Modular Tensor Categories and Topological Quantum Computing

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and
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- A tall order!

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- The topology will keep the configuration from coming apart.
- Where do we find quantum braids or knots?

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- Depending on the type of particle the answer could be 1/3 (bosons) or 0 (fermions).

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- $$\rho : G \to GL(H)$$

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- We have never seen parastatistics in nature.

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- If the state vector changes sign under an interchange of identical particles, but must also look the same if they are in the same state, we have v = -v; where v is the state vector describing two identical particles in the same state.
- In short v = 0!
- With fermions two particles cannot be in exactly the same state: Pauli exclusion principle. The reason for chemistry!!

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- The explanation of black-body radiation, lasers, superconductivity, BE condensation and many other collective phenomena.

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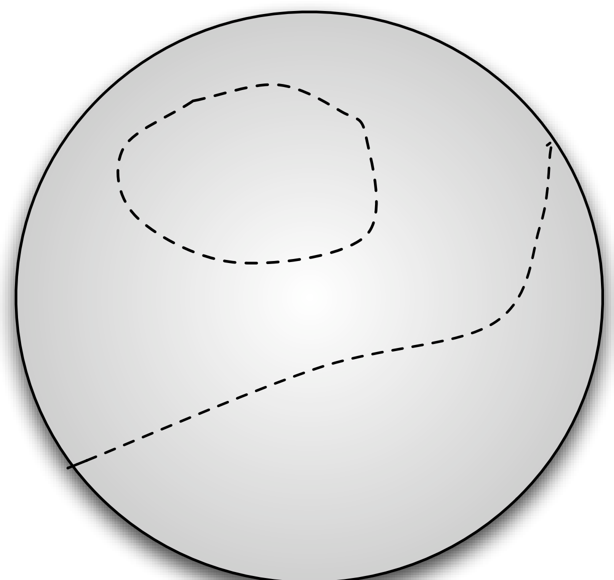
- The resulting group is not simply connected: there are loops that cannot be continuously deformed to a point.

A picture of $SO(3)$ showing a loop that can be shrunk to a point and one that cannot.



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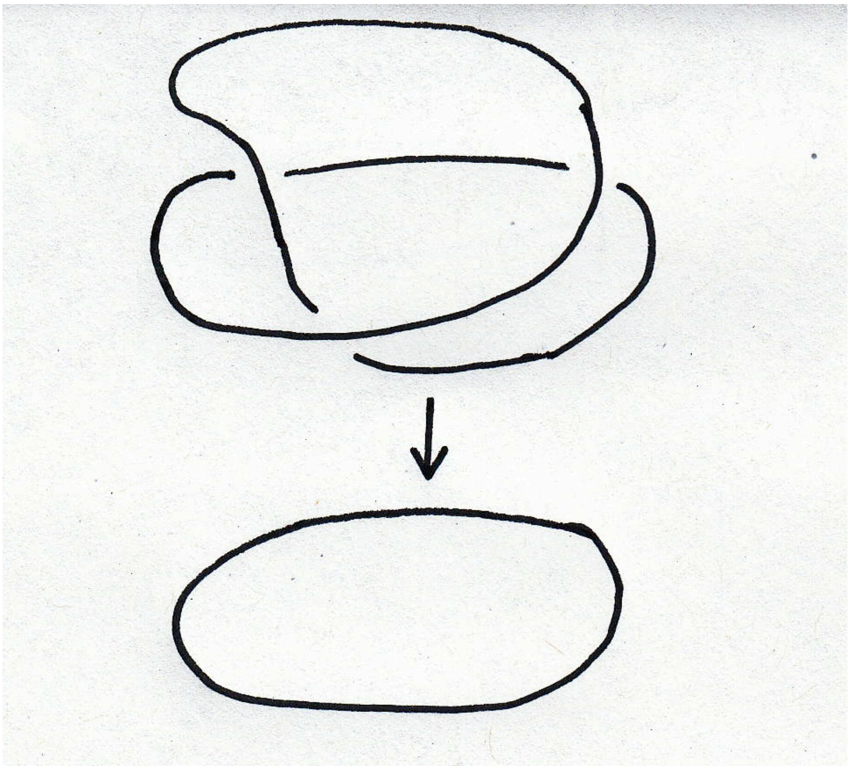

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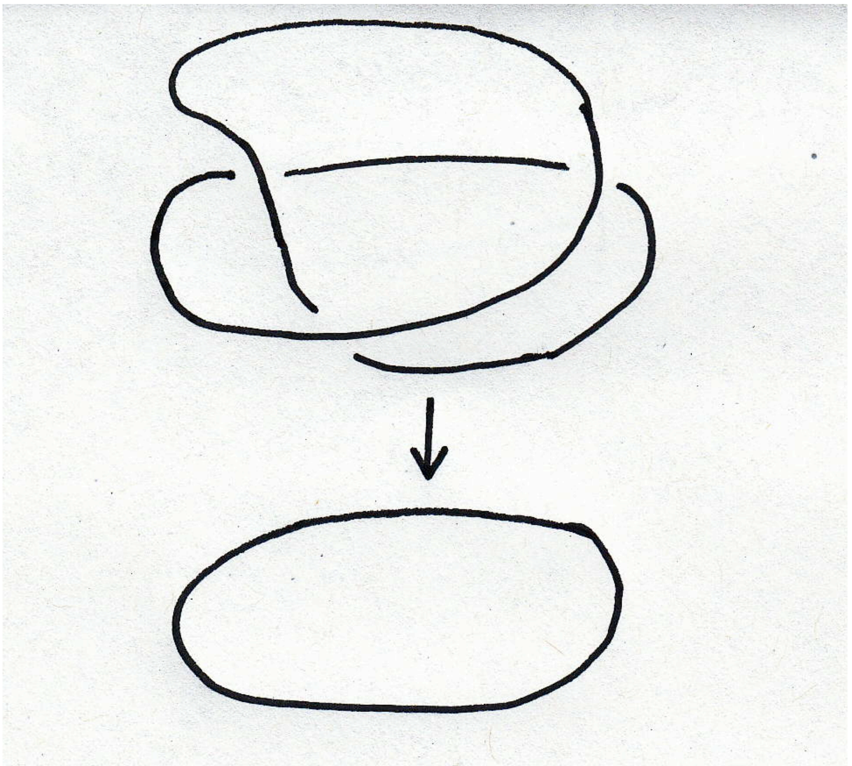
Now which is the relevant symmetry group for quantum mechanics?



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Nature has two types of particles: those for which a 2π rotation is the identity and those for which a 4π rotation is the identity.

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What happens in two dimensions?

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Such entities are called *anyons*.

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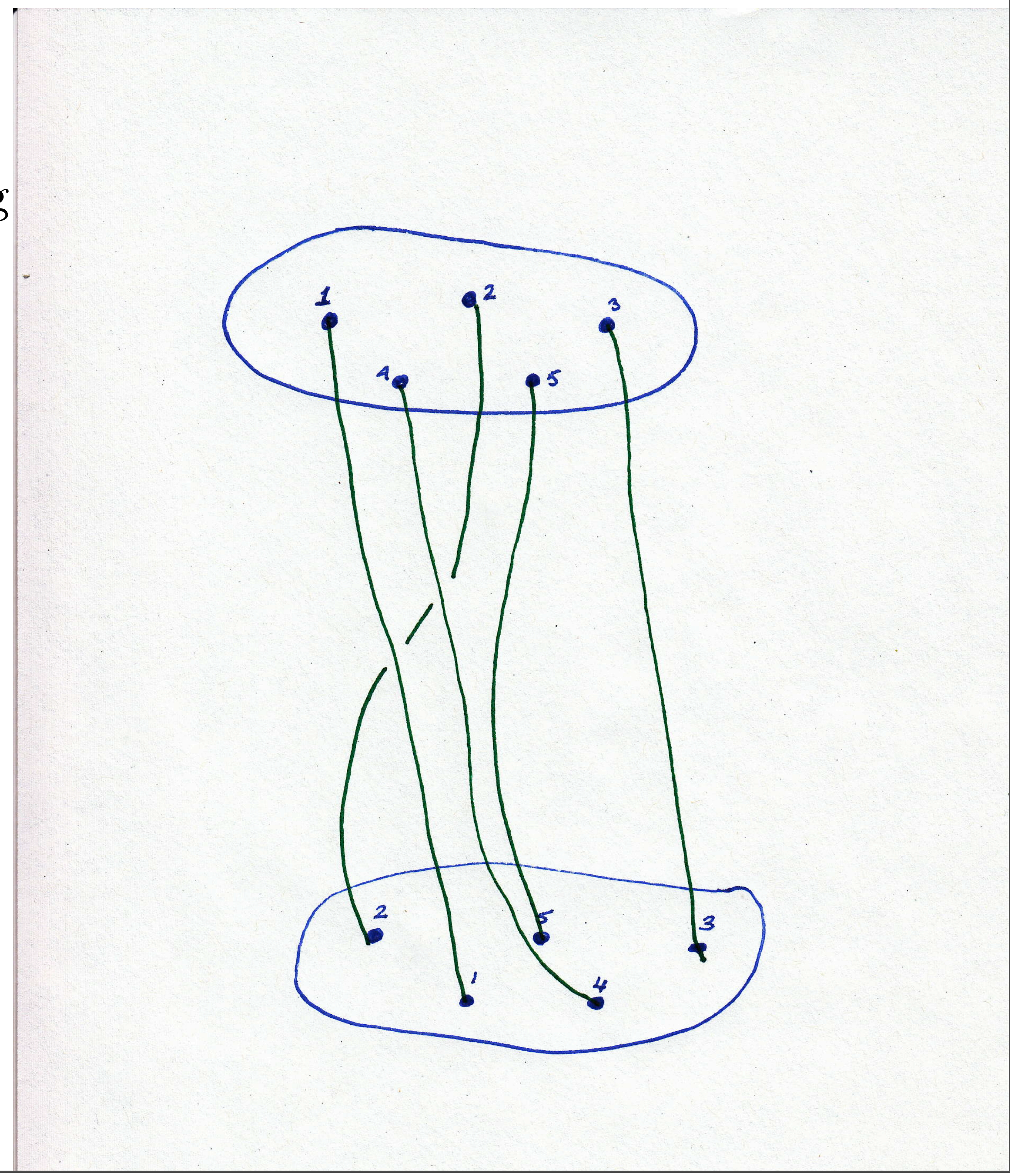
Excited states of this system are not electrons but *virtual* particles with strange properties.

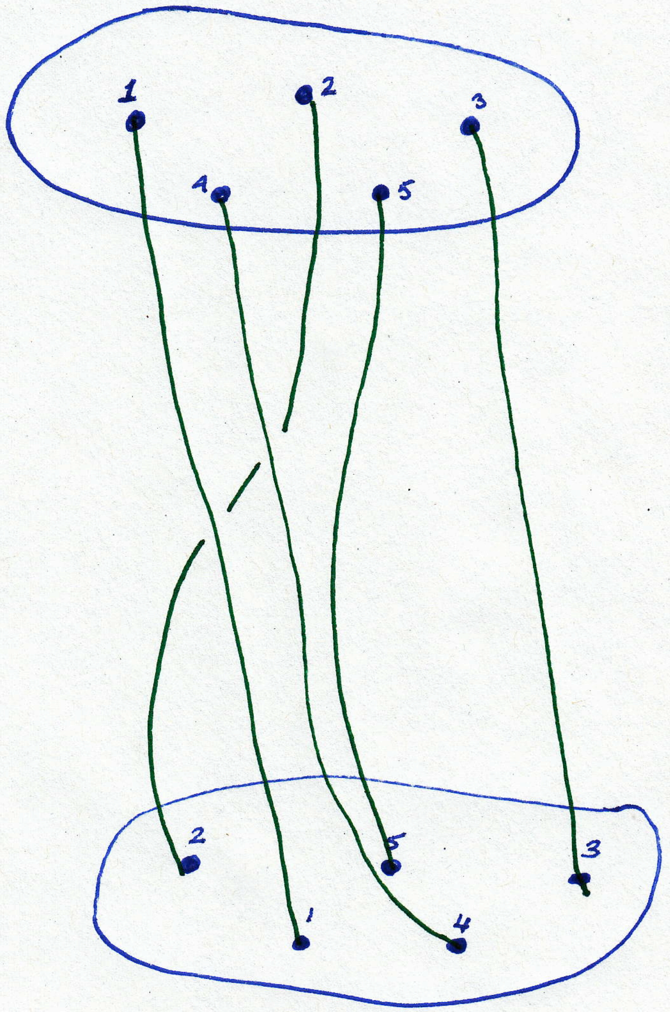
Imagine some (5 in the picture) particles and consider what happens when some of them are exchanged.

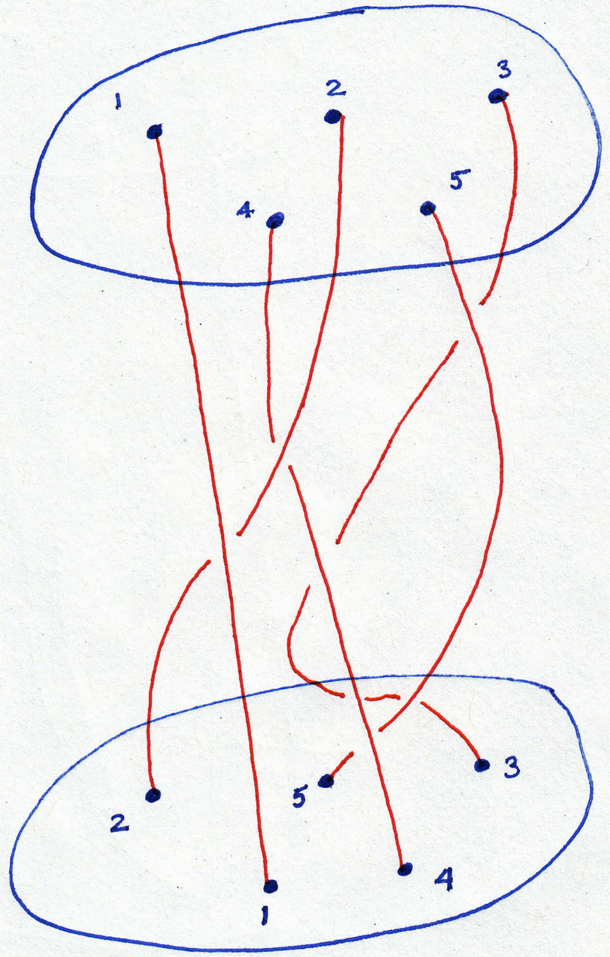
Here $1 \mapsto 4, 2 \mapsto 1, 3 \mapsto 3, 4 \mapsto 5$ and $5 \mapsto 2$

In 3D the strands can always be disentangled; the only thing that matters is the start and end point. So we can describe the effect just by giving a permutation.

In 2D the entangling matters. One has to distinguish between different braidings.



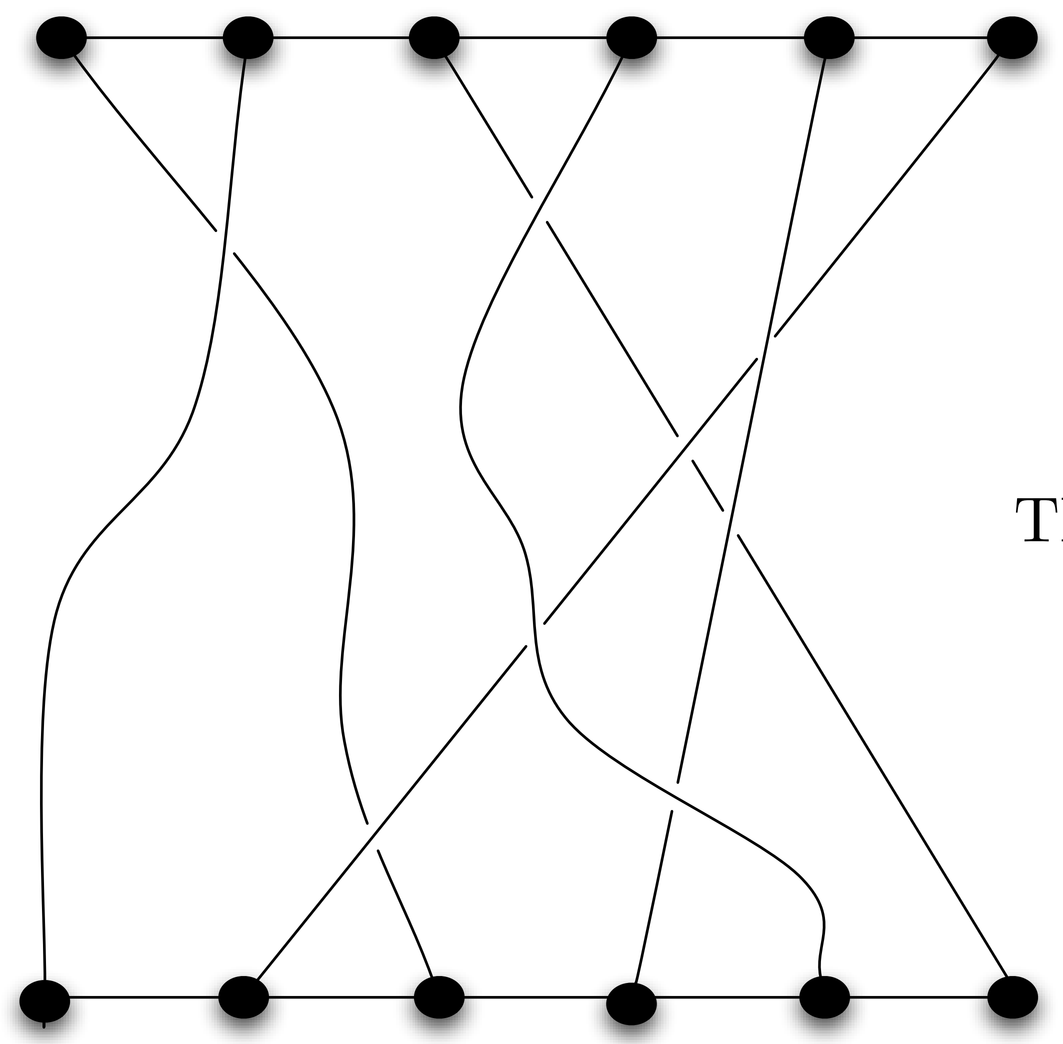




Here the permutations are the same but the braiding is different.

The Braid Group

Fix n and consider n points on a line with another n points on a line below. We connect them with strands. The generators of the group are interchanges of adjacent strands.



This is an element of B_6.

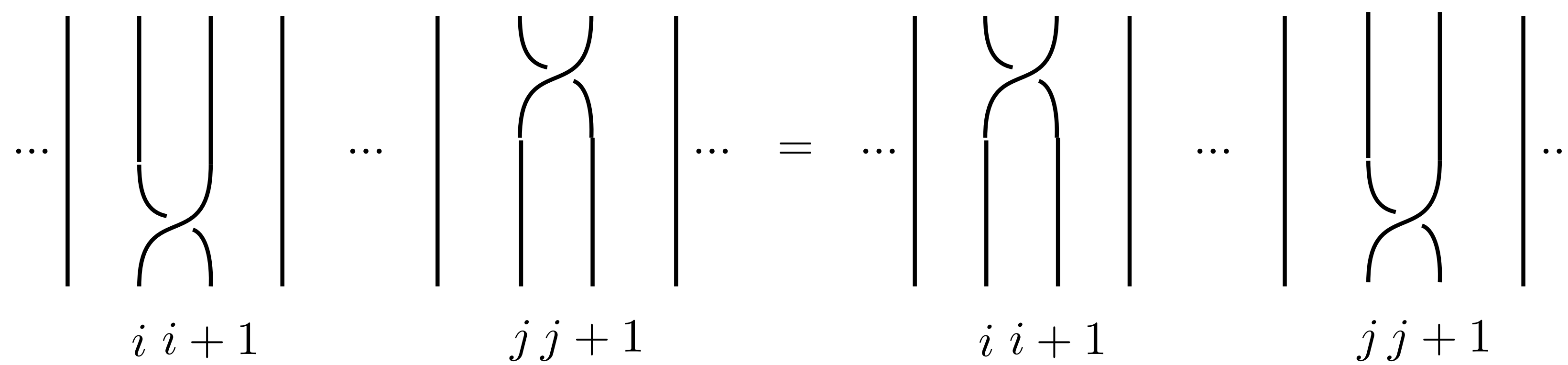
Much richer theory than the permutation group.

For n points the generators are b_1 to b_{n-1} and their inverses. The generators obey the following equations:

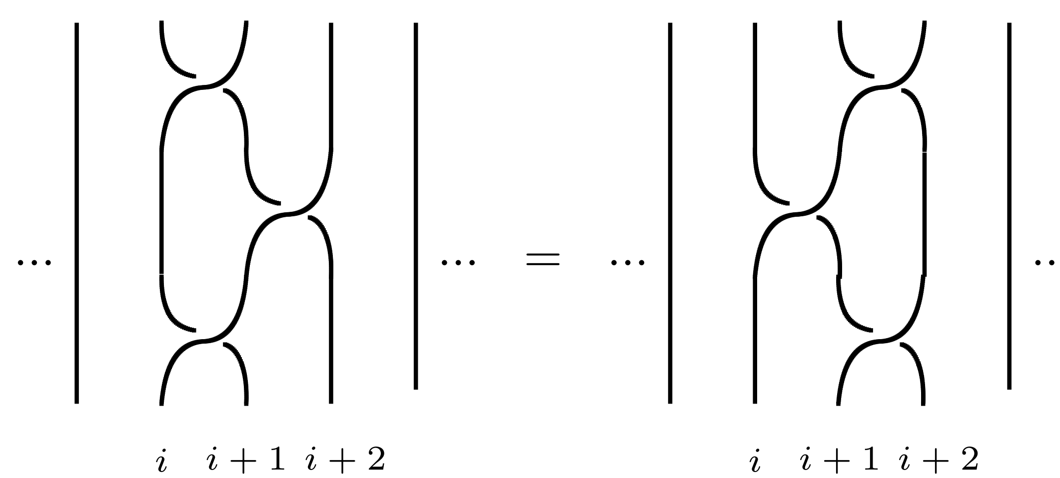
$$b_i b_j = b_j b_i \quad \text{for} \quad |i-j| \geq 2 \tag{1}$$

$$b_i b_{i+1} b_i = b_{i+1} b_i b_{i+1} \quad \text{for} \quad 1 \leq i \leq n-1. \tag{2}$$

which respectively depicts as:



and



Generalized Spin-Statistics theorem holds in dimensions 2 and 3.

See the paper by Froelich and Gabbiani : Local Quantum Theory and Braid Group Statistics.

There is a lot more to be said about knots, braids, physics and related things but we need to get on with the main story.

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Not all anyons are so simple!

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However, there are more interesting representations.

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There are candidates but there are no definite laboratory demonstrations of non-abelian anyons.

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It is the space of fusion possibilities that describes the qubits! If $[a, b] = 2c$ we use the 2D fusion space of the resulting c anyon to encode a qubit.

Fusing non-abelian anyons

Now the *type* of an anyon is not just a complex number but a matrix.

What happens when we combine anyons of different types? Write $[a, b]$ for the combination of a type-a anyon and a type-b anyon.

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How do we describe all this complicated algebra? There are different types of things that combine in non-trivial ways. We have essentially an exotic type theory.

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To accomodate everything we use what are called *modular tensor categories.*

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A condition that says everything is a combination of the basic charges: semi-simplicity.

A way of capturing braiding and twisting: ribbon structure.

A way of capturing "anti-particles": conjugation, rigid structure.

In particular, semisimplicity captures the following ideas:

- The charge of an anyon is elementary i.e., it cannot be decomposed into other elementary entities. In categorical terms, the charge of an anyon has no other subobjects other than 0 and itself.

- The set of endomorphisms of a charge (a simple object) is isomorphic to the complex field.

Finally, this structure entails that given two different simple charges S_1 and S_2, $\mathrm{Hom}(S_1, S_2) = \{0\}$.

We consider a special class of semisimple ribbon categories called **modular tensor categories**. Such categories have only a finite number of simple objects i.e. possible charges for an anyon. Moreover, its defining conditions ensure that the braids are not degenerate.

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or it could be higher-dimensional, in which case we have non-abelian anyons.

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Think of $a + b \to c$ as a reaction, but do not think of $N^c ab$ as the *number* of copies of c produced.

Rather, it is the number of ways in which a c can be produced. Thus, there is a vector space of possible c states and N_{ab}^c is its dimension.

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The columns of this matrix are eigenvectors of the fusion matrices and the dimensions of the simple objects are the eigenvalues.

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If we combine spin l_1 and l_2 the spectrum of possible total spins is: $|l_1 - l_2|, \ldots, l_1 + l_2$ in steps of 1.

If we are coupling l_1, l_2, l_3 we can combine l_1, l_2 and then l_3 or l_1 with the result of l_2, l_3. We get isomorphic ("same") spaces but different bases. The coefficients of the transformation are called $3j$ symbols or Wigner-Racah coefficients.

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There are infinitely many irreps of $SU(2)$; so definitely not a modular tensor category.

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These notations are very confusing: the "chemical reaction" plus symbol is not a direct sum nor even a tensor product. We are combining *labels* not vector spaces.

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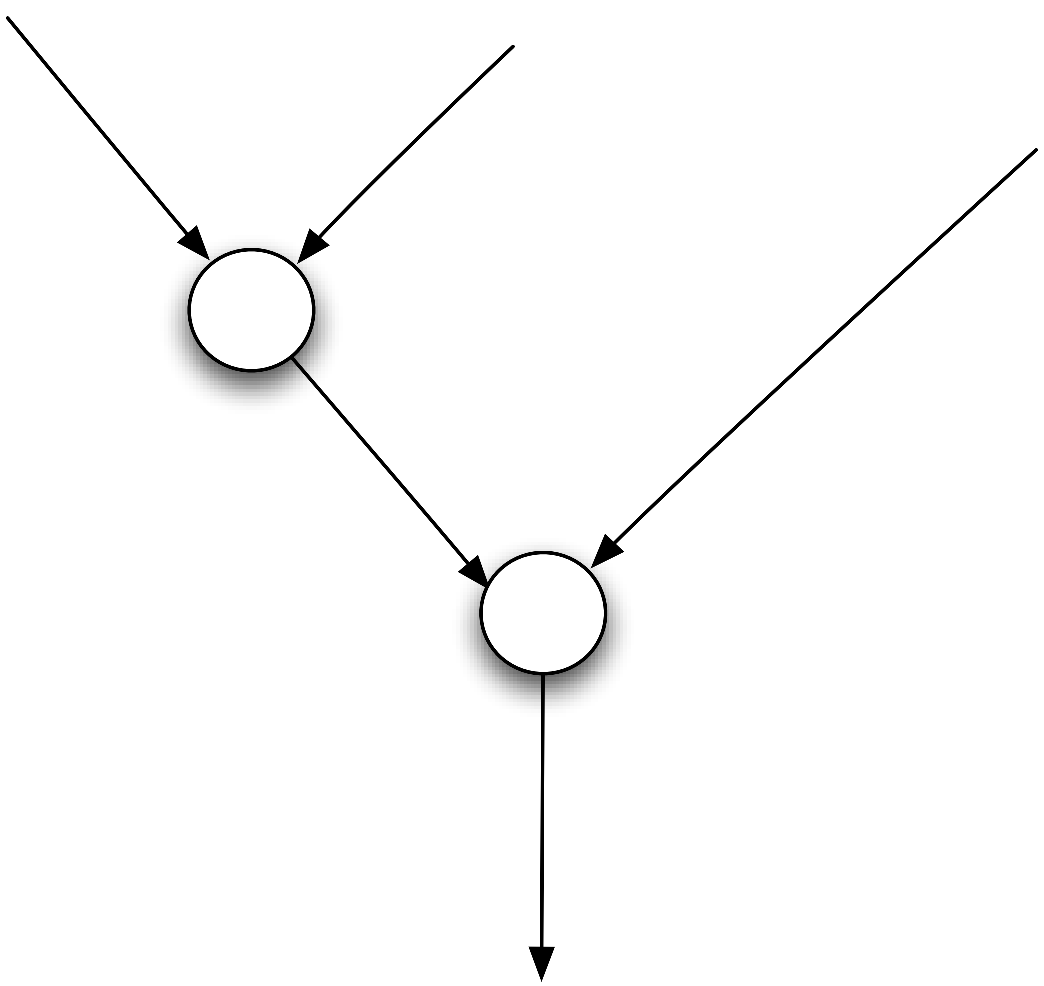

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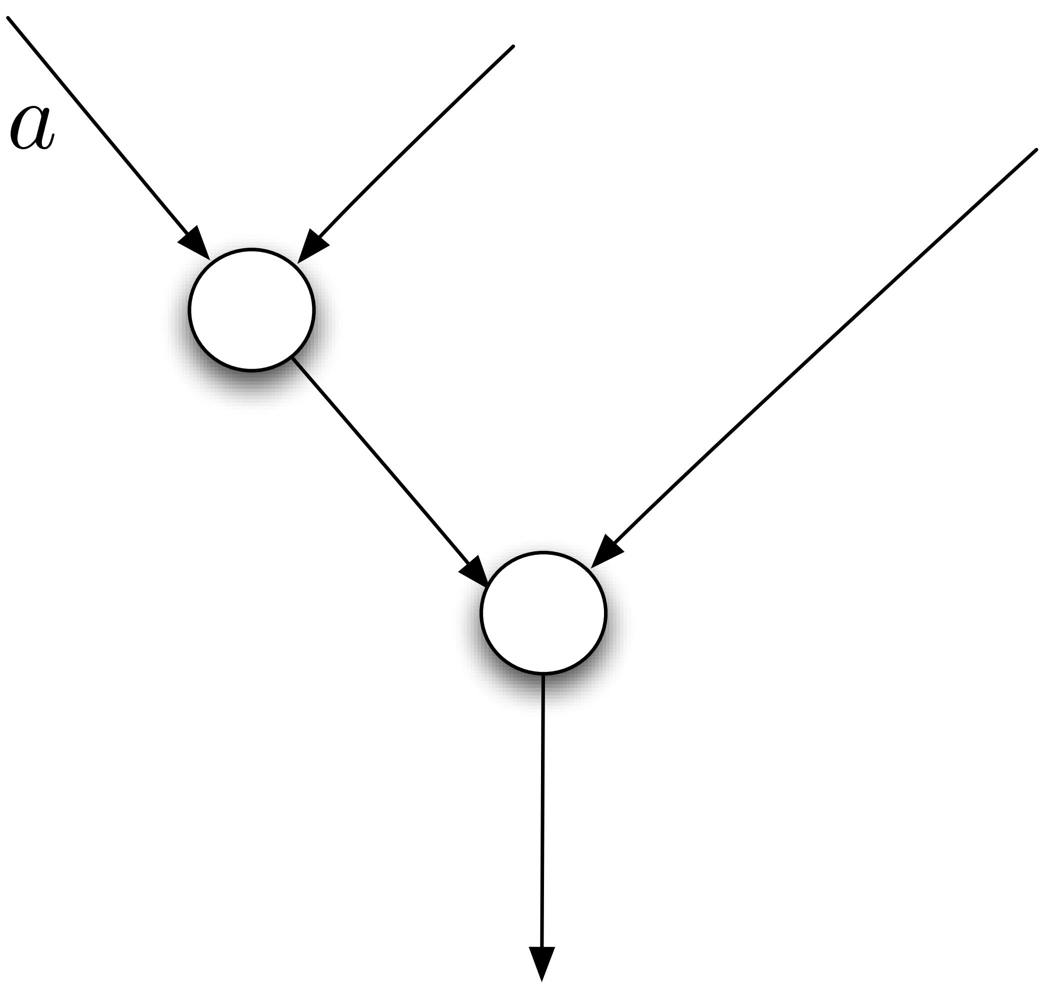

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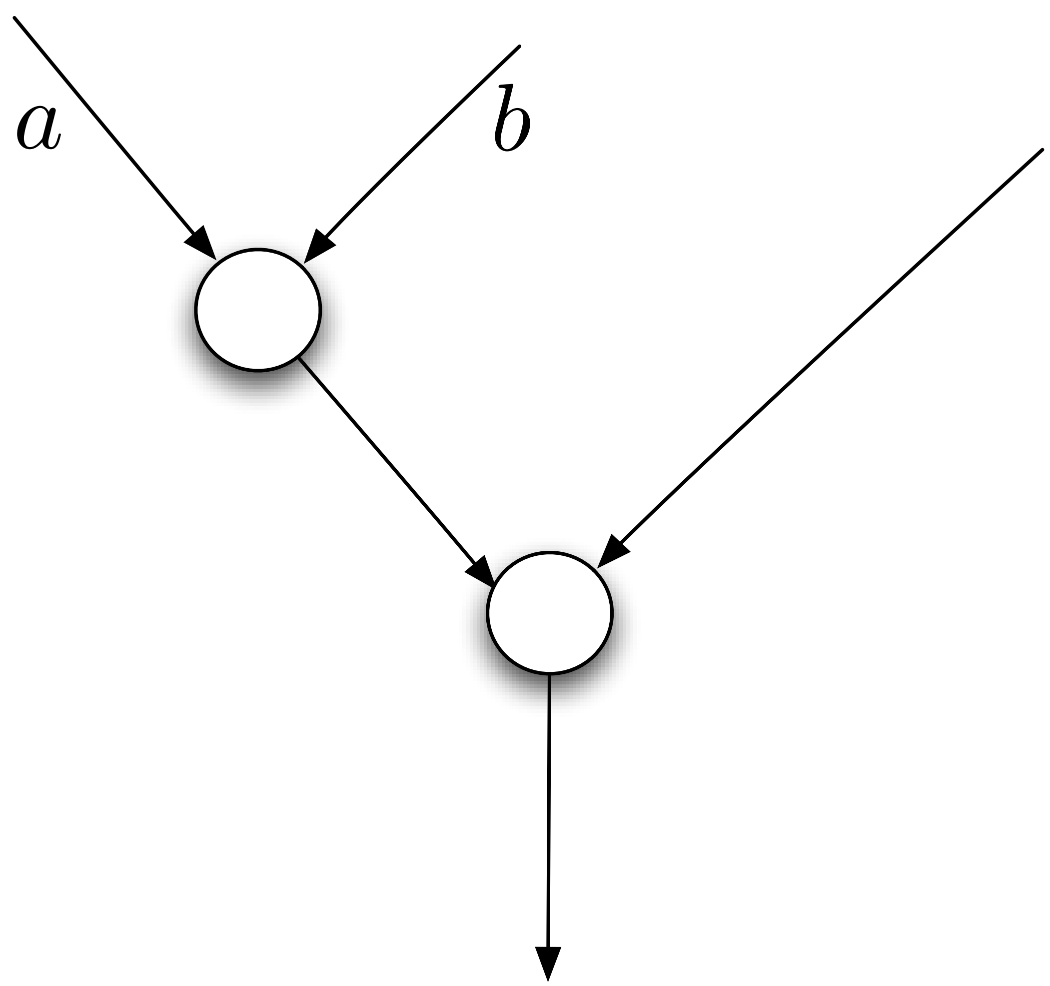

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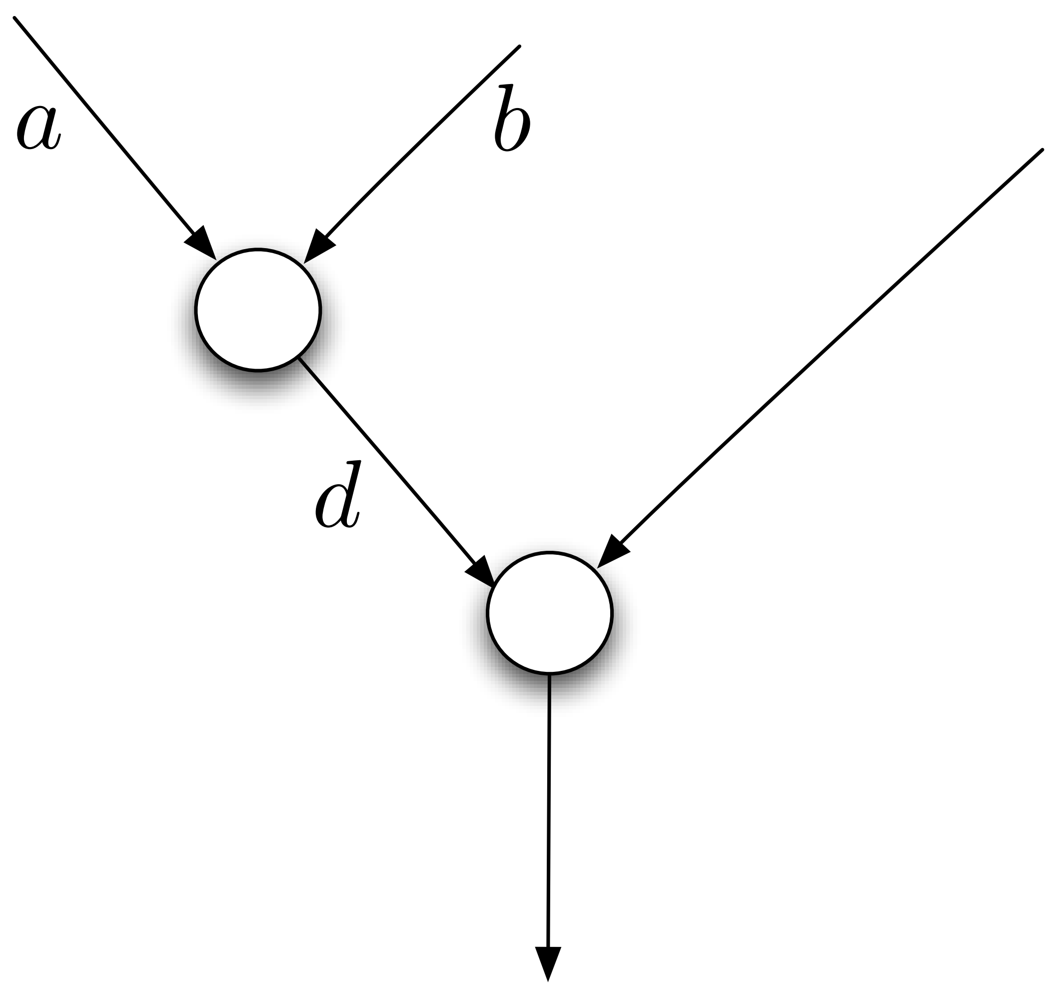

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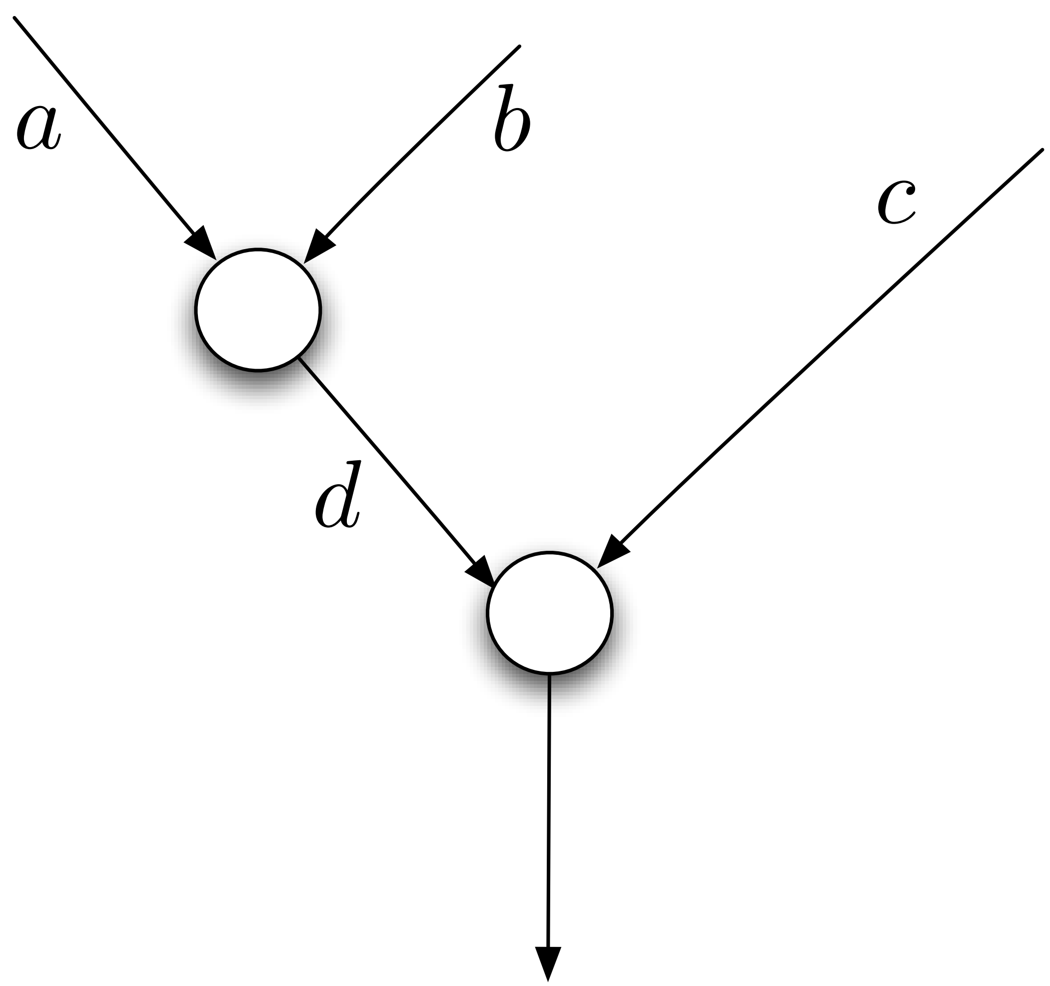

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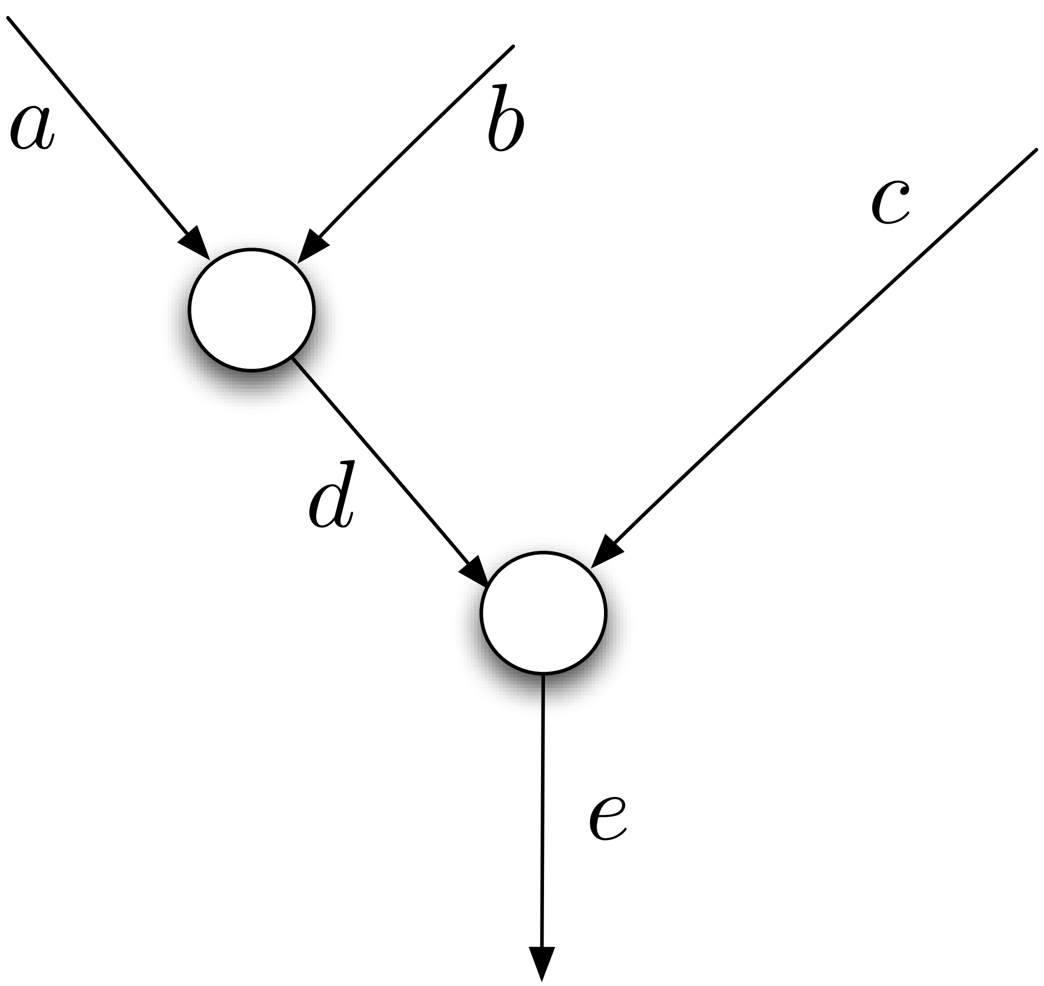

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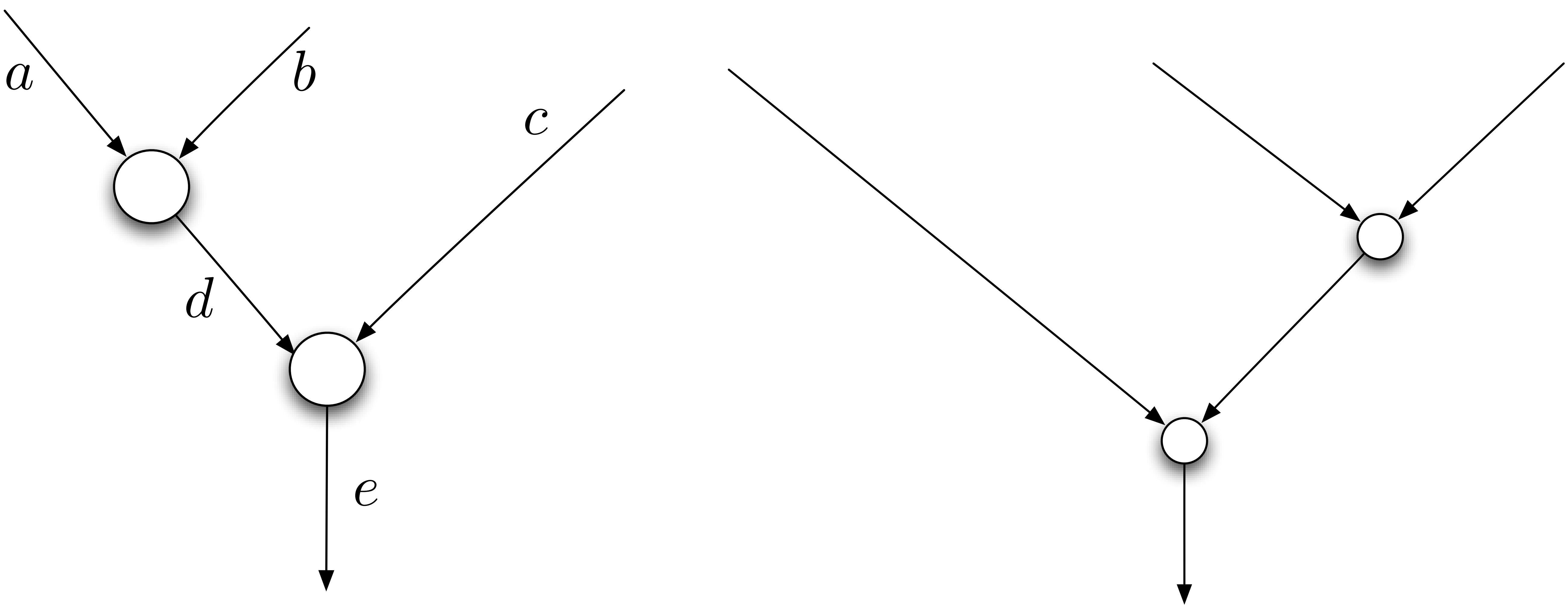

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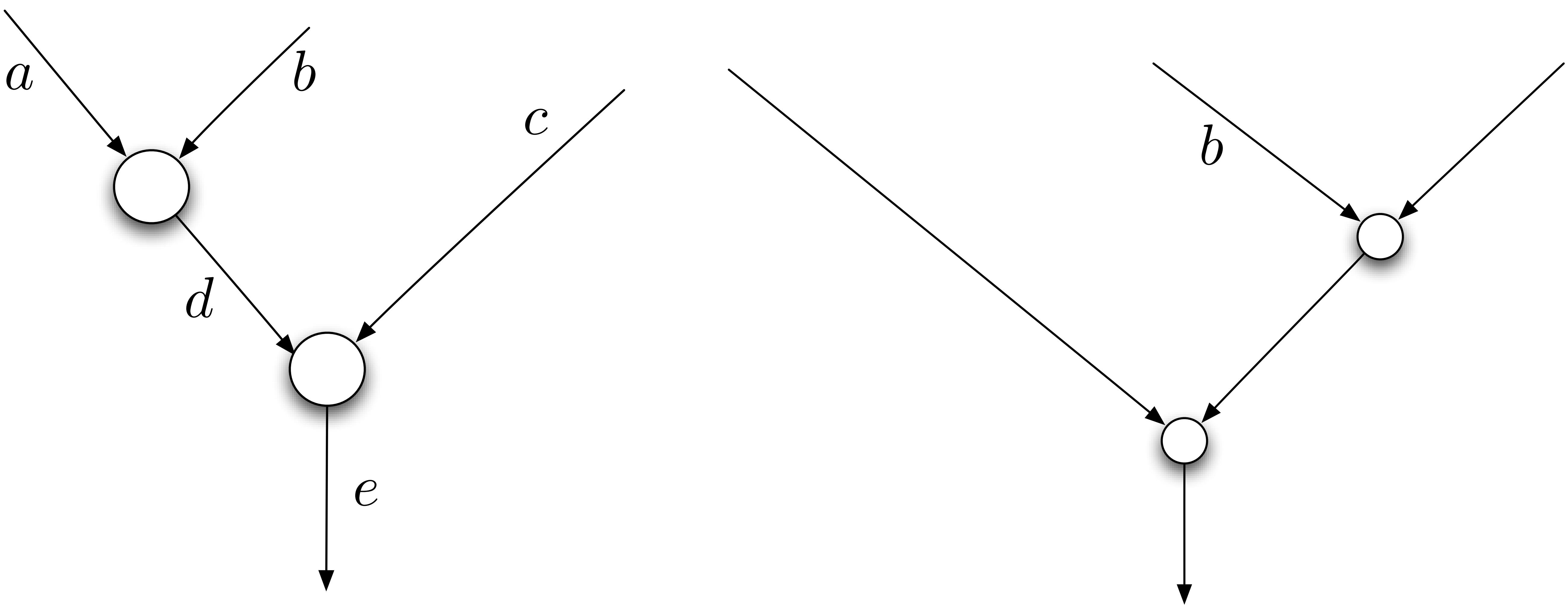

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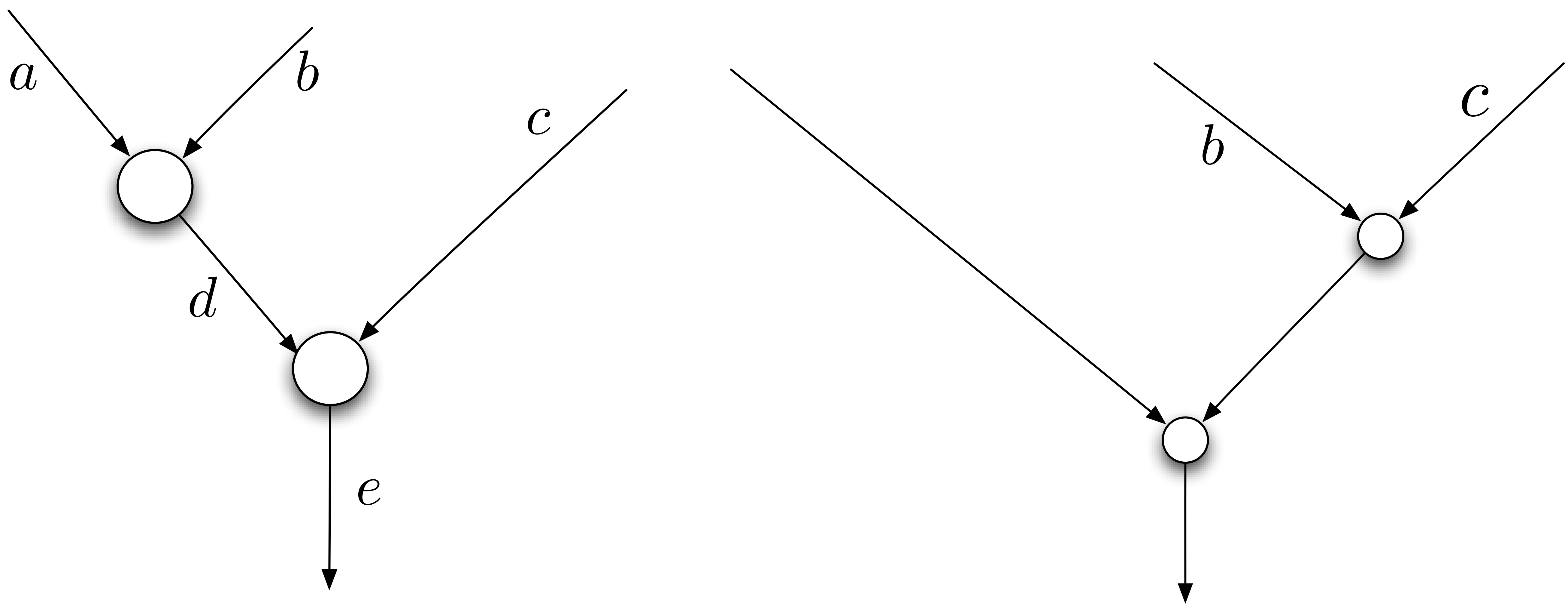

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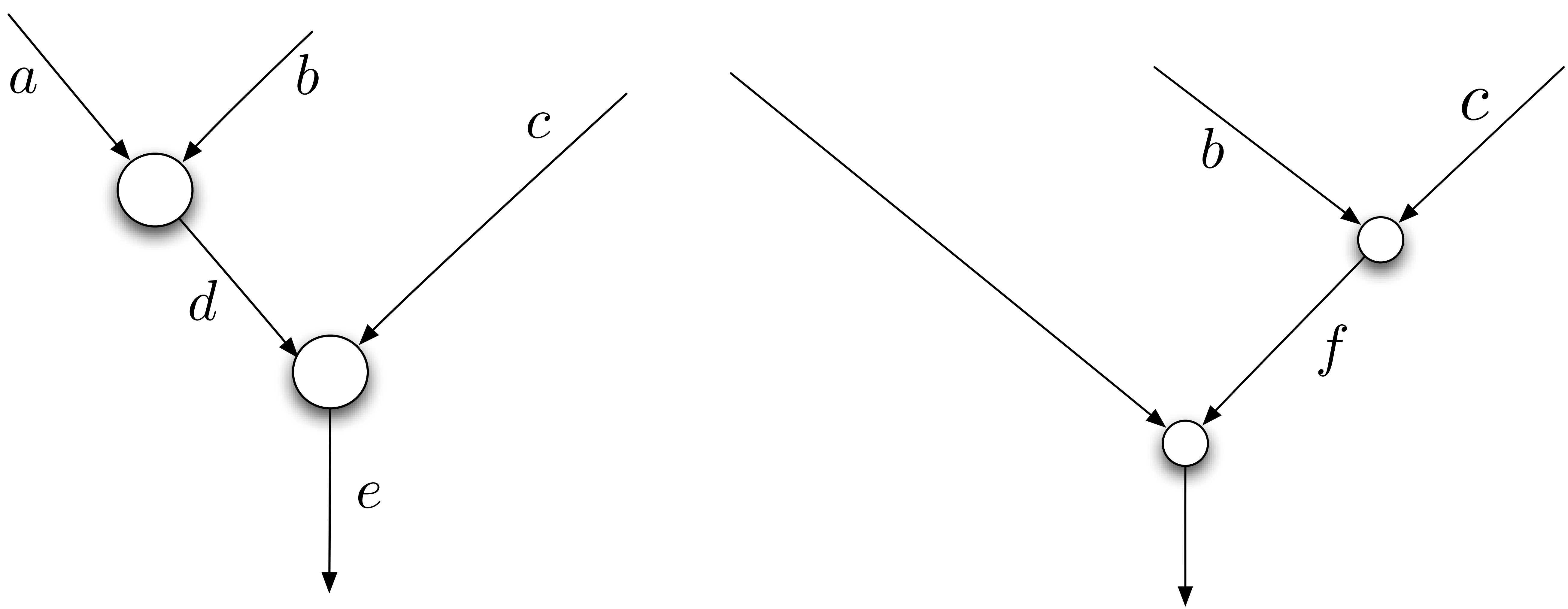

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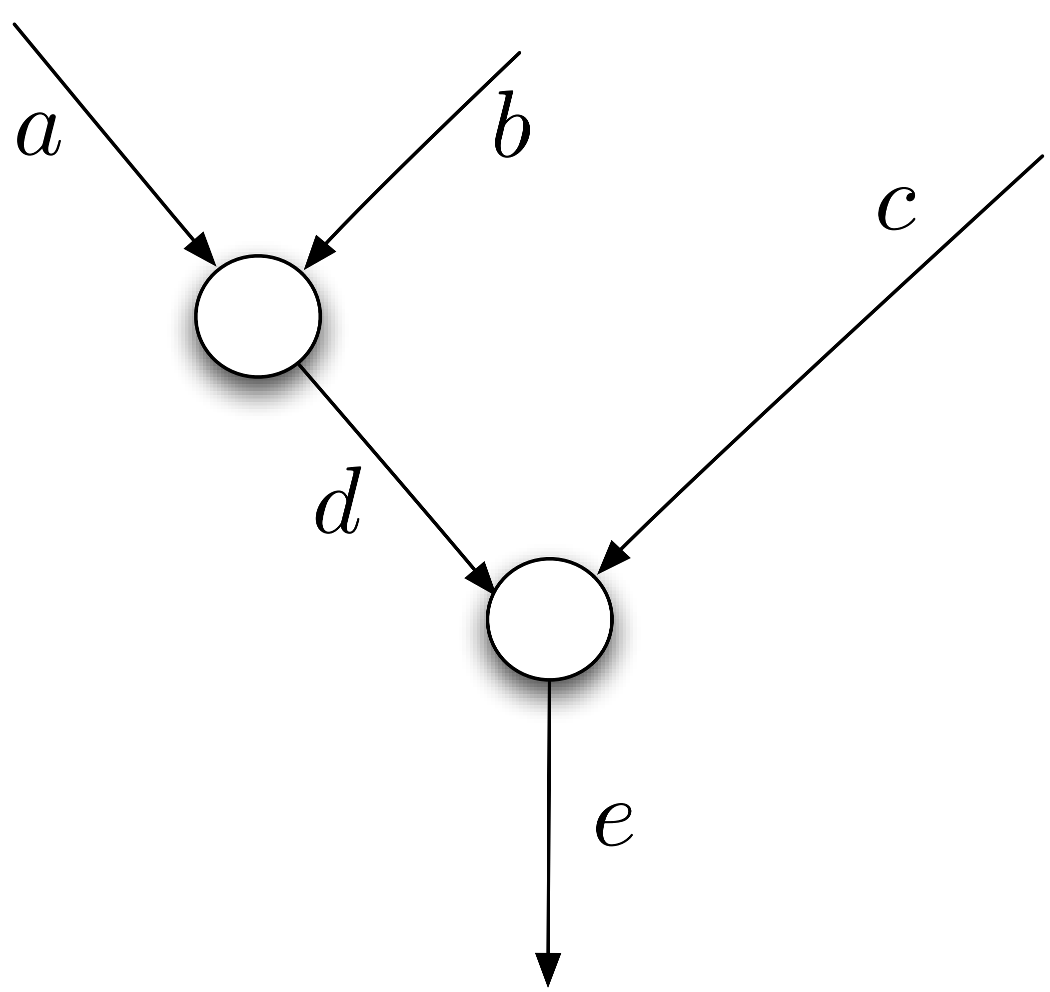

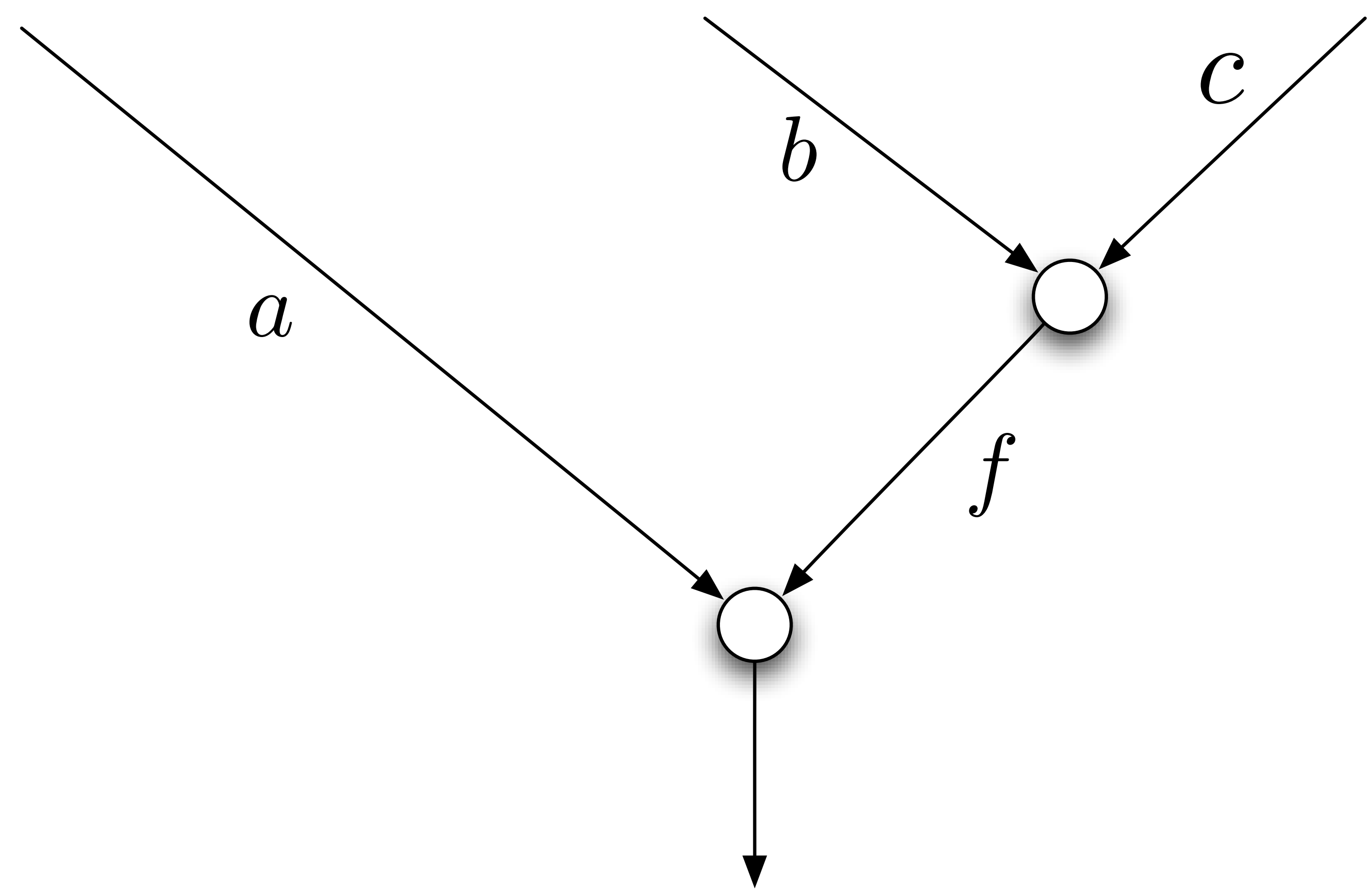

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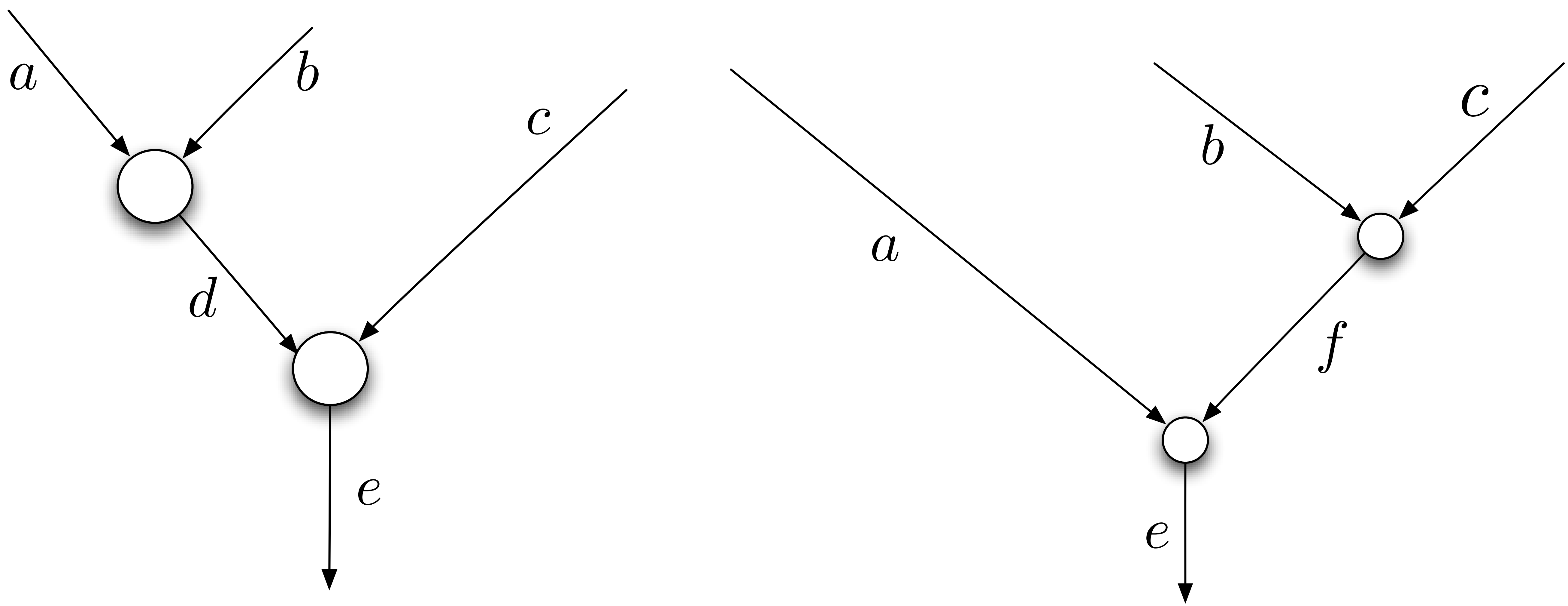

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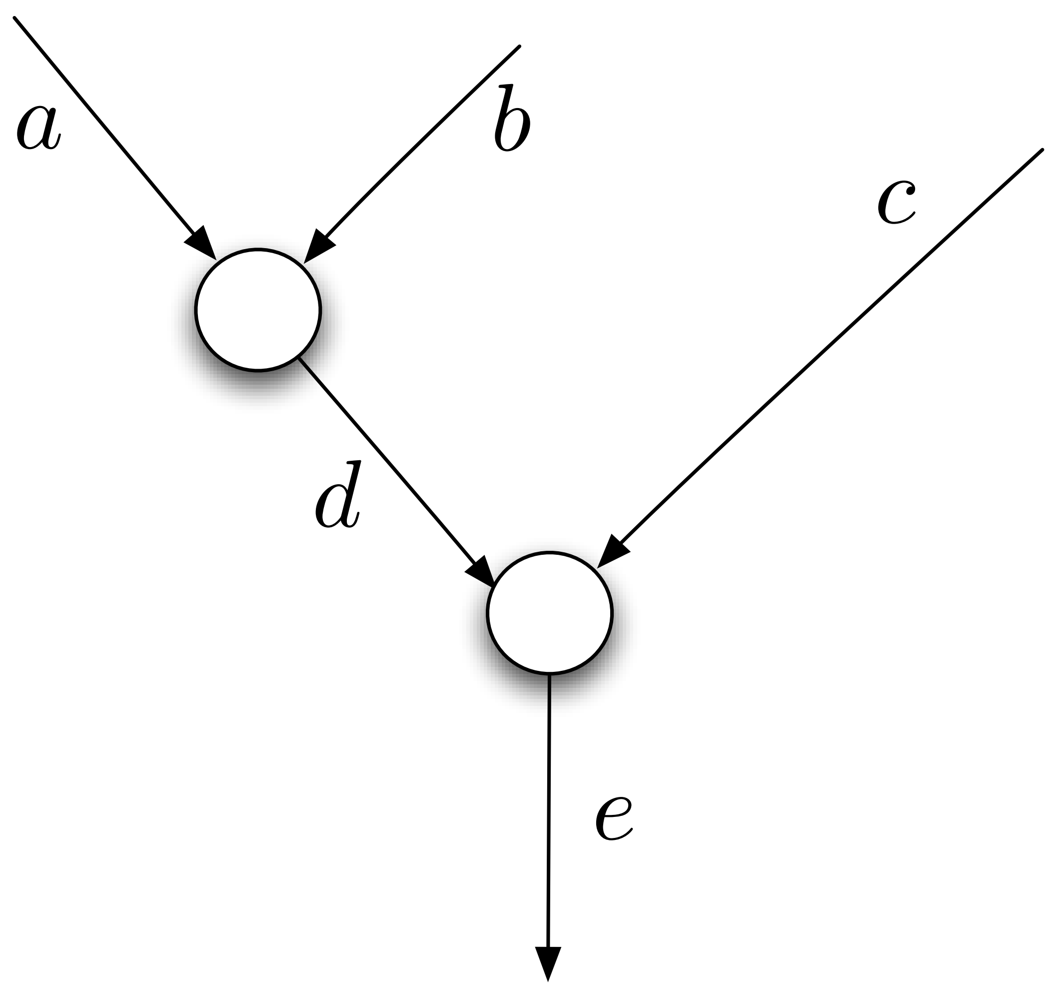

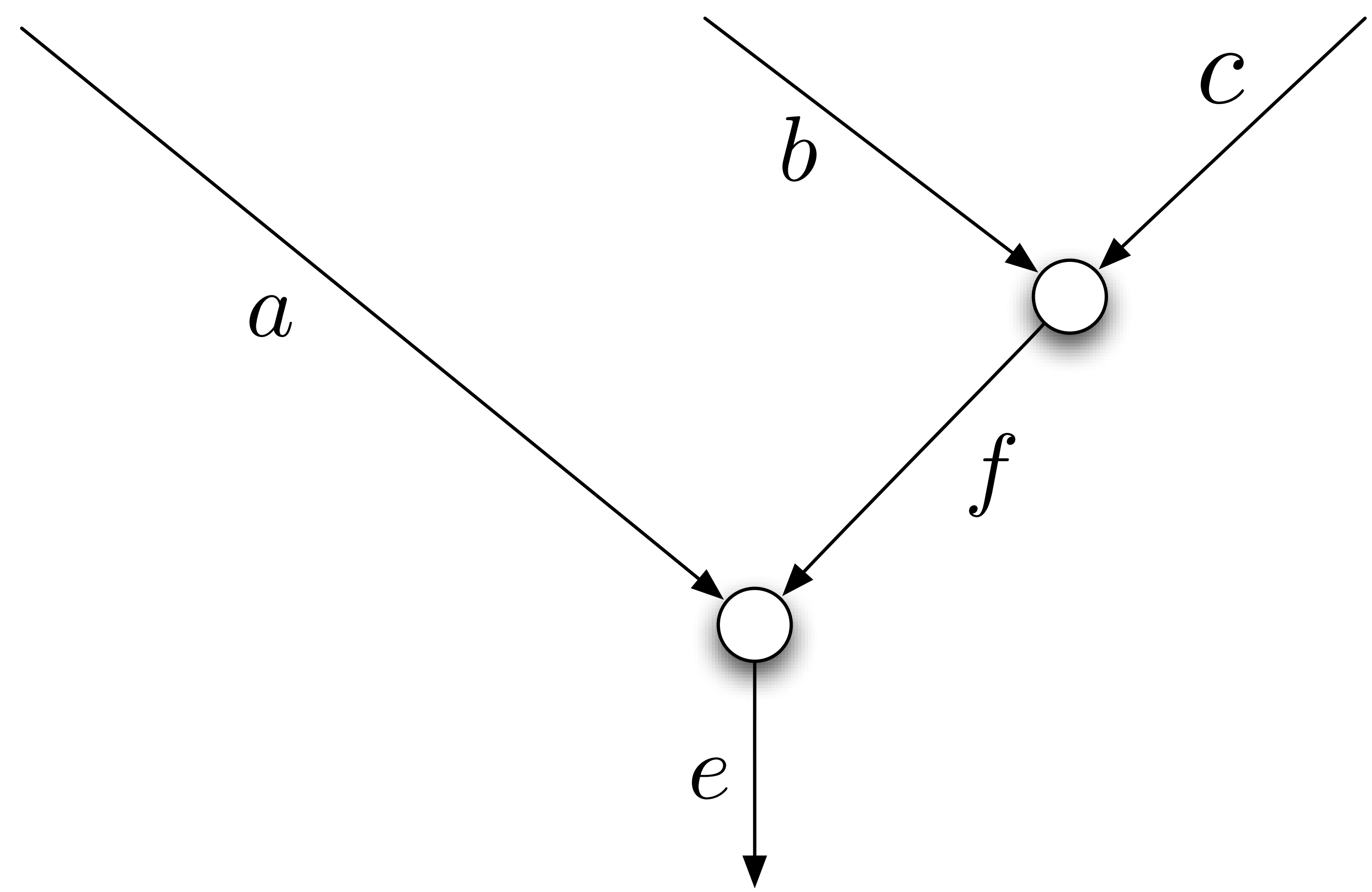

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The pentagon equation gives an equation for F that can (sometimes) be solved to obtain F explicitly.

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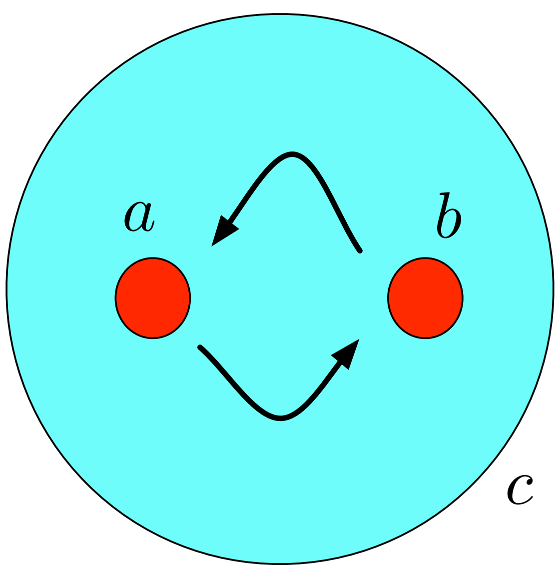

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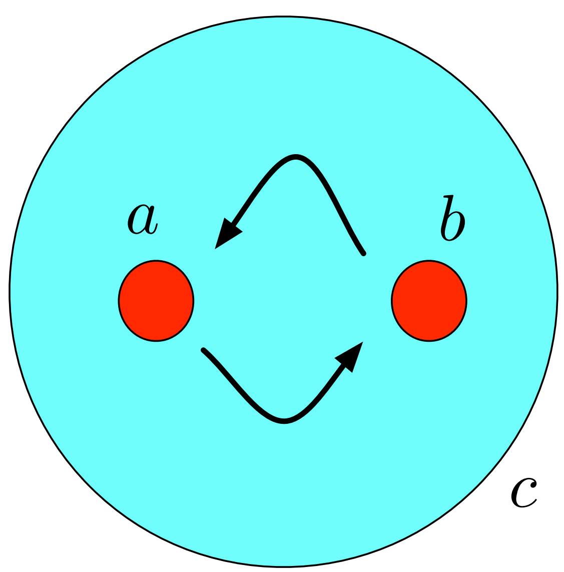

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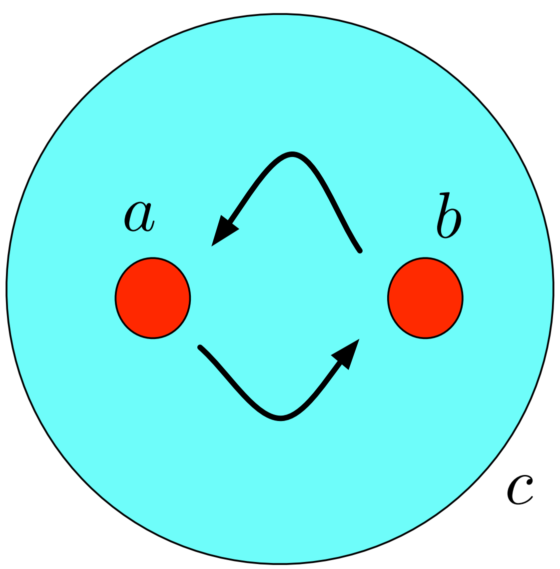

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When we exchange two anyons, the fusion spaces are iso, the isomorphism is given by a "matrix" called the R-matrix.

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Consider the following calculation:

An example: Fibonacci anyons

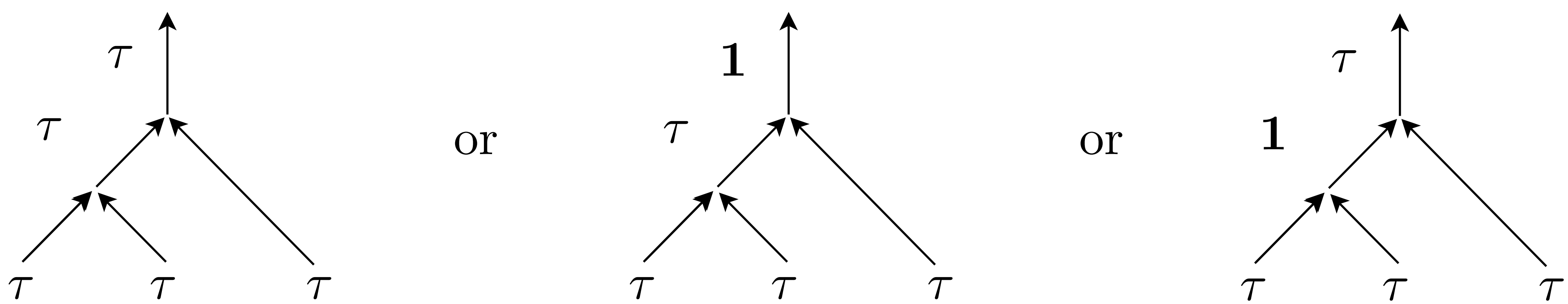
Two basic types: 1 and τ. Each anyon type is self-dual.

$$\left.\begin{array}{l} \mathbf{1} \otimes \mathbf{1} \simeq \mathbf{1} \\ \mathbf{1} \otimes \tau \simeq \tau \otimes \mathbf{1} \simeq \tau \\ \tau \otimes \tau \simeq \mathbf{1} \oplus \tau \end{array}\right\} \quad \text{Here are the fusion rules.}$$

Consider the following calculation:

$$\begin{aligned} (\tau \otimes \tau) \otimes \tau &\simeq (\mathbf{1} \oplus \tau) \otimes \tau \\ &\simeq (\mathbf{1} \otimes \tau) \oplus (\tau \otimes \tau) \\ &\simeq \tau \oplus (\mathbf{1} \oplus \tau) \\ &\simeq \mathbf{1} \oplus 2 \cdot \tau. \end{aligned}$$

In pictures



So when we fuse three τ anyons, we get a 2-dimensional space of τ-anyon states; we do *not* get two τ anyons.

We may also get a 1 anyon which represents "leakage" or "loss."

How many dimensions in the quantum state space for n anyons?

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n	dim	anyons		fusion result
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1	1	τ	$=$	τ
2	2	$\tau \otimes \tau$	$=$	$\mathbf{1} + \tau$
3	3	$\tau \otimes \tau \otimes \tau$	$=$	$\mathbf{1} + 2 \cdot \tau$
4	5	$\tau^{4\otimes}$	$=$	$2 \cdot \mathbf{1} + 3 \cdot \tau$
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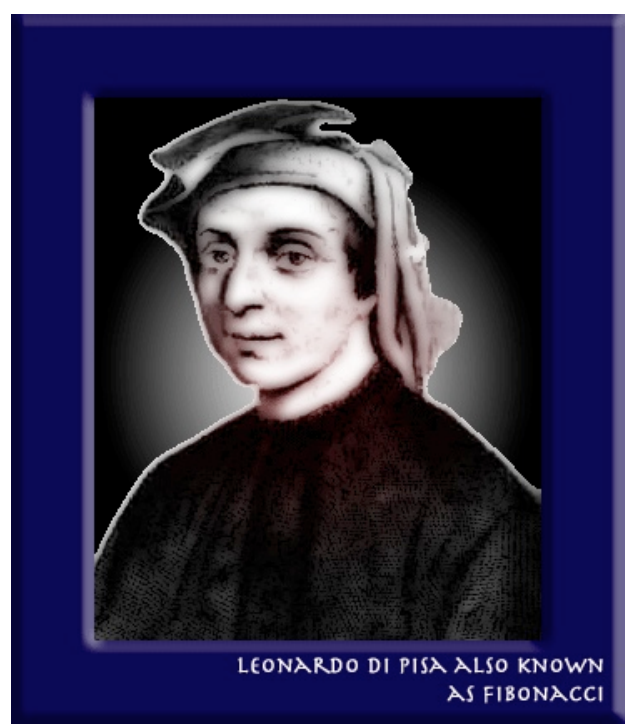

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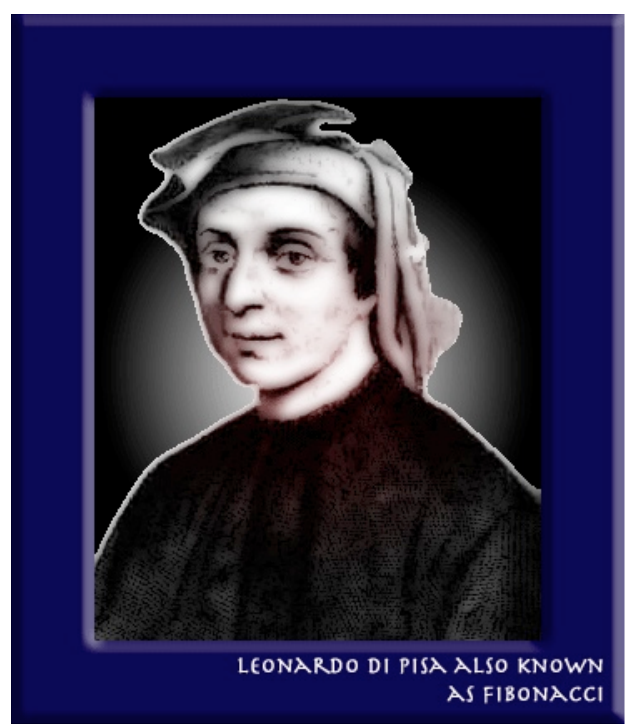

Consider

$$\begin{aligned}
\mathrm{Hom}(b, (\tau \otimes \tau) \otimes \tau) &\simeq \mathrm{Hom}(b, \mathbf{1} \oplus 2 \cdot \tau) \\
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Now, since for different charges S_1 and S_2 we have $\mathrm{Hom}(S_1,S_2)=\{0\}$ and since for any $S\in\{\mathbf{1},\tau\}$, $\mathrm{End}(S)\simeq\mathbb{C}$; if we set $S=\mathbf{1}$, then the last expression is isomorphic to $\mathbb{C}\oplus 2\cdot 0$. Conversely if $S=\tau$, then it is isomorphic to $0\oplus 2\cdot\mathbb{C}$.

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It is within the two-dimensional complex vector space (i.e. with $S = \tau$ as global charge) that we will simulate our qubit. Indeed, with this global charge, we are left with two degrees of freedom which are the two possible outputs of the second splitting.

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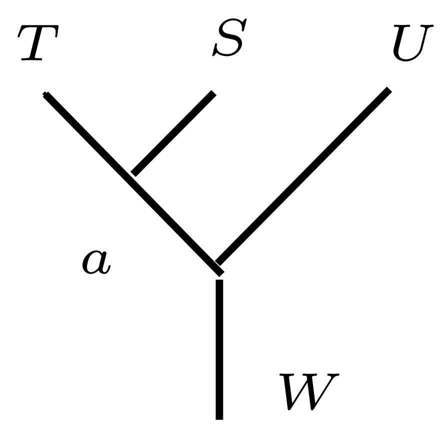
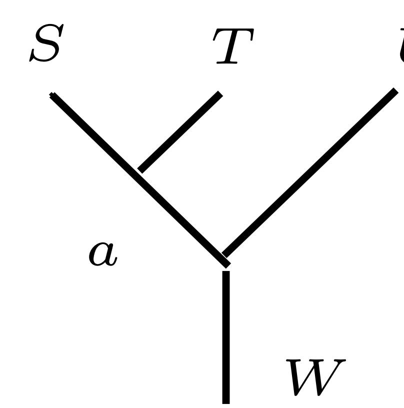
This is why we call them Fibonacci anyons:
they were invented by Lee and Yang.

The R matrix is described, using splitting diagrams:



$$= \quad [R_a^{ST}]_{aa}$$



We need to find an R-matrix explicitly to do calculations.

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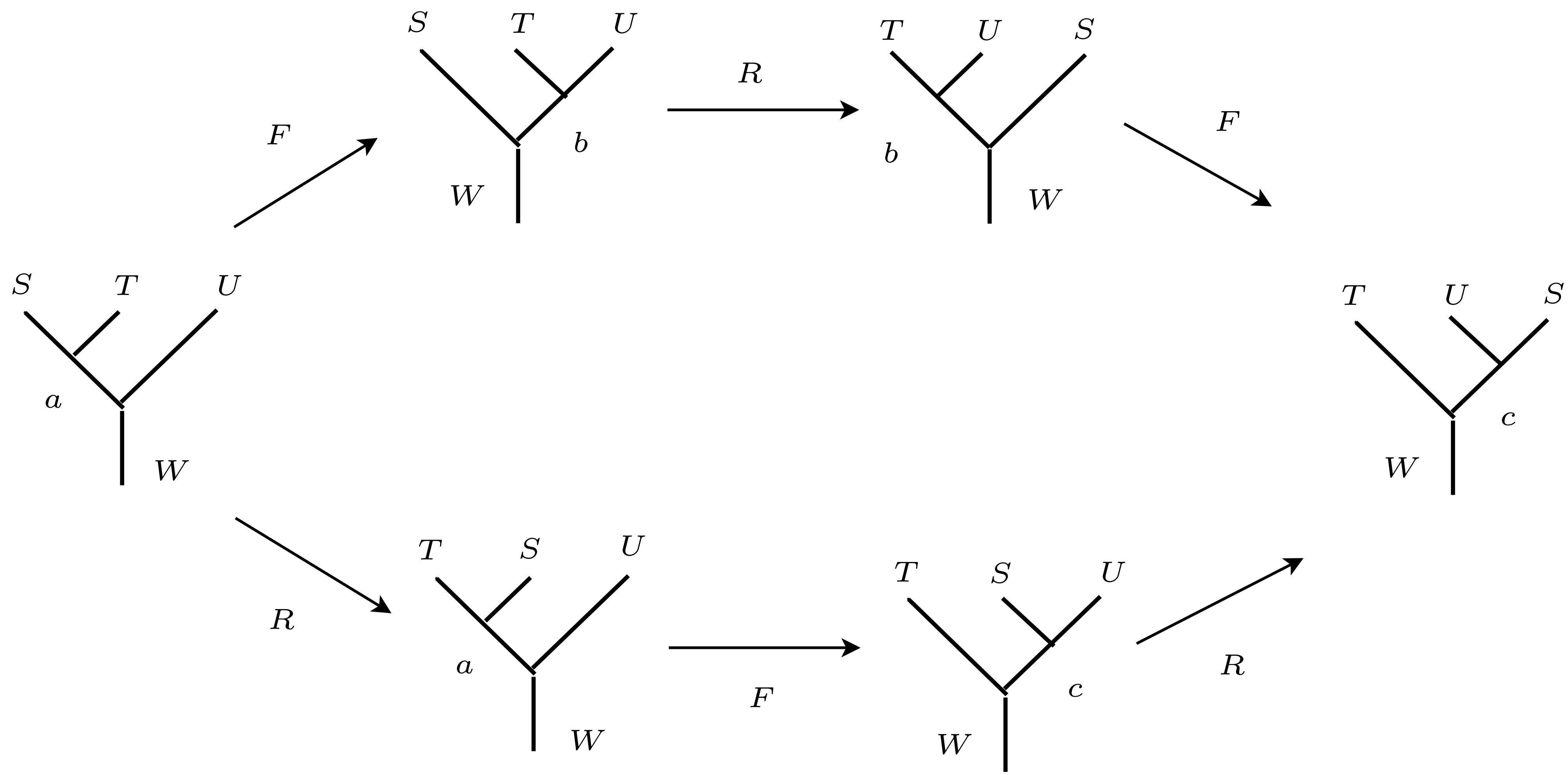

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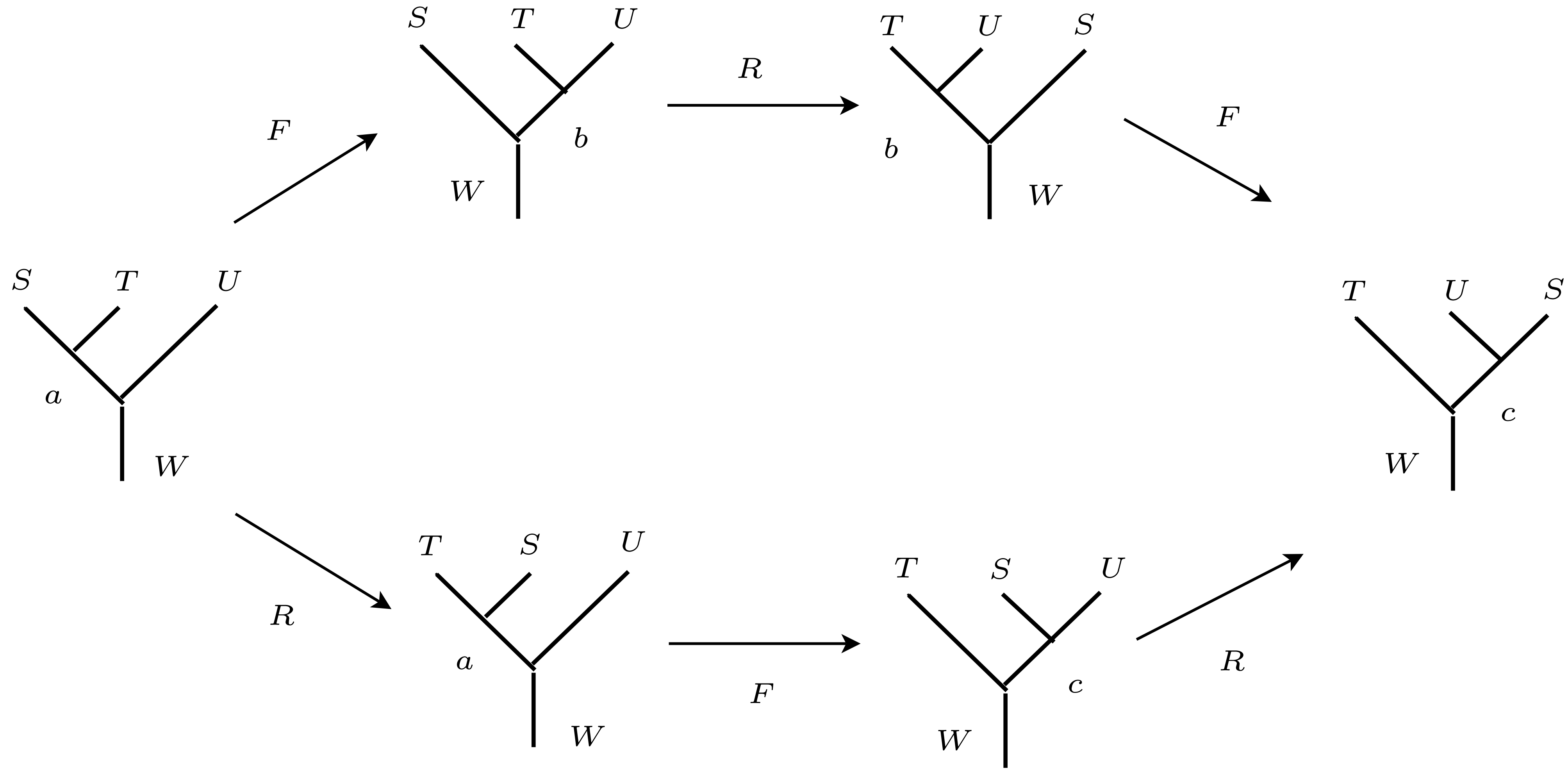

Writing it as a matrix equation yields

$$R_c^{SU}(F_W^{TSU})_{ca}R_a^{ST} = \sum_b (F_W^{TUS})_{bc}R_W^{Sb}(F_W^{STU})_{ba}.$$

For a triple of anyons with charge τ, explicit calculations of the R-matrix yields:

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$$B := F^{-1}RF$$

The basic idea to simulate quantum computation with anyons is given by the following steps:

1. Consider a compound system of anyons. We initialise a state in the splitting space by fixing the charges the subsets of anyons according to the way they will fuse. This determines the basis state in which the computation starts.
2. We braid the anyons together, it will induce a unitary action on the chosen splitting space.
3. Finally, we let the anyon fuse together and the way they fuse determines which state is measured and this constitutes the output of our computation.

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In fact it is possible to show that the Fibonacci anyons are *universal* for quantum computation.

In fact, we are lucky. The set of R- and B-matrices and their inverses (the representation of the inverse braiding) is dense in $SU(2)$ thus satisfies the condition of Sovolay-Kitaev theorem. Thus, to get an approximate universal set of gates, it just remains to construct a controlled-**NOT** gate. We will do so by following the works of Bonesteel et al.

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The idea is relatively simple: We start with two triplets of anyons. We need to intertwine a pair of quasi-particles from the first triplet – the control pair – with the target triplet *without disturbing it*; as the braid operators are dense in $SU(2)$, we will arrange such an intertwining so that its representation in $SU(2)$ is close enough to the identity. The next thing is to implement a **NOT** – actually a $i \cdot \mathbf{NOT}$ – by braiding our two anyons of the control pair with those of the target triple. Finally, we extract the control pair from the second triplet – again – without disturbing it.

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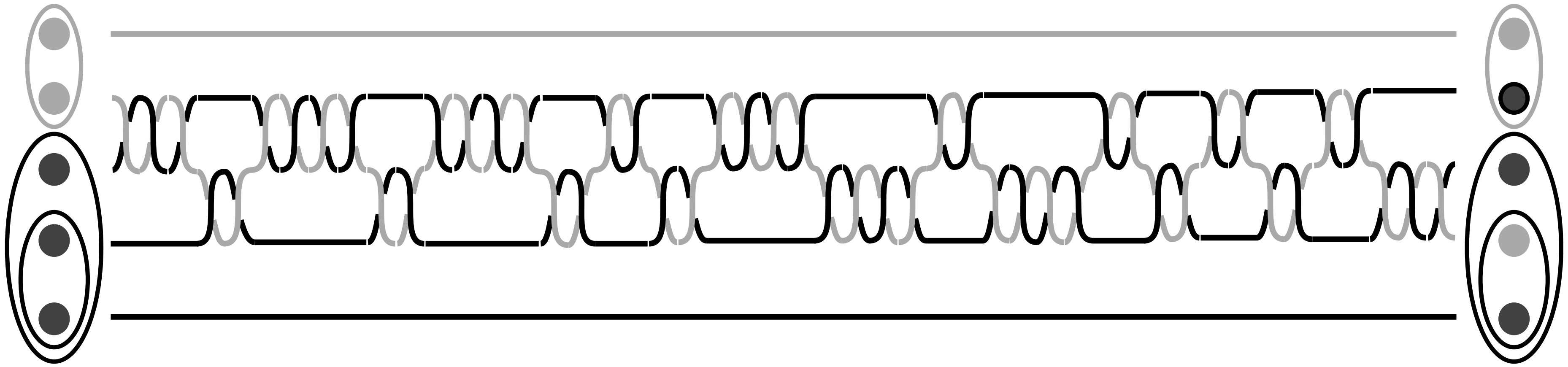
The key point is the following: a braiding involving the trivial charge $\mathbf{1}$ with an anyon of arbitrary charge does not change anything. Thus, when measuring the control pair, the $i \cdot \mathbf{NOT}$ will occur if and only if the two anyons from the control pair fuse as an anyon of charge τ; otherwise the control pair only induces a trivial change on the system.

Consider the following braiding:

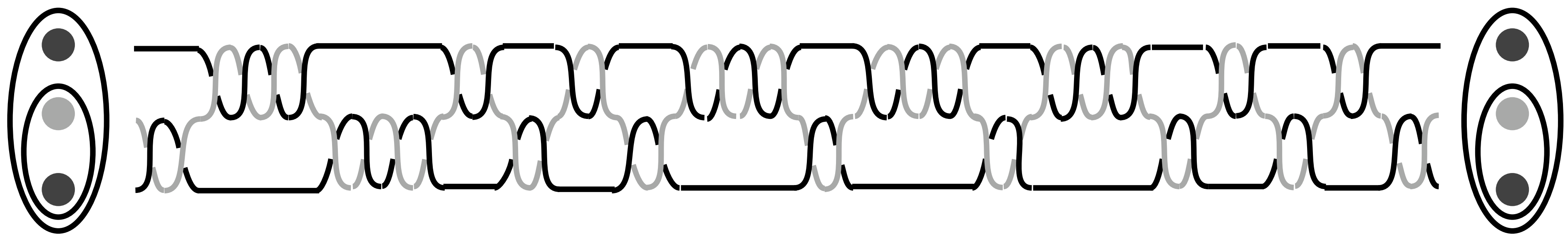


As an action on the splitting space of the three anyons involved, this is, in the same order as depicted in the picture:

$$B^3 R^{-2} B^{-4} R^2 B^4 R^2 B^{-2} R^{-2} B^{-4} R^{-4} B^{-2} R^4 B^2 R^{-2} B^2 R^2 B^{-2} R^3 \sim \left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \hline 0 & 0 & 1 \end{array}\right)$$

This tells us how the given braid inserts an anyon into a given triplet without disturbing it.

Now, we implement an $i \cdot \mathbf{NOT}$ as the following braid:



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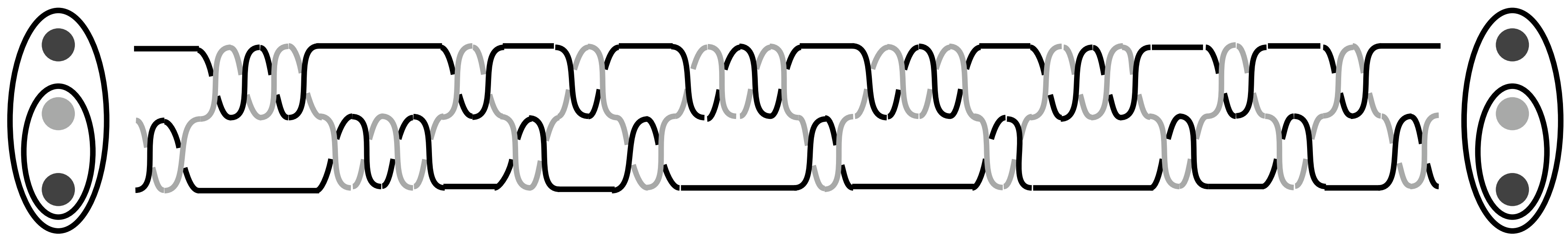

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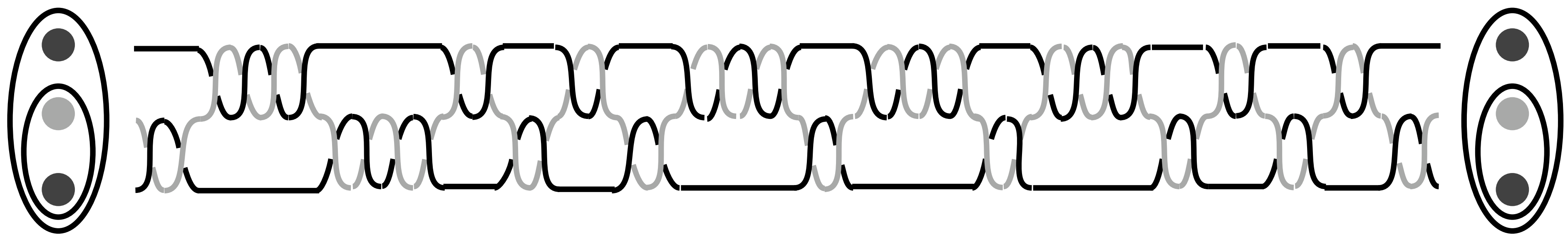

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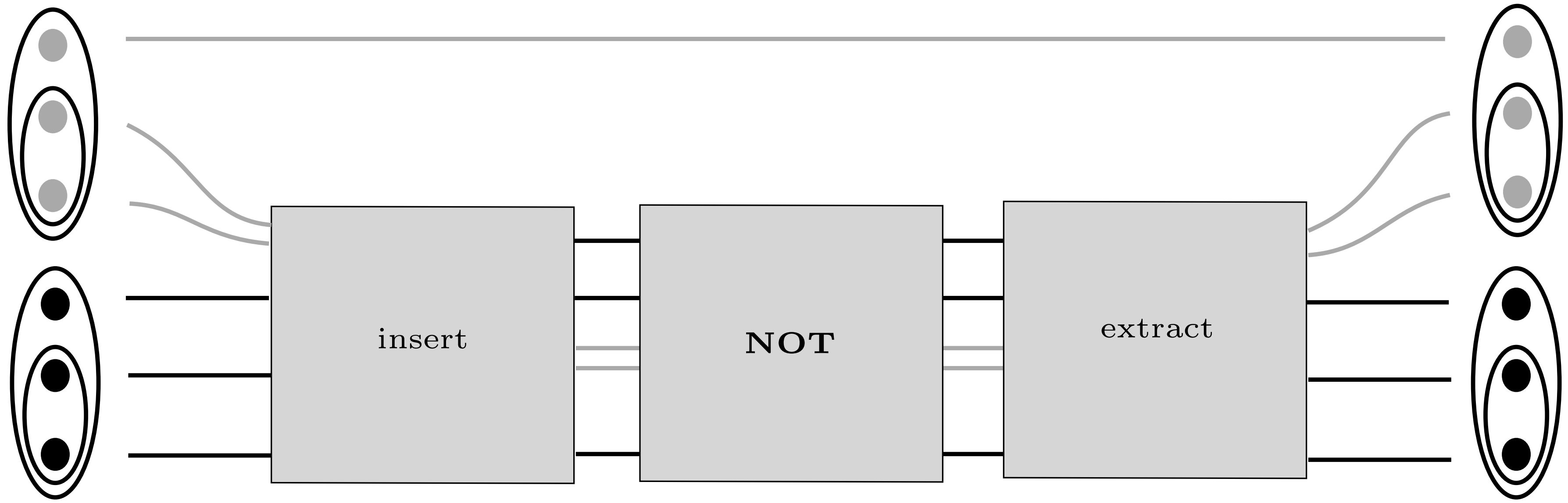
$$R^{-2}B^{-4}R^{4}B^{-2}R^{2}B^{2}R^{-2}B^{4}R^{-2}B^{4}R^{2}B^{-4}R^{2}B^{-2}R^{2}B^{-2}R^{-2} \sim \left(\begin{array}{cc|c} 0 & i & 0 \\ i & 0 & 0 \\ \hline 0 & 0 & 1 \end{array}\right)$$

Finally, the $i \cdot \mathbf{CNOT}$ gate acting on two topological qubits is realised as follows:



Note that instead of inserting 1 anyon, we insert a couple that will be used as a test couple.

We claim that this implements a $\mathbf{CNOT}$. Indeed, the test couple can fuse in two ways. If it fuses as $\mathbf{1}$, then nothing happens as $\mathbf{1}$ is the trivial charge. If it fuses as τ, then we effectively apply the $i \cdot \mathbf{NOT}$ gate.

In conclusion, using a set of anyons we can:

- Simulate a qubit,
- Approximate any unitary tranformation on a set of qubits and
- Measure the system using fusion,

from which we can simulate quantum computation with anyons.

Category Theory is a Pain!

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We need to understand the category of representations.

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Representations of quantum groups!

Question: What fusion rules will give one universal quantum computation?

The Yang-Baxter Equation

Given a vector space V and $c \in End(V \otimes V)$:

$$(c \otimes id_V)(id_V \otimes c)(c \otimes id_V) = (id_v \otimes c)(c \otimes id_V)(id_V \otimes c)$$

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Categories of representations of *quasi-triangular* Hopf algebras.

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From such representations we get knot invariants by taking traces.

What is to be done?

What does it take to have universal quantum computation? We know that something as simple as the Fibonacci anyon gives UCC.

The knot invariants defined by the MTC describing the Fibonacci anyon cannot be that easy to compute. What are they?

What is the topological "signal" or algebraic structure needed to guarantee that one has UCC?