Proof Nets as Formal Feynman Diagrams

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Outline

- Proof nets as an algebra under cut (LRA)
- Feynman diagrams
- The phi-calculus
- Does it mean anything?

Linear Logic

Resource-sensitive logic

with a constructive reading and marvelous symmetries.

$$\frac{}{\vdash A, A^{\perp}}\ \mathrm{id} \qquad \frac{\vdash A, \Gamma \quad \vdash A^{\perp}, \Delta}{\vdash \Gamma, \Delta}\ \mathrm{cut}$$

$$\frac{\vdash A, \Gamma \quad \vdash B, \Delta}{\vdash A \otimes B, \Gamma, \Delta}\ \otimes \qquad \frac{\vdash A, \Gamma \quad \vdash B, \Gamma}{\vdash A \,\&\, B, \Gamma}\ \&$$

$$\frac{A, B, \Gamma}{A \parr B, \Gamma}\ (\parr) \qquad \frac{\vdash A, \Gamma}{\vdash A \oplus B, \Gamma}\ \oplus_1 \qquad \frac{\vdash B, \Gamma}{\vdash A \oplus B, \Gamma}\ \oplus_2$$

Exponentials

$$\frac{\vdash A, ?\Gamma}{\vdash !A, ?\Gamma}\,! \qquad \frac{\vdash A, \Gamma}{\vdash ?A, \Gamma}\,?\mathrm{d} \qquad \frac{\vdash ?A, ?A, \Gamma}{\vdash ?A, \Gamma}\,?\mathrm{c} \qquad \frac{\vdash \Gamma}{\vdash ?A, \Gamma}\,?\mathrm{w}$$

Renewable resources modelled as modalities.

Why are they called exponentials?

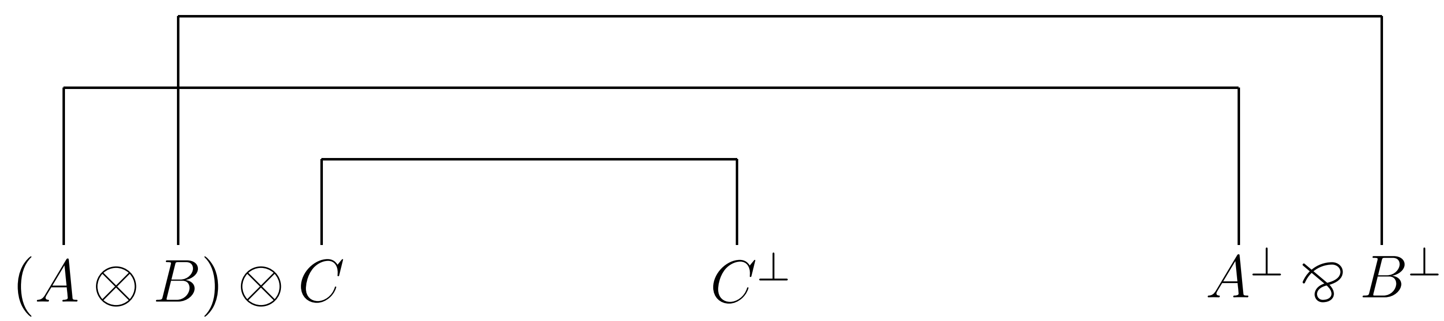
Proof Nets

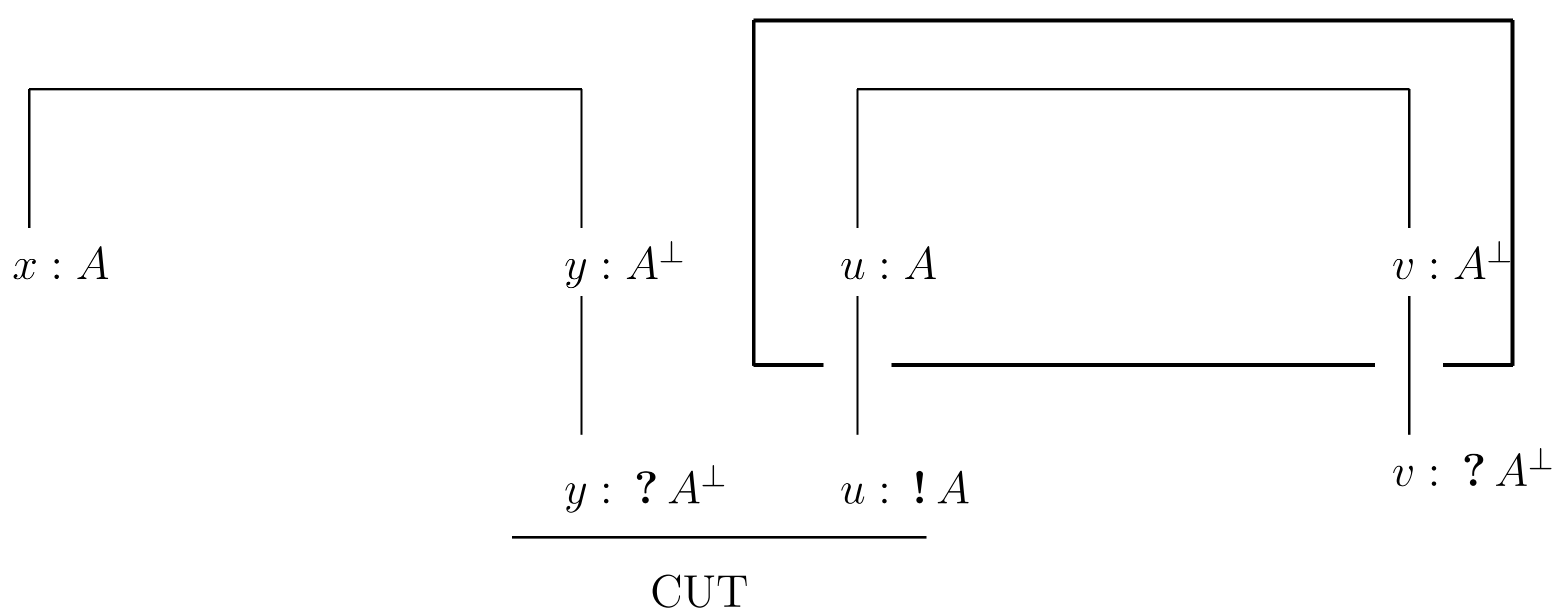
$$\frac{}{\vdash A, A^\perp}\,\mathrm{id}$$

$$\frac{\vdash A, \Gamma \qquad \vdash A^\perp, \Delta}{\vdash \Gamma, \Delta}\,\mathrm{cut}$$

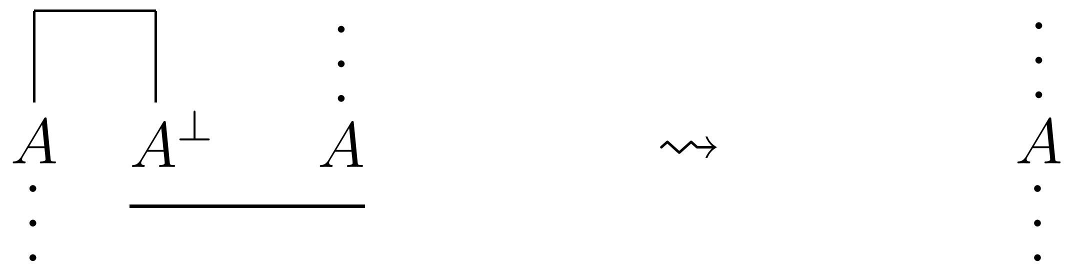
$$\frac{A \qquad B}{A \otimes B} \qquad\qquad \frac{A \qquad B}{A \parr B}$$

More Proof Nets

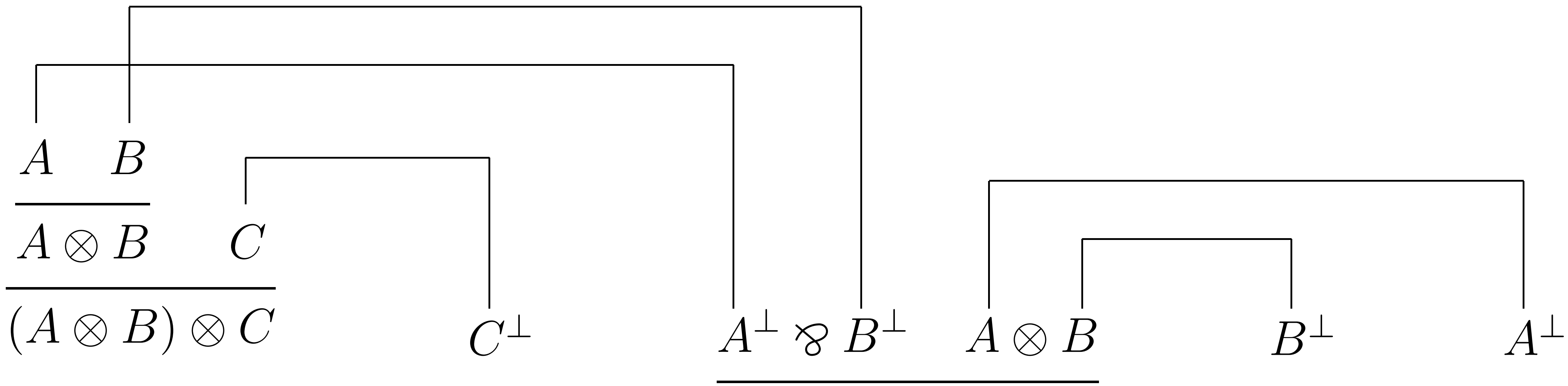




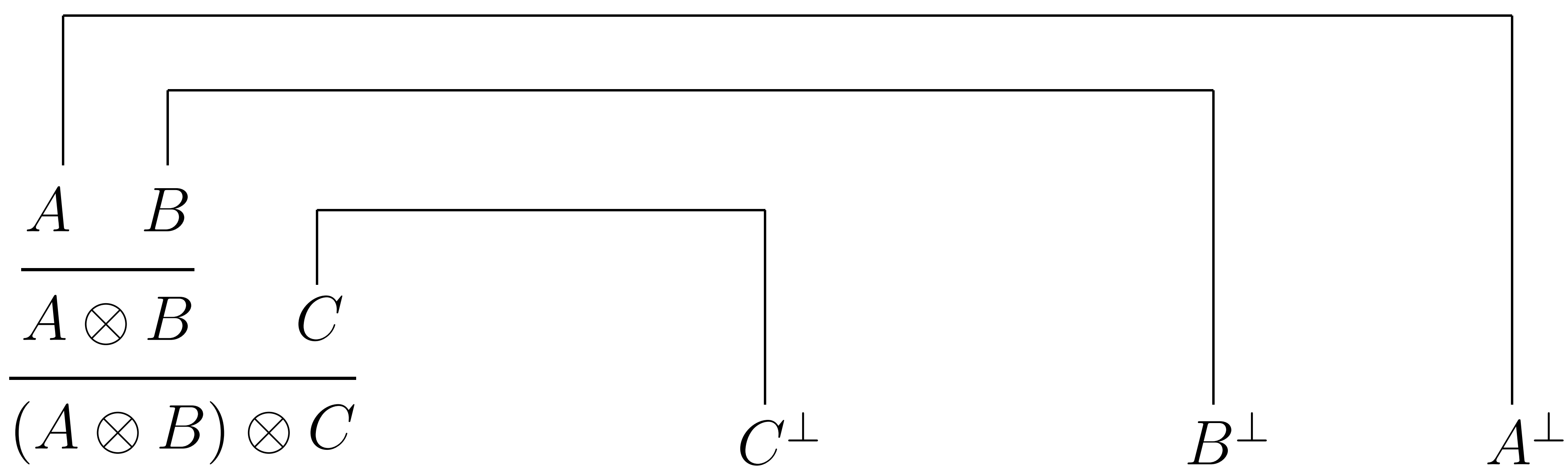
Cut Elimination



$$\dfrac{\dfrac{\overset{\vdots}{A} \qquad \overset{\vdots}{B}}{A \otimes B} \qquad \dfrac{\overset{\vdots}{A^{\perp}} \qquad \overset{\vdots}{B^{\perp}}}{A^{\perp} \parr B^{\perp}}}{} \quad \rightsquigarrow \quad \dfrac{\overset{\vdots}{A} \qquad \overset{\vdots}{A^{\perp}}}{} \quad \dfrac{\overset{\vdots}{B} \qquad \overset{\vdots}{B^{\perp}}}{}$$



reduces (in three steps) to



Proof Rule	**Operation**	**Constraint**	**Sort**
Axiom	$I_{x,y}$		$\{x, y\}$
Cut	$P \cdot_x Q$	$\mathtt{FN}(P) \cap \mathtt{FN}(Q) = \{x\}$	$\mathtt{FN}(P) \cup \mathtt{FN}(Q) \setminus \{x\}$
Unit	U_x		$\{x\}$
Perp	$\perp_x (P)$	$x \notin \mathtt{FN}(P)$	$\mathtt{FN}(P) \cup \{x\}$
Times	$\otimes_z^{x,y}(P, Q)$	$x \in \mathtt{FN}(P), y \in \mathtt{FN}(Q)$ $\mathtt{FN}(P) \cap \mathtt{FN}(Q) = \emptyset$ $z \notin \mathtt{FN}(P) \cup \mathtt{FN}(Q)$	$\mathtt{FN}(P) \cup \mathtt{FN}(Q) \setminus \{x, y\} \cup \{z\}$
Par	$\invamp_z^{x,y}(P)$	$x, y \in \mathtt{FN}(P)$ $x \neq y$ $z \notin \mathtt{FN}(P)$	$\mathtt{FN}(P) \setminus \{x, y\} \cup \{z\}$
Plus Left	$L_z^x(P)$	$x \in \mathtt{FN}(P), z \notin \mathtt{FN}(P)$	$\mathtt{FN}(P) \setminus \{x\} \cup \{z\}$
Plus Right	$R_z^x(P)$	$x \in \mathtt{FN}(P), z \notin \mathtt{FN}(P)$	$\mathtt{FN}(P) \setminus \{x\} \cup \{z\}$
With	$\&_z^{x,y}(P)$	$x \in \mathtt{FN}(P), y \in \mathtt{FN}(Q)$ $\mathtt{FN}(P) \setminus \{x\} = \mathtt{FN}(Q) \setminus \{y\}$	$\mathtt{FN}(P) \setminus \{x\} \cup \{z\}$
Dereliction	$D_z^x(P)$	$x \in \mathtt{FN}(P), z \notin \mathtt{FN}(P)$	$\mathtt{FN}(P) \setminus \{x\} \cup \{z\}$
Weakening	$W_z(P)$	$z \notin \mathtt{FN}(P)$	$\mathtt{FN}(P) \cup \{z\}$
Contraction	$C_z^{x,y}(P)$	$x, y \in \mathtt{FN}(P)$ $x \neq y$ $z \notin \mathtt{FN}(P)$	$\mathtt{FN}(P) \setminus \{x, y\} \cup \{z\}$
Of course	$\mathbf{!}_z^x(P)$	$x \in \mathtt{FN}(P), z \notin \mathtt{FN}(P)$ $\forall u \in FN(P) \setminus \{x\}.u$ is introduced by dereliction, weakening or contraction.	$\mathtt{FN}(P) \setminus \{x\} \cup \{z\}$

Identity Group	$\dfrac{}{I_{x,y} \vdash x : A^{\perp}, y : A}$	$\dfrac{P \vdash \Gamma', x : A \qquad Q \vdash \Gamma'', x : A^{\perp}}{P \cdot_x Q \vdash \Gamma', \Gamma''}$
Multiplicative Units	$\dfrac{}{U_x \vdash x : I}$	$\dfrac{P \vdash \Gamma}{\perp_x \vdash x : \perp, \Gamma}$
Multiplicatives	$\dfrac{P \vdash \Gamma', x : A \qquad Q \vdash \Gamma'', y : B}{\otimes_z^{x,y}(P, Q) \vdash \Gamma', \Gamma'', z : A \otimes B}$	$\dfrac{P \vdash \Gamma', x : A, y : B}{⅋_z^{x,y}(P) \vdash \Gamma', z : A ⅋ B}$
Additives	$\dfrac{P \vdash \Gamma, x : A}{L_z^x(P) \vdash \Gamma, z : A \oplus B}$ $\dfrac{P \vdash \Gamma, x : B}{R_z^x(P) \vdash \Gamma, z : A \oplus B}$	$\dfrac{P \vdash \Gamma, x : A \qquad Q \vdash \Gamma, y : B}{\&(P, Q) \vdash \Gamma, z : A \& B}$
Exponentials	$\dfrac{P \vdash \Gamma, x : A}{D_z^x(P) \vdash \Gamma, z : ?A}$ $\dfrac{P \vdash \Gamma}{W_z(P) \vdash \Gamma, z : ?A}$ $\dfrac{P \vdash \Gamma, x : ?A, y : ?A}{C_z^{x,y}(P) \vdash \Gamma, z : ?A}$	$\dfrac{P \vdash ?\Gamma, x : A}{!_z^x(P) \vdash ?\Gamma, z : !A}$

Reduction Rules

(R1) $P \cdot_x I_{x,y} \rightarrow P[y/x]$.

(R3) $⅋_z^{x,y}(P) \cdot_z \otimes_z^{x,y}(Q, R) \rightarrow P \cdot_x Q \cdot_y R$.

(R4) $L_z^x(P) \cdot_z \&_z^{x,y}(Q, R) \rightarrow P \cdot_x Q$.

(R5) $R_z^x(P) \cdot_z \&_z^{x,y}(Q, R) \rightarrow P \cdot_x R$.

(R6) $D_z^x(P) \cdot_z \,!_z^x(Q) \rightarrow P \cdot_x Q$.

(R7) $W_z(P) \cdot_z \,!_z^x(Q) \rightarrow W_{\boldsymbol{x}}(P)$, where $\mathtt{FN}(Q) \setminus \{x\} = \boldsymbol{x}$.

(R8) $C_z^{z',z''}(P) \cdot_z \,!_z^x(Q) \rightarrow C_{\boldsymbol{x}}^{\boldsymbol{x'},\boldsymbol{x''}}(P \cdot_{z'} \,!_{z'}^x(Q[\boldsymbol{x'}/\boldsymbol{x}]) \cdot_{z''} \,!_{z''}^x(Q[\boldsymbol{x''}/\boldsymbol{x}]))$,
where $\mathtt{FN}(Q) \setminus \{x\} = \boldsymbol{x}$.

(R9) $!_z^x(P) \cdot_u \,!_u^v(Q) \rightarrow \,!_z^x(P \cdot_u \,!_u^v(Q))$, if $u \in \mathtt{FN}(P)$.

These reductions can be applied in any context.

$$\frac{P \rightarrow Q}{C[P] \rightarrow C[Q]}$$

and are performed modulo structural congruence.

$$\frac{P' \equiv P \quad P \rightarrow Q \quad Q' \equiv Q}{P \rightarrow Q}$$

Feynman's Brilliant Intuition

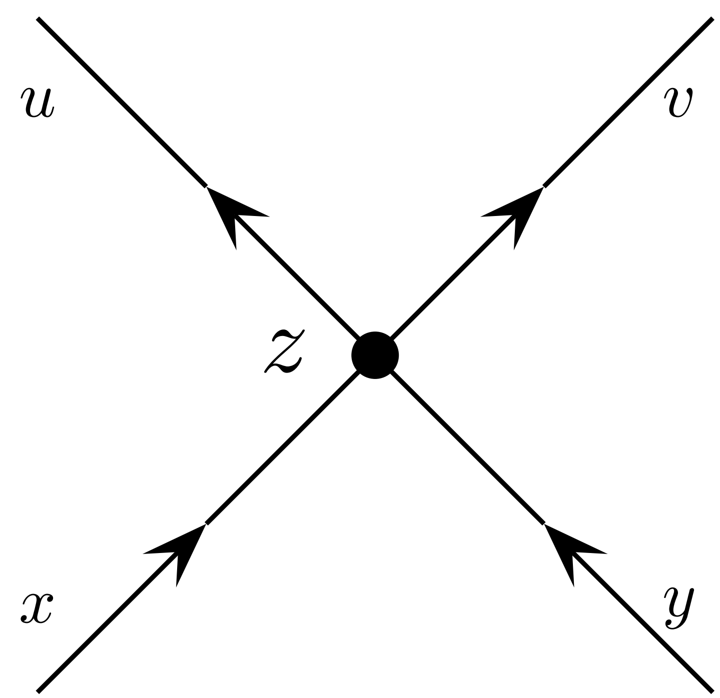
Think in terms of particles and their trajectories.

Particles coast freely until they interact. For a given type of theory the interaction is always the same.

Coasting particles are represented by straight lines; interactions by vertices.

The pictures define integrals that express the probability (amplitude) for the process shown.

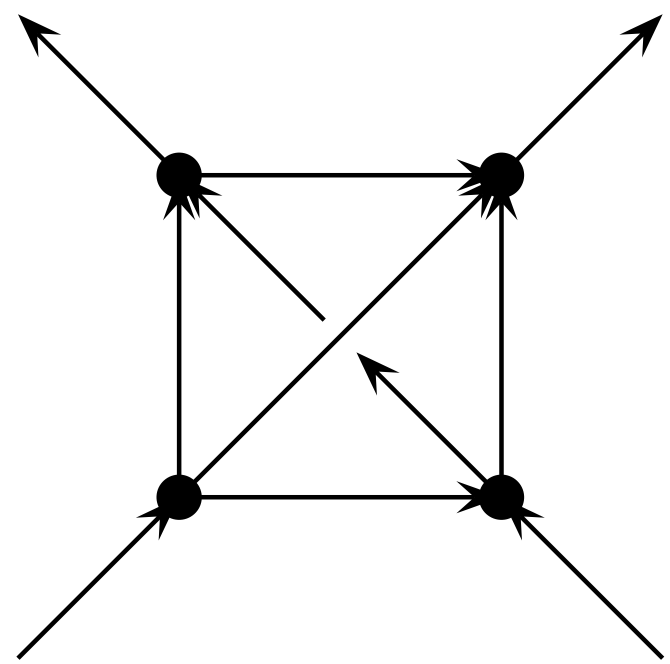
A Typical Feynman Diagram



Two particles enter at x and y they interact at z and scatter to u and v.

In this diagram every vertex has degree 4.
This is a *first-order* diagram (one vertex) of $\lambda\phi^4$ theory.

A More Complex Feynman Diagram



In this diagram every vertex has degree 4.
This is a *fourth-order* diagram (four vertices) of $\lambda\phi^4$ theory.

The nature of the theory determines the type and degree of the vertices.

Feynman Propagators

The pictures are just mnemonics for certain integrals that arise in QFT.

The lines are functions that
describe how particles are propagated.

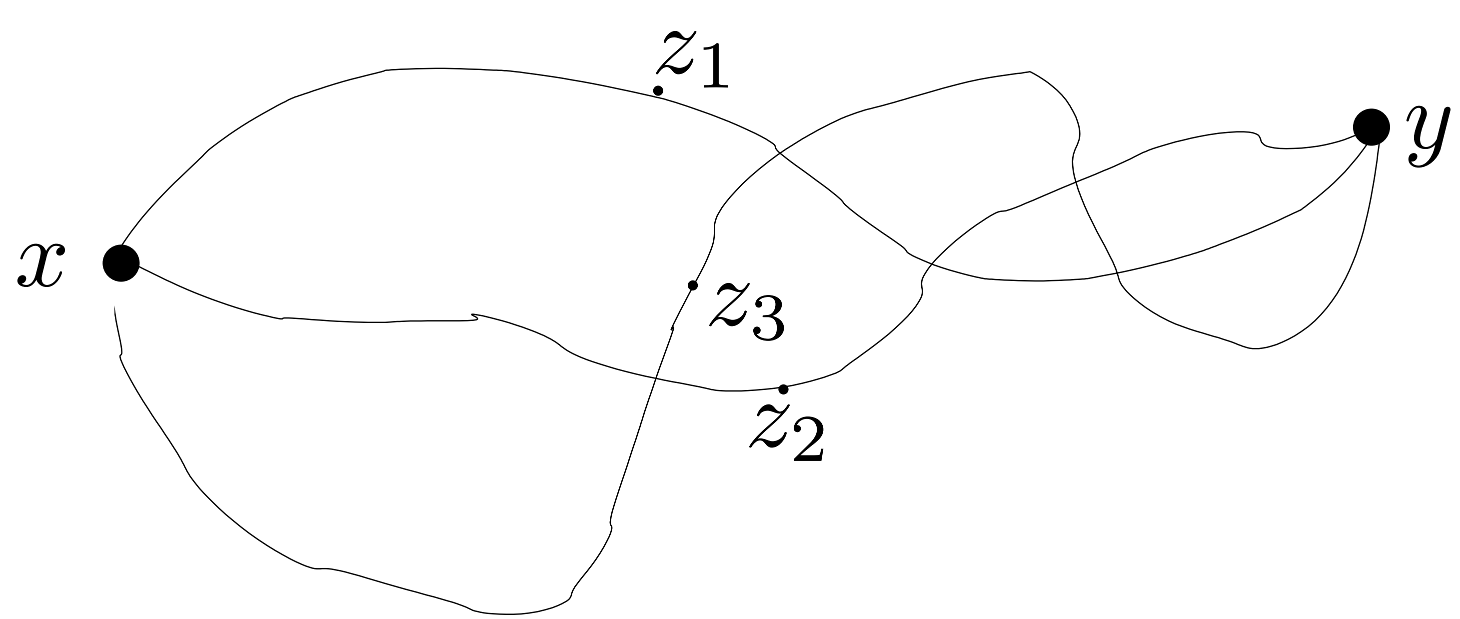
The vertices represent integrals.

I will not describe how these are calculated here.

Sum Over Paths

$G(x, y)$ is obtained by summing over all paths from x to y.

$$G(x, y) = \int G(x, z) G(z, y) \mathrm{d}z$$



Functional Integrals

In QFT one uses an integration over *all field configurations*.

This is not well defined and is used as a *formal device.*

I will use formal integrals, but

they will be analogues of ordinary integrals.

Variational Derivatives

These are derivatives of *functionals* with respect to *functions.*

They are perfectly well defined and have been used since the 17th century in the calculus of variations.

I will use formal analogues of them.

The ϕ-calculus

Locations: which play the same role as the locations in the located sequents of LRA.

Definition 1. *We assume that there are countably many distinct symbols, called* **locations** *for each basic type. We assume that there are the following operations on locations: if x and y are locations of types A and B respectively, then $\langle x, y\rangle$ and $[x, y]$ are locations of type $A \otimes B$ and $A \parr B$ respectively. We use the usual sequent notation $x : A, y : B \vdash \langle x, y\rangle : A \otimes B$ and $x : A, y : B \vdash [x, y] : A \parr B$ to express this.*

Basic terms: which, for the multiplicative fragment, play the role of LRA terms.

Operators: Which act on basic terms and which play the role of terms in the full LRA.

Expressions

Definition 2. *The collection of* **expressions** *is given by the following inductive definition. We also define, at the same time, the notion of the* **sort** *of an expression, which is the set of free locations, and their types, that appear in the expression.*

1. *Any real number* r *is an expression of sort* $\emptyset$.
2. *Given any two* distinct *locations,* $x : A$ *and* $y : A^{\perp}$ $\delta(x, y)$ *is an expression of sort* $\{x : A, y : A^{\perp}\}$.
3. *Given any two expressions* P *and* Q, PQ *and* $P + Q$ *are expressions of sort* $\mathsf{S}(P) \cup \mathsf{S}(Q)$.
4. *Given any expression* P *and any location* $x : A$ *in* P, *the expression* $\int P dx$ *is an expression of sort* $\mathsf{S}(P) \setminus \{x : A\}$.

Equations

1. $\delta(x, y) = \delta(y, x)$

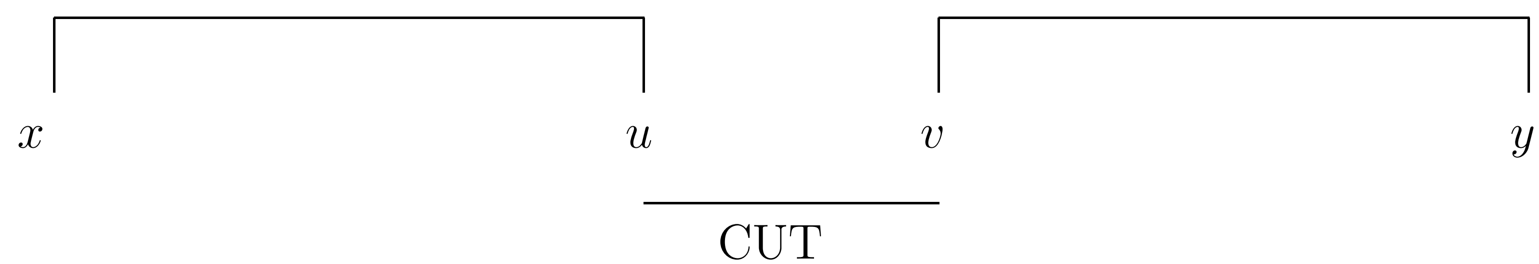
Associativity and commutativity for $+$ and $\cdot$.

11. $\int P(\ldots, x, \ldots)\delta(x, y)dx = P(\ldots, y/x, \ldots)$

12. $\delta([x, y], \langle u, v\rangle) = \delta(x, u)\delta(y, v)$.

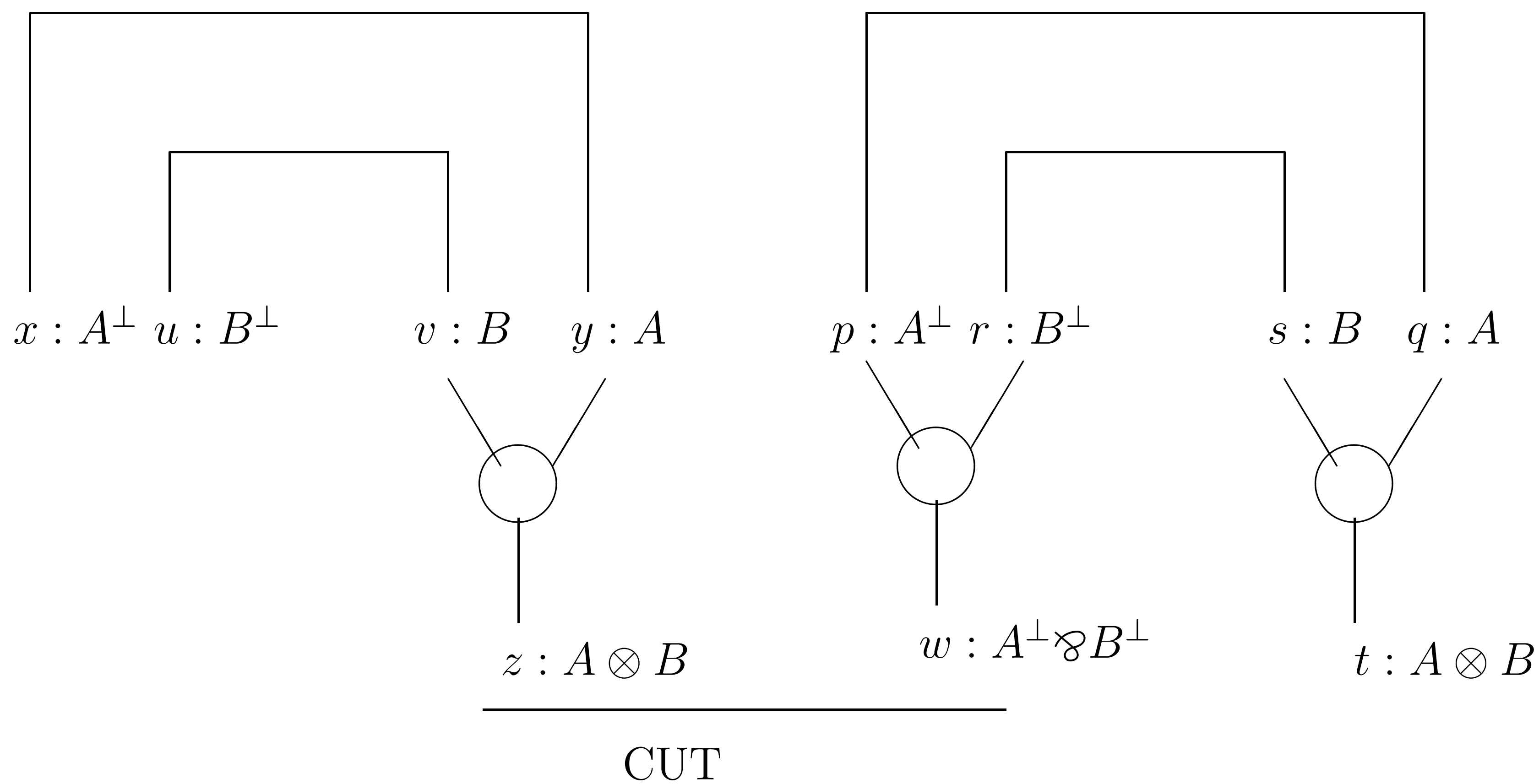
Interpreting LRA in ϕ

Proof Rule	**LRA Term**		**Φ-Calculus**
Axiom	$[\![I_{xy}]\!]$	$=$	$\delta(x,y)$
Cut	$[\![P \cdot_x Q]\!]$	$=$	$\int [\![P]\!][\![Q]\!]dx$
Tensor	$[\![\otimes_z^{x,y}(P,Q)]\!]$	$=$	$\int [\![P]\!][\![Q]\!]\delta(z,\langle x,y\rangle)dxdy$
Par	$[\![\parr_z^{x,y}(P)]\!]$	$=$	$\int [\![P]\!]\delta(z,[x,y])dxdy$



$$I_{x,u} \cdot_{u,v} I_{v,y} \quad = \quad \int \delta(x,u)\delta(u,v)\delta(v,y)dudv$$

Using rule 11 we get $\delta(x,y)$ which is $I_{x,y}$.



$$\int \quad \begin{array}{l} [\int \delta_A(x,y)\delta_B(u,v)\delta(z,\langle y,v\rangle)dvdy] \\ [\int \delta_A(p,q)\delta_B(r,s)\delta(t,\langle q,s\rangle)\delta(w,[p,r])dpdqdrds] \\ \qquad\qquad\qquad \delta(w,z)dwdz. \end{array}$$

Simplifies to $$\int \delta_A(x,q)\delta_B(u,s)\delta(t,\langle q,s\rangle)dqds$$

Exponentials

Intuition: model the box by the exponential power series!

Model dereliction by a derivative probing an exponential.

Key analogies: $\frac{\mathrm{d}}{\mathrm{d}x}\mathsf{e}^{ax}|_{x=0} = \mathsf{a}$

$$\exp(a\tfrac{\mathrm{d}}{\mathrm{d}x})e^{bx}|_{x=0} = \exp(a\tfrac{\mathrm{d}}{\mathrm{d}x}e^{bx})|_{x=0}$$

This is "nesting of boxes"!

Operators

1. *If* M *is any expression* $\hat{M}$ *is an operator of the same sort as* M.
2. *If* x $:A$ *is a location then* (. $]_{\alpha(x)=0}$)*is an operator of sort* x $:A$.
3. *If* x $:A$ *is a location then* $\frac{\delta}{\delta\alpha(x)}$ *is an operator of sort* x $:A$.
4. *If* P *and* Q *are operators then so are* $P + Q$ *and* $P \circ Q$ *their sort is the union of the individual sorts.*
5. *If* P *is an operator then so is* $\int P dx$; *its sort is* $\mathsf{S}(P) \setminus \{x\}$.

1. *If* x *and* y *are distinct locations then* $\frac{\delta}{\delta\alpha(x)}\alpha(y) = 0$.
2. *If* $\alpha(x)$ *does not occur in the expression* M *then* $\frac{\delta}{\delta\alpha(x)}M = 0$.
3. $\frac{\delta}{\delta\alpha(x)}\alpha(x) = 1$.
4. $\frac{\delta}{\delta\alpha(x)}\ M \quad N \qquad \frac{\delta}{\delta\alpha(x)}M \qquad \frac{\delta}{\delta\alpha(x)}N$.
5. $\frac{\delta}{\delta\alpha(x)}\ MN \qquad M\frac{\delta}{\delta\alpha(x)}N \qquad \frac{\delta}{\delta\alpha(x)}M\ N$

Exponential Series

$$\Sigma_{k\geq 0} M^k/k!$$

Lemma 4.2 *If the expression M contains no occurrence of $\alpha(x)$ then:*

1. $\frac{\delta}{\delta\alpha(x)}(MN) = M\frac{\delta}{\delta\alpha(x)}(N)$;
2. $(([.]|_{\alpha(x)=0}) \circ \frac{\delta}{\delta\alpha(x)})\exp(M\alpha(x)) = M$;
3. $(([.]|_{\alpha(x)=0}) \circ \frac{\delta}{\delta\alpha(x)} \circ \ldots n \ldots \circ \frac{\delta}{\delta\alpha(x)})\exp(M\alpha(x)) = M^n$.

The combination $\frac{\delta}{\delta\alpha(x)} \circ \ldots n \ldots \circ \frac{\delta}{\delta\alpha(x)}$ is often written $\frac{\delta^n}{\delta\alpha(x)^n}$.

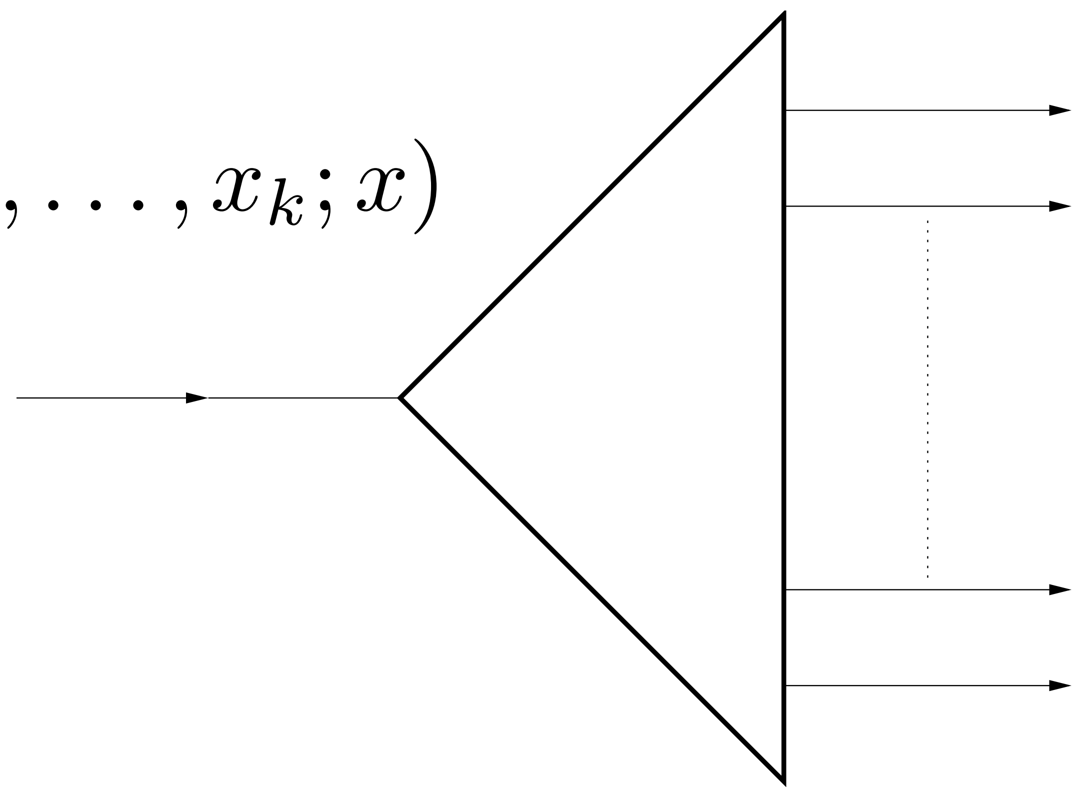
Lemma 4.3 Suppose that M is an expression, the following equations hold.

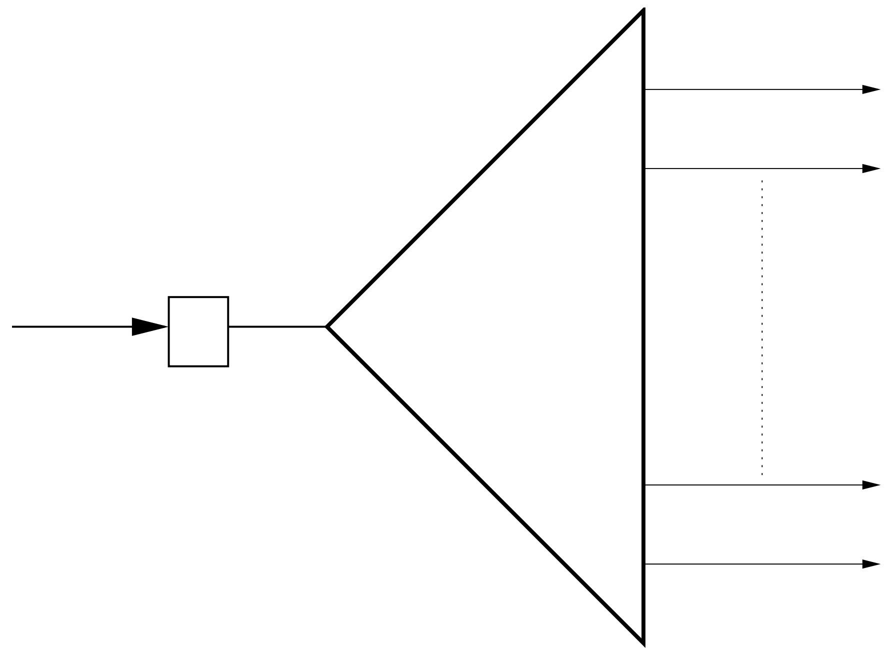
1. $\frac{\delta}{\delta\alpha(x)} \exp(M\alpha(x)) = M \cdot \exp(M\alpha(x))$.
2. $([\exp(M)]|_{\alpha(x)=0}) = \exp(([M]|_{\alpha(x)=0}))$.
3. $\exp(0) = 1$.

Symmetrization

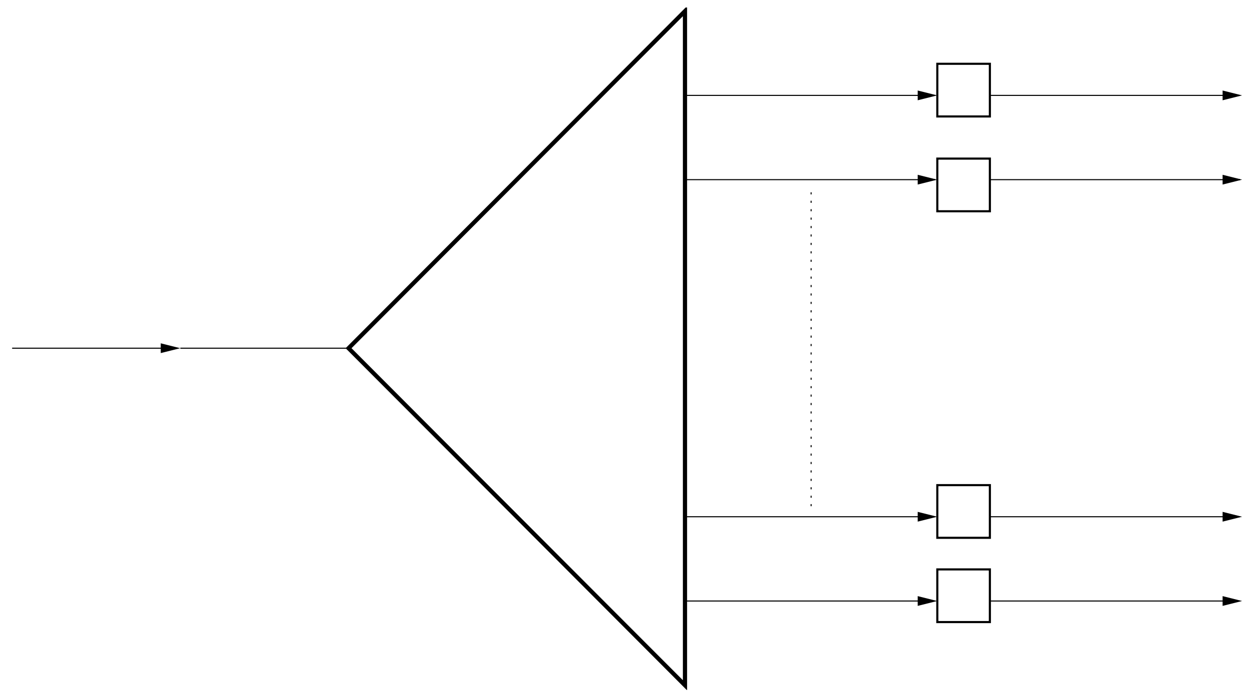
In fact the above definition of exponentials overlooks a subtlety which makes a difference as soon as we exponentiate operators. The factors of the form $1/(n!)$ are not just numerical factors, they indicate symmetrization. This is the key ingredient needed to model contraction in linear logic. We introduce a new syntactic primitive for symmetrization and give its rules.

$$\Delta^{(k)}(x_1, \ldots, x_k; x)$$





equals



1. $\int \Delta^{(k)}(x_1, \ldots, x_k; x) M(x, y_1, \ldots, y_l)\, dx =$
$\int \prod_{i=1}^{k} M[x_i/x, y_1^i/y_1, \ldots, y_l^i/y_l] \prod_{j=1}^{l} \Delta^{(k)}(y_j^1, \ldots, y_j^k; y_j)\, dy_1^1 \ldots\, dy_l^k.$

2. $\int \Delta^{(k)}(x_1, \ldots, x_k; x) \Delta^{(m+1)}(x, x_{k+1}, \ldots, x_{k+m}; y)\ dx$
$= \Delta^{(k+m)}(x_1, \ldots, x_k, x_{k+1}, \ldots, x_{k+m}; y).$

3. $\int \Delta^{(k)}(u_1, \ldots, u_k; x) \Delta^{(k)}(u_1, \ldots, u_k; y)\ du_1 \ldots du_k = \delta(x, y).$

Connecting the principal ports

$$\int \Delta^{(k)}(\mathsf{x}_1, \ldots, \mathsf{x}_k; \mathsf{x}) \Delta^{(k)}(\mathsf{y}_1, \ldots, \mathsf{y}_k; \mathsf{x}) \mathsf{dx}$$

$$= \Sigma_{\sigma \in perm\{1,\ldots,k\}} \delta(\mathsf{x}_1, \mathsf{y}_{\sigma(1)}) \ldots \delta(\mathsf{x}_k, \mathsf{y}_{\sigma(end)})$$

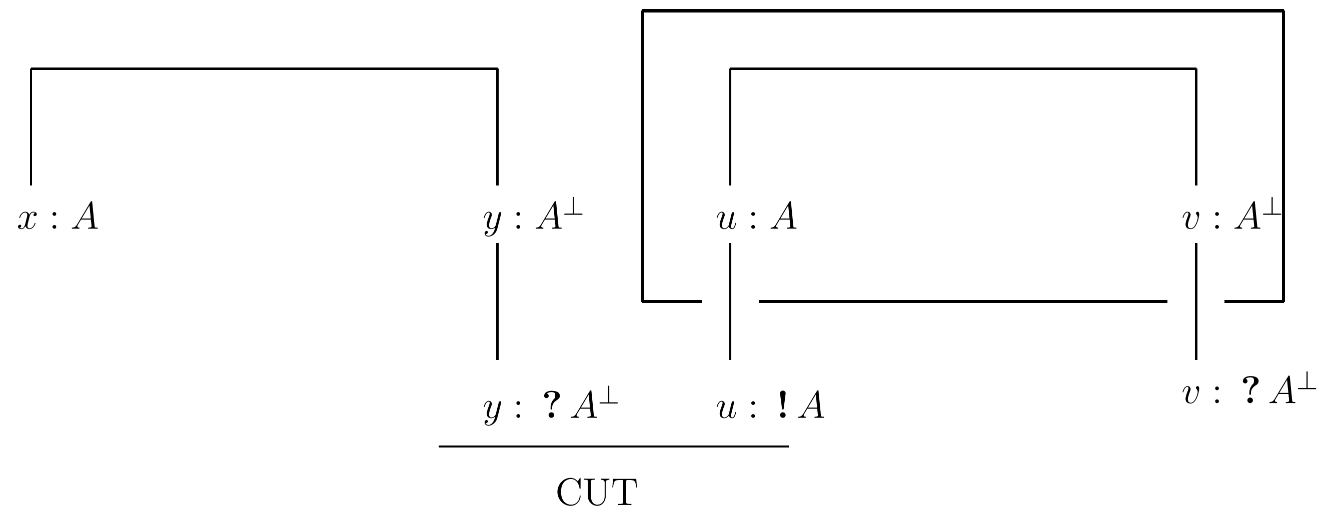
$$
\begin{aligned}
&\exp(Q) = 1 + Q(x_1, \ldots, x_m) + \\
&(1/2) \int Q(x_1', \ldots, x_m')Q(x_1'', \ldots, x_m'')\Delta(x_1', x_1''; x_1) \ldots \Delta(x_m', x_m''; x_m) \\
&\qquad dx_1' \ldots dx_m' dx_1'' \ldots dx_m'' \\
&\qquad + \ldots \\
&+ (1/(k!)) \int Q(x_1^{(1)}, \ldots, x_m^{(k)}) \ldots Q(x_1^{(k)}, \ldots, x_m^{(k)}) \\
&\Delta^k(x_1^{(1)}, \ldots, x_1^{(k)}; x_1) \ldots \Delta^k(x_m^{(1)}, \ldots, x_m^{(k)}; x_m) \\
&\qquad dx_1^{(1)} \ldots dx_m^{(1)} \ldots dx_1^{(k)} \ldots dx_m^{(k)} + \ldots
\end{aligned}
$$

We prove all the analogues of the exponential identities for these formal power series and our formal derivatives.

Interpreting Exponentials

Dereliction	$[\![D_z^x(P)]\!]$	$=$	$[\![P[z/x]]\!]([.]\|_{\alpha(z)=0}) \circ \frac{\delta}{\delta\alpha(z)}$
Weakening	$[\![W_z(P)]\!]$	$=$	$[\![P]\!] \circ W(z)([.]\|_{\alpha(z)=0})\frac{\delta}{\delta\alpha(z)}$
Contraction	$[\![C_z^{x,y}]\!]$	$=$	$\int [\![P]\!]\Delta(x,y;z)dxdy$
Exponentiation	$[\![\,!\,_y^x(P)]\!]$	$=$	$\exp([\![P[y/x]]\!]\alpha_A(y))$

The exponential identities immediately show that the LRA equations hold.



$$\int \delta(x,y)([.]|_{\alpha(y)=0})\frac{\delta}{\delta\alpha(y)} \exp[\alpha(u)\delta(u,v)([.]|_{\alpha(v)=0})\frac{\delta}{\delta\alpha(v)}]\delta(y,u)dydu.$$

Now the last integral can be done with the convolution property of δ and we get

$$\delta(x,v)([.]|_{\alpha(v)=0})\frac{\delta}{\delta\alpha(v)}$$

which is what we expect from the cut-free proof.

What does it all mean?

Combinatorial coincidence?

I think there are interesting suggestive analogies that are more than coincidence.

The combinatorics of exponentials in linear logic are based on the same intuitions as the use of exponential power series.

There are too many artificial syntactic devices for my taste.

It was a big mistake to be so obsessed with linear logic!

The real question:
what is the “logic” of Feynman diagrams as they are?