $\top \sqsubseteq F_r \top$

(AA3) $F_r a \sqsubseteq \neg F_s \neg a$, $r + s > 1$

(AA4) $F_r(a \wedge b) \wedge F_s(a \wedge \neg b) \sqsubseteq F_{r+s} a$, $r + s \leq 1$

(AA5) $\neg F_r(a \wedge b) \wedge \neg F_s(a \wedge \neg b) \sqsubseteq \neg F_{r+s} a$, $r + s \leq 1$

(AA6) $a \sqsubseteq b \Rightarrow F_r a \sqsubseteq F_r b$

(AA7) $\left(\bigwedge_{r<s} F_{r_1\cdots r_n r} a\right) = F_{r_1\cdots r_n s} a$

Comments on the axioms

1. The operator F_r is the algebraic counterpart of the logical modality L_r. The first two axioms state tautologies, while the third captures the way F_r interacts with negation. Axioms (AA4) and (AA5) assert finite additivity, while (AA6) asserts monotonicity.

Comments on the axioms

1. The operator F_r is the algebraic counterpart of the logical modality L_r. The first two axioms state tautologies, while the third captures the way F_r interacts with negation. Axioms (AA4) and (AA5) assert finite additivity, while (AA6) asserts monotonicity.
2. The most interesting axiom is the infinitary axiom (AA7). It asserts that $F_{r_1 \cdots r_n s} a$ is the greatest lower bound of the set $\mathbf{Set} F_{r_1 \cdots r_n r} a r < s$ with respect to the natural order $\leq$. We will use it to establish countable additivity when we prove duality.

Comments on the axioms

1. The operator F_r is the algebraic counterpart of the logical modality L_r. The first two axioms state tautologies, while the third captures the way F_r interacts with negation. Axioms (AA4) and (AA5) assert finite additivity, while (AA6) asserts monotonicity.
2. The most interesting axiom is the infinitary axiom (AA7). It asserts that $F_{r_1 \cdots r_n s} a$ is the greatest lower bound of the set $\mathbf{Set} F_{r_1 \cdots r_n r} a r < s$ with respect to the natural order $\leq$. We will use it to establish countable additivity when we prove duality.
3. There are only countably many instances of (AA7).

Basic Lemmas

Let $\mathcal{A} = (A, \Rightarrow, \bot, \{F_r\}_{r\in\mathbb{Q}_0}, \sqsubseteq)$ be an Aumann algebra. For all $a, b \in A$ and $r, s \in \mathbb{Q}_0$,

1. $F_r\bot = \bot$ for $r > 0$;
2. if $r \leq s$, then $F_s a \sqsubseteq F_r a$;
3. if $a \sqsubseteq \neg b$ and $r + s > 1$, then $F_r a \sqsubseteq \neg F_s b$.

The logic yields an Aumann algebra

Let $[\phi]$ denote the equivalence class of ϕ modulo $\equiv$, and let $\mathcal{L}/{\equiv} = \{[\phi] \mid \phi \in \mathcal{L}\}$.

Theorem

The structure

$$(\mathcal{L}/{\equiv}, \Rightarrow, [\bot], \{L_r\}_{r \in \mathbb{Q}_0}, \leq)$$

is an Aumann algebra, where $[\phi] \leq [\psi]$ iff $\vdash \phi \Rightarrow \psi$.

Soundness for Aumann Algebras

Theorem

Let $\mathcal{A}$ be an Aumann algebra and $a \in \mathcal{A}$. If $\top \sqsubseteq a$, then for any Markov process $\mathcal{M} = (M, \Sigma, \tau)$ and any interpretation $[\![\cdot]\!]$ of terms in the language of Aumann algebras as measurable sets in M with the properties listed above, $[\![a]\!] = M$.

Preliminary Remarks

- We need to combine an LMP with a Stone space. It is natural to use the Borel algebra from the Stone topology as the σ-algebra.

Preliminary Remarks

- We need to combine an LMP with a Stone space. It is natural to use the Borel algebra from the Stone topology as the σ-algebra.
- Thus we have Markov processes defined not on arbitrary measure spaces but on “Stone-like” spaces.

Preliminary Remarks

- We need to combine an LMP with a Stone space. It is natural to use the Borel algebra from the Stone topology as the σ-algebra.
- Thus we have Markov processes defined not on arbitrary measure spaces but on "Stone-like" spaces.
- However, it is not just a simple combination of the definitions of Markov processes and Stone spaces.

Preliminary Remarks

- We need to combine an LMP with a Stone space. It is natural to use the Borel algebra from the Stone topology as the σ-algebra.
- Thus we have Markov processes defined not on arbitrary measure spaces but on "Stone-like" spaces.
- However, it is not just a simple combination of the definitions of Markov processes and Stone spaces.
- Our spaces will not be compact and this will cause (and cure!) some headaches.

Preliminary Remarks

- We need to combine an LMP with a Stone space. It is natural to use the Borel algebra from the Stone topology as the σ-algebra.
- Thus we have Markov processes defined not on arbitrary measure spaces but on "Stone-like" spaces.
- However, it is not just a simple combination of the definitions of Markov processes and Stone spaces.
- Our spaces will not be compact and this will cause (and cure!) some headaches.
- We need to have spaces where the F_r operations of the Aumann algebra can be sensibly interpreted.

Distinguished Base

- We restrict attention to Markov processes $(M, \mathcal{D}, \tau)$ where $\mathcal{D}$ is a *distinguished* countable base of clopen sets that is closed under the set-theoretic Bolean operations and

Distinguished Base

- We restrict attention to Markov processes $(M, \mathcal{D}, \tau)$ where $\mathcal{D}$ is a *distinguished* countable base of clopen sets that is closed under the set-theoretic Bolean operations and
- also under the operation

$$F_r(D) = \{m \mid \tau(m)(D) \geq r\}, r \in \mathbb{Q}_0.$$

Distinguished Base

- We restrict attention to Markov processes $(M, \mathcal{D}, \tau)$ where $\mathcal{D}$ is a *distinguished* countable base of clopen sets that is closed under the set-theoretic Bolean operations and
- also under the operation

$$F_r(D) = \{m \mid \tau(m)(D) \geq r\}, r \in \mathbb{Q}_0.$$

- The measurable sets are the Borel sets of the topology generated by $\mathcal{D}$.

Distinguished Base

- We restrict attention to Markov processes $(M, \mathcal{D}, \tau)$ where $\mathcal{D}$ is a *distinguished* countable base of clopen sets that is closed under the set-theoretic Bolean operations and
- also under the operation

$$F_r(D) = \{m \mid \tau(m)(D) \geq r\}, r \in \mathbb{Q}_0.$$

- The measurable sets are the Borel sets of the topology generated by $\mathcal{D}$.
- Morphisms of such MPs are *continuous* function $f : \mathcal{M} \longrightarrow \mathcal{N}$ such that
  1. $\forall m \in M$ and $B \in \Sigma_{\mathcal{N}}$, $\tau_{\mathcal{M}}(m)(f^{-1}(B)) = \tau_{\mathcal{N}}(f(m))(B)$;
  2. $\forall D \in \mathcal{D}_{\mathcal{N}}, f^{-1}(D) \in \mathcal{D}_{\mathcal{M}}$.

Saturation

- In model theory there is a concept called "saturation" which means roughly speaking that the space is maximal in some sense; it is the semantic counterpart to the proof theoretic notion of maximally consistent.

Saturation

- In model theory there is a concept called “saturation” which means roughly speaking that the space is maximal in some sense; it is the semantic counterpart to the proof theoretic notion of maximally consistent.
- We introduce a similar concept for our Markov processes.

Saturation

- In model theory there is a concept called “saturation” which means roughly speaking that the space is maximal in some sense; it is the semantic counterpart to the proof theoretic notion of maximally consistent.
- We introduce a similar concept for our Markov processes.
- Intuitively, one adds points to the structure without changing the represented algebra. An MP is *saturated* if it is maximal with respect to this operation.

Saturation definition

- Formally, consider MP morphisms $f : \mathcal{M} \longrightarrow \mathcal{N}$ such that
  - f is a homeomorphism between $\mathcal{M}$ and its image in $\mathcal{N}$;
  - the image $f(M)$ is dense in N; and
  - f preserves the distinguished base in the forward direction as well as the backward; that is, if $D \in \mathcal{D}_{\mathcal{M}}$, then there exists $B \in \mathcal{D}_{\mathcal{N}}$ such that $A = f^{-1}(B)$.

Saturation definition

- Formally, consider MP morphisms $f : \mathcal{M} \to \mathcal{N}$ such that
  - f is a homeomorphism between $\mathcal{M}$ and its image in $\mathcal{N}$;
  - the image $f(M)$ is dense in N; and
  - f preserves the distinguished base in the forward direction as well as the backward; that is, if $D \in \mathcal{D}_{\mathcal{M}}$, then there exists $B \in \mathcal{D}_{\mathcal{N}}$ such that $A = f^{-1}(B)$.
- Call such a morphism a *strict embedding*. The collection of all $\mathcal{N}$ such that there exists a strict embedding $\mathcal{M} \to \mathcal{N}$ contains a final object, which is the colimit of the strict embeddings $\mathcal{M} \to \mathcal{N}$. This is the saturation of $\mathcal{M}$.

Saturation definition

- Formally, consider MP morphisms $f : \mathcal{M} \to \mathcal{N}$ such that
  - f is a homeomorphism between $\mathcal{M}$ and its image in $\mathcal{N}$;
  - the image $f(M)$ is dense in N; and
  - f preserves the distinguished base in the forward direction as well as the backward; that is, if $D \in \mathcal{D}_{\mathcal{M}}$, then there exists $B \in \mathcal{D}_{\mathcal{N}}$ such that $A = f^{-1}(B)$.
- Call such a morphism a *strict embedding*. The collection of all $\mathcal{N}$ such that there exists a strict embedding $\mathcal{M} \to \mathcal{N}$ contains a final object, which is the colimit of the strict embeddings $\mathcal{M} \to \mathcal{N}$. This is the saturation of $\mathcal{M}$.
- The collection of *all* such $\mathcal{N}$ might be too big for the colimit to exist. But we can take the collection of all the embeddings of $\mathcal{M}$ into a suitably large space (e.g. the Stone-Cech compactification of $\mathcal{M}$).

Saturation definition

- Formally, consider MP morphisms $f : \mathcal{M} \to \mathcal{N}$ such that
  - f is a homeomorphism between $\mathcal{M}$ and its image in $\mathcal{N}$;
  - the image $f(M)$ is dense in N; and
  - f preserves the distinguished base in the forward direction as well as the backward; that is, if $D \in \mathcal{D}_{\mathcal{M}}$, then there exists $B \in \mathcal{D}_{\mathcal{N}}$ such that $A = f^{-1}(B)$.
- Call such a morphism a *strict embedding*. The collection of all $\mathcal{N}$ such that there exists a strict embedding $\mathcal{M} \to \mathcal{N}$ contains a final object, which is the colimit of the strict embeddings $\mathcal{M} \to \mathcal{N}$. This is the saturation of $\mathcal{M}$.
- The collection of *all* such $\mathcal{N}$ might be too big for the colimit to exist. But we can take the collection of all the embeddings of $\mathcal{M}$ into a suitably large space (e.g. the Stone-Cech compactification of $\mathcal{M}$).
- One can construct the saturation by adding suitable ultrafilters.

Formal Definition of Stone-Markov Process

Stone-Markov Processes

- A Markov process $\mathcal{M} = (M, \mathcal{D}, \tau)$ with distinguished base is a *Stone–Markov process (SMP)* if it is saturated.
- The morphisms of SMPs are just the morphisms of MPs with distinguished base as defined above.
- The category of SMPs and SMP morphisms is denoted $\mathbf{SMP}$.

Recall that we are not requiring the spaces to be compact, they are just zero-dimensional Hausdorff spaces.

Where we start

- Fix an arbitrary countable Aumann algebra

$$\mathcal{A} = (A, \Rightarrow, \bot, \{F_r\}_{r \in \mathbb{Q}_0}, \sqsubseteq).$$

Where we start

- Fix an arbitrary countable Aumann algebra

$$\mathcal{A} = (A, \Rightarrow, \bot, \{F_r\}_{r \in \mathbb{Q}_0}, \sqsubseteq).$$

- Let $\mathcal{U}^*$ be the set of all ultrafilters of $\mathcal{A}$. The classical Stone construction gives a Boolean algebra of sets isomorphic to $\mathcal{A}$ with elements

Where we start

- Fix an arbitrary countable Aumann algebra

$$\mathcal{A} = (A, \Rightarrow, \bot, \{F_r\}_{r\in\mathbb{Q}_0}, \sqsubseteq).$$

- Let $\mathcal{U}^*$ be the set of all ultrafilters of $\mathcal{A}$. The classical Stone construction gives a Boolean algebra of sets isomorphic to $\mathcal{A}$ with elements
-

$$(\!|a|\!)^* = \{u \in \mathcal{U}^* \mid a \in u\},\ a \in \mathcal{A}$$
$$(\!|\mathcal{A}|\!)^* = \{(\!|a|\!)^* \mid a \in A\}.$$

Where we start

- Fix an arbitrary countable Aumann algebra

$$\mathcal{A} = (A, \Rightarrow, \bot, \{F_r\}_{r\in\mathbb{Q}_0}, \sqsubseteq).$$

- Let $\mathcal{U}^*$ be the set of all ultrafilters of $\mathcal{A}$. The classical Stone construction gives a Boolean algebra of sets isomorphic to $\mathcal{A}$ with elements
-

$$\begin{aligned} (\!|a|\!)^* &= \{u \in \mathcal{U}^* \mid a \in u\},\ a \in \mathcal{A} \\ (\!|\mathcal{A}|\!)^* &= \{(\!|a|\!)^* \mid a \in A\}. \end{aligned}$$

- The sets $(\!|a|\!)^*$ generate a Stone topology τ^* on $\mathcal{U}^*$,

Where we start

- Fix an arbitrary countable Aumann algebra

$$\mathcal{A} = (A, \Rightarrow, \bot, \{F_r\}_{r\in\mathbb{Q}_0}, \sqsubseteq).$$

- Let $\mathcal{U}^*$ be the set of all ultrafilters of $\mathcal{A}$. The classical Stone construction gives a Boolean algebra of sets isomorphic to $\mathcal{A}$ with elements
-

$$(\!|a|\!)^* = \{u \in \mathcal{U}^* \mid a \in u\},\ a \in \mathcal{A}$$
$$(\!|\mathcal{A}|\!)^* = \{(\!|a|\!)^* \mid a \in A\}.$$

- The sets $(\!|a|\!)^*$ generate a Stone topology τ^* on $\mathcal{U}^*$,
- and the $(\!|a|\!)^*$ are exactly the clopen sets of the topology.

Ultrafilters: Good and Bad

- Let $\mathcal{F}$ be the set of elements of the form $\alpha^r = F_{t_1 \cdots t_n r} a$ for $a \in A$ and $t_1, \ldots, t_n, r \in \mathbb{Q}_0$.

Ultrafilters: Good and Bad

- Let $\mathcal{F}$ be the set of elements of the form $\alpha^r = F_{t_1\cdots t_n r}a$ for $a \in A$ and $t_1, \ldots, t_n, r \in \mathbb{Q}_0$.
- Notation: we view the term $F_{t_1\cdots t_n r}a$ as parametrized by r, so, *e.g.* α^s means $F_{t_1\cdots t_n s}a$.

Ultrafilters: Good and Bad

- Let $\mathcal{F}$ be the set of elements of the form $\alpha^r = F_{t_1 \cdots t_n r} a$ for $a \in A$ and $t_1, \ldots, t_n, r \in \mathbb{Q}_0$.
- Notation: we view the term $F_{t_1 \cdots t_n r} a$ as parametrized by r, so, *e.g.* α^s means $F_{t_1 \cdots t_n s} a$.
- The set $\mathcal{F}$ is countable since A is.

Ultrafilters: Good and Bad

- Let $\mathcal{F}$ be the set of elements of the form $\alpha^r = F_{t_1 \cdots t_n r} a$ for $a \in A$ and $t_1, \ldots, t_n, r \in \mathbb{Q}_0$.
- Notation: we view the term $F_{t_1 \cdots t_n r} a$ as parametrized by r, so, *e.g.* α^s means $F_{t_1 \cdots t_n s} a$.
- The set $\mathcal{F}$ is countable since A is.
- Axiom (AA7) asserts all infinitary equations of the form

$$\alpha^s = \bigwedge_{r<s} \alpha^r. \tag{1}$$

for $\alpha^s \in \mathcal{F}$.

Ultrafilters: Good and Bad

- Let $\mathcal{F}$ be the set of elements of the form $\alpha^r = F_{t_1 \cdots t_n r} a$ for $a \in A$ and $t_1, \ldots, t_n, r \in \mathbb{Q}_0$.
- Notation: we view the term $F_{t_1 \cdots t_n r} a$ as parametrized by r, so, *e.g.* α^s means $F_{t_1 \cdots t_n s} a$.
- The set $\mathcal{F}$ is countable since A is.
- Axiom (AA7) asserts all infinitary equations of the form

$$\alpha^s = \bigwedge_{r<s} \alpha^r. \tag{1}$$

  for $\alpha^s \in \mathcal{F}$.
- Let us call an ultrafilter u *bad* if it violates one or more of these equations, *i.e.* for some $\alpha^s \in \mathcal{F}$, $\alpha^r \in u$ for all $r < s$ but $\alpha^s \notin u$.

Ultrafilters: Good and Bad

- Let $\mathcal{F}$ be the set of elements of the form $\alpha^r = F_{t_1 \cdots t_n r} a$ for $a \in A$ and $t_1, \ldots, t_n, r \in \mathbb{Q}_0$.
- Notation: we view the term $F_{t_1 \cdots t_n r} a$ as parametrized by r, so, *e.g.* α^s means $F_{t_1 \cdots t_n s} a$.
- The set $\mathcal{F}$ is countable since A is.
- Axiom (AA7) asserts all infinitary equations of the form

$$\alpha^s = \bigwedge_{r<s} \alpha^r. \tag{1}$$

  for $\alpha^s \in \mathcal{F}$.
- Let us call an ultrafilter u *bad* if it violates one or more of these equations, *i.e.* for some $\alpha^s \in \mathcal{F}$, $\alpha^r \in u$ for all $r < s$ but $\alpha^s \notin u$.
- Otherwise, u is called *good*. Let $\mathcal{U}$ be the set of good ultrafilters of $\mathcal{A}$.

The space of good ultrafilters

1. The set of bad ultrafilters is meager in the Stone topology.

The space of good ultrafilters

1. The set of bad ultrafilters is meager in the Stone topology.
2. The set of good ultrafilters is a zero-dimensional Hausdorff space.

The space of good ultrafilters

1. The set of bad ultrafilters is meager in the Stone topology.
2. The set of good ultrafilters is a zero-dimensional Hausdorff space.
3. By removing the bad ultrafilters we have a space that is not compact any more.

The space of good ultrafilters

1. The set of bad ultrafilters is meager in the Stone topology.
2. The set of good ultrafilters is a zero-dimensional Hausdorff space.
3. By removing the bad ultrafilters we have a space that is not compact any more.
4. The good ultrafilters are dense in the set of all ultrafilters.

Construction of the Markov process - I

1. We are going to build an SMP on the space of good ultrafilters.

Construction of the Markov process - I

1. We are going to build an SMP on the space of good ultrafilters.
2. For any ultrafilter u and any a in the Aumann Algebra, the set $\{r \in \mathbb{Q}_0 \mid F_r a \in u\}$ is non-empty and downward closed.

Construction of the Markov process - I

1. We are going to build an SMP on the space of good ultrafilters.
2. For any ultrafilter u and any a in the Aumann Algebra, the set $\{r \in \mathbb{Q}_0 \mid F_r a \in u\}$ is non-empty and downward closed.
3. Using the axioms one can show that

$$\sup\{r \mid F_r a \in u\} = \inf\{r \mid \neg F_r a \in u\}.$$

Construction of the Markov process - I

1. We are going to build an SMP on the space of good ultrafilters.
2. For any ultrafilter u and any a in the Aumann Algebra, the set $\{r \in \mathbb{Q}_0 \mid F_r a \in u\}$ is non-empty and downward closed.
3. Using the axioms one can show that

$$\sup\{r \mid F_r a \in u\} = \inf\{r \mid \neg F_r a \in u\}.$$

4. Thus one can define $\tau(u)(\llparenthesis a \rrparenthesis) = \sup\{\ldots\} = \inf\{\ldots\}$.

Construction of the Markov process - II

1. The collection $(\!|\mathcal{A}|\!)$ is a field of sets.

Construction of the Markov process - II

1. The collection $(\!|\mathcal{A}|\!)$ is a field of sets.
2. The function $\tau(u)$ is finitely additive on the field of sets $(\!|\mathcal{A}|\!)$.

Construction of the Markov process - II

1. The collection $(\!|\mathcal{A}|\!)$ is a field of sets.
2. The function $\tau(u)$ is finitely additive on the field of sets $(\!|\mathcal{A}|\!)$.
3. For all u, $\tau(u)$ is continuous from above at $\emptyset$: – Hardest part of the paper!

Construction of the Markov process - II

1. The collection $(\!|\mathcal{A}|\!)$ is a field of sets.
2. The function $\tau(u)$ is finitely additive on the field of sets $(\!|\mathcal{A}|\!)$.
3. For all u, $\tau(u)$ is continuous from above at $\emptyset$: – Hardest part of the paper!
4. This argument uses – in a crucial way – the fact that the bad ultrafilters have been removed.

Construction of the Markov process - II

1. The collection $(\!|\mathcal{A}|\!)$ is a field of sets.
2. The function $\tau(u)$ is finitely additive on the field of sets $(\!|\mathcal{A}|\!)$.
3. For all u, $\tau(u)$ is continuous from above at $\emptyset$: – Hardest part of the paper!
4. This argument uses – in a crucial way – the fact that the bad ultrafilters have been removed.
5. One can use standard measure extension theorems to define τ as a measure on the space of good ultrafilters.

Construction of the Markov process - II

1. The collection $(\!|\mathcal{A}|\!)$ is a field of sets.
2. The function $\tau(u)$ is finitely additive on the field of sets $(\!|\mathcal{A}|\!)$.
3. For all u, $\tau(u)$ is continuous from above at $\emptyset$: – Hardest part of the paper!
4. This argument uses – in a crucial way – the fact that the bad ultrafilters have been removed.
5. One can use standard measure extension theorems to define τ as a measure on the space of good ultrafilters.
6. Basic result: If $\mathcal{A}$ is a countable Aumann algebra then we can construct a countably-generated SMP, $\mathbb{M}(\mathcal{A})$, on the space of good ultrafilters.

Construction of the Markov process - II

1. The collection $(\!|\mathcal{A}|\!)$ is a field of sets.
2. The function $\tau(u)$ is finitely additive on the field of sets $(\!|\mathcal{A}|\!)$.
3. For all u, $\tau(u)$ is continuous from above at $\emptyset$: – Hardest part of the paper!
4. This argument uses – in a crucial way – the fact that the bad ultrafilters have been removed.
5. One can use standard measure extension theorems to define τ as a measure on the space of good ultrafilters.
6. Basic result: If $\mathcal{A}$ is a countable Aumann algebra then we can construct a countably-generated SMP, $\mathbb{M}(\mathcal{A})$, on the space of good ultrafilters.
7. It is straighforward to extend $\mathbb{M}(\cdot)$ to a functor.

No difficulties in the reverse direction

1. Let $\mathcal{M} = (M, \mathcal{B}, \tau)$ be a Stone Markov process with distinguished base $\mathcal{B}$. By definition, $\mathcal{B}$ is a field of clopen sets closed under the operations
$$F_r(A) = \{m \in M \mid \tau(m)(A) \geq r\}.$$

No difficulties in the reverse direction

1. Let $\mathcal{M} = (M, \mathcal{B}, \tau)$ be a Stone Markov process with distinguished base $\mathcal{B}$. By definition, $\mathcal{B}$ is a field of clopen sets closed under the operations
$$F_r(A) = \{m \in M \mid \tau(m)(A) \geq r\}.$$
2. The structure $\mathcal{B}$ with the set-theoretic Boolean operations and the operations F_r, $r \in \mathbb{Q}_0$ is a countable Aumann algebra.

No difficulties in the reverse direction

1. Let $\mathcal{M} = (M, \mathcal{B}, \tau)$ be a Stone Markov process with distinguished base $\mathcal{B}$. By definition, $\mathcal{B}$ is a field of clopen sets closed under the operations
$$F_r(A) = \{m \in M \mid \tau(m)(A) \geq r\}.$$
2. The structure $\mathcal{B}$ with the set-theoretic Boolean operations and the operations F_r, $r \in \mathbb{Q}_0$ is a countable Aumann algebra.
3. The verification is routine.

No difficulties in the reverse direction

1. Let $\mathcal{M} = (M, \mathcal{B}, \tau)$ be a Stone Markov process with distinguished base $\mathcal{B}$. By definition, $\mathcal{B}$ is a field of clopen sets closed under the operations
$$F_r(A) = \{m \in M \mid \tau(m)(A) \geq r\}.$$
2. The structure $\mathcal{B}$ with the set-theoretic Boolean operations and the operations F_r, $r \in \mathbb{Q}_0$ is a countable Aumann algebra.
3. The verification is routine.
4. We denote this algebra by $\mathbb{A}(\mathcal{M})$.

No difficulties in the reverse direction

1. Let $\mathcal{M} = (M, \mathcal{B}, \tau)$ be a Stone Markov process with distinguished base $\mathcal{B}$. By definition, $\mathcal{B}$ is a field of clopen sets closed under the operations
$$F_r(A) = \{m \in M \mid \tau(m)(A) \geq r\}.$$
2. The structure $\mathcal{B}$ with the set-theoretic Boolean operations and the operations F_r, $r \in \mathbb{Q}_0$ is a countable Aumann algebra.
3. The verification is routine.
4. We denote this algebra by $\mathbb{A}(\mathcal{M})$.
5. Once again, we can make $\mathbb{A}(\cdot)$ a functor.

The non-categorical version

1. Any countable Aumann algebra $\mathcal{A}$
$\mathcal{A} = (A, \top, \bot, \neg, \vee, \wedge, \{F_r\}_{r \in \mathbb{Q}^+}, \sqsubseteq)$ is isomorphic to $\mathbb{A}(\mathbb{M}(\mathcal{A}))$ via the map $\beta : \mathcal{A} \to \mathbb{A}(\mathbb{M}(\mathcal{A}))$ defined by

$$\beta(a) = \{u \in \mathrm{supp}(\mathbb{M}(\mathcal{A})) \mid a \in u\} = (\!|a|\!).$$

2. Any Stone Markov process $\mathcal{M} = (M, \mathcal{A}, \theta)$ is homeomorphic to $\mathbb{M}(\mathbb{A}(\mathcal{M}))$ via the map $\alpha : \mathcal{M} \to \mathbb{M}(\mathbb{A}(\mathcal{M}))$ defined by

$$\alpha(m) = \{A \in \mathcal{A} \mid m \in A\}.$$

Defining the arrow part of $\mathbb{A}(\cdot)$

We define a contravariant functor $\mathbb{A} : \mathbf{SMP} \longrightarrow \mathbf{AA}^{\mathrm{op}}$:

$\mathbb{A}(\cdot)$

On arrows $f : \mathcal{M} \longrightarrow \mathcal{N}$ we define $\mathbb{A}(f) = f^{-1} : \mathbb{A}(\mathcal{N}) \longrightarrow \mathbb{A}(\mathcal{M})$.

It is well known that this is a Boolean algebra homomorphism. It is also easy to verify that it is an Aumann algebra homomorphism.

Defining the arrow part of $\mathbb{M}(\cdot)$

We define the arrow part of $\mathbb{M} : \mathbf{AA} \longrightarrow \mathbf{SMP}^{\mathrm{op}}$.

$\mathbb{M}(\cdot)$

On morphisms $h : \mathcal{A} \to \mathcal{B}$, $\mathbb{M}(h) = h^{-1} : \mathbb{M}(\mathcal{B}) \to \mathbb{M}(\mathcal{A})$, explicitly

$$\mathbb{M}(h)(u) = h^{-1}(u) = \{A \in \mathcal{A}_{\mathcal{N}} \mid h(A) \in u\}.$$

Duality: categorical form

The functors $\mathbb{M}$ and $\mathbb{A}$ define a dual equivalence of categories.

$$\begin{tikzcd} \mathbf{SMP} \arrow[r, bend left, "\mathbb{A}"] & \mathbf{AA}^{\mathrm{op}} \arrow[l, bend left, "\mathbb{M}"] \end{tikzcd}$$