Duality for Transition Systems

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- Cricket prediction for Australia v India: 1-1.

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- Special case of this construction known since 1962 to Brzozowski.
- Works for probabilistic automata.
- Seems interesting for learning and planning.

Joint work with Doina Precup, Joelle Pineau at the RL Lab at McGill and Chris Hundt now working for Google. More recently with Nick Bezhanishvili and Clemens Kupke. Now also with Helle Hvid Hansen, Alexandra Silva, Jan Rutten, Dexter Kozen, Marcello Bonsangue and Filippo Bonchi.

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- Thus, one has two completely different sets of theorems that one can use.

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- Vector spaces and vector spaces.

Deterministic Automata

- $\mathcal{M} = (S, \mathcal{A}, \mathcal{O}, \delta, \gamma)$: a deterministic finite automaton. S is the set of **states**, $\mathcal{A}$ an **input alphabet** (actions), $\mathcal{O}$ is a set of **observations**.

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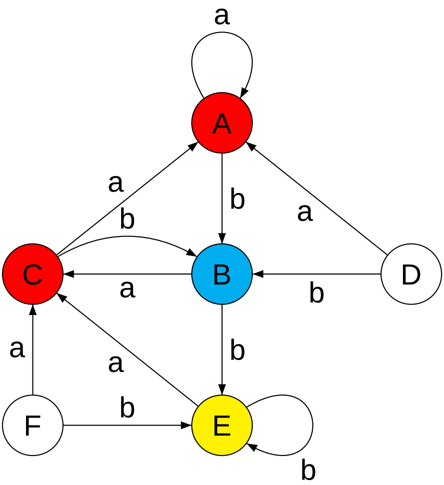
- *except* that we do not have a start state,
- but this can be easily added to the framework.

An Example

States: $\{A, B, C, D, E, F\}$ Observations: $\{\textbf{Blue}, \textbf{Red}, \textbf{Yellow}\}$.



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- We can ask whether *after some fixed sequence of transitions* a particular light is on.

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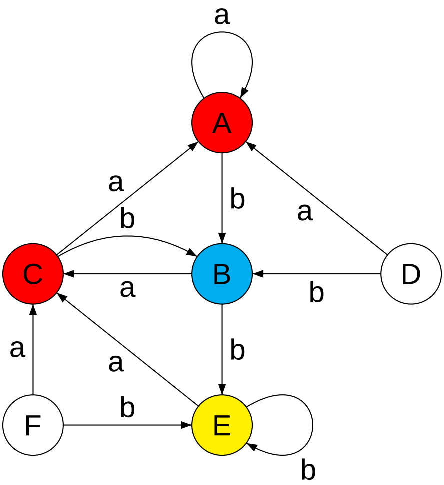
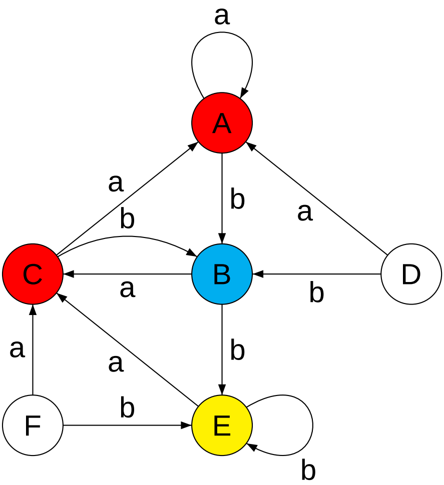
- After b, **yellow** is on, is satisfied by $\{B, E, F\}$ and no other states.
- After $abab$, **blue** is on is satisfied by $\{A, B, C, D, E, F\}$, i.e. by *all* states.

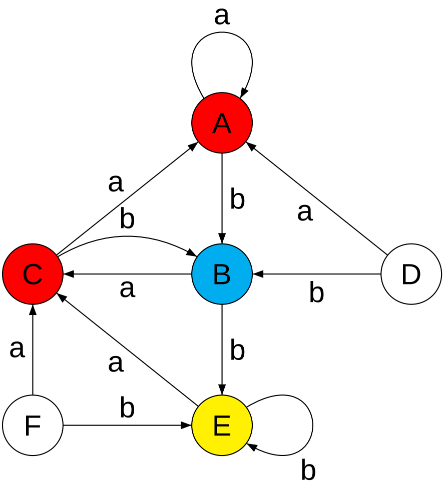
red is satisfied by $\{A, C\}$



b-Yellow is satisfied by $\{B, E, F\}$



abab-Blue is always satisfied



A Simple Modal Logic

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- Now we define $[\![\varphi]\!]_{\mathcal{M}} = \{s \in S | s \models \varphi\}$.

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- Note that another way of defining this equivalence relations is

$$\varphi \sim_{\mathcal{M}} \varphi' := \forall s \in S. s \models \varphi \iff s \models \varphi'.$$

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- Other equivalent formulas are all formulas of the form $b^m\mathbf{yellow}$ for $m > 1$.
- There are a lot of formulas in this equivalence class!
- But there are only finitely many equivalence classes.

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- Since there is more than just one proposition in general the relation $\equiv$ is finer than the usual equivalence of automata theory.

A Dual Automaton

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- $\gamma'([\![\varphi]\!]_{\mathcal{M}}) = [\![\varphi]\!]_{\mathcal{M}}$ or $\gamma'([\![\varphi]\!]_{\mathcal{A}}, s) = (s \models \varphi)$.

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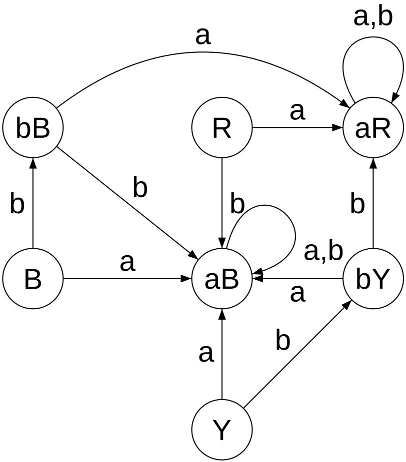
- but that is just a set of states of the original machine!

The intuition

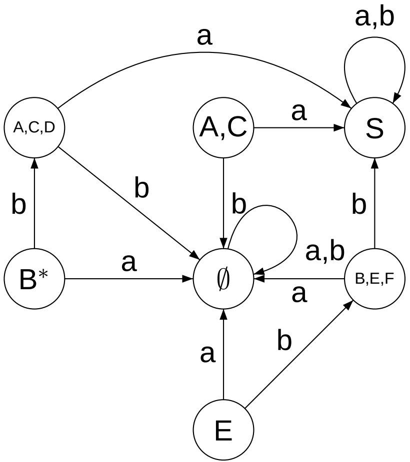
We have interchanged the states and the observations or propositions; more precisely we have interchanged equivalence classes of formulas - based on the observations - with the states. We have made the states of the old machine the observations of the dual machine.

The Dual Machine For Our Example



Note that $aB \sim abY \sim bR \sim$ **false** and that $aR \sim bbY \sim abB \sim$ **true**.

The Dual Machine Labelled with Observations (aka States)



$*$: This means the state B, not the colour Blue!
S stands for the set of all states.

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- Not every possible set of states is the denotation of some formula.
- If it were the case that every possible set of states is described by some formula then we would indeed have exponential blow up in the size.
- If we had a richer logic then more sets of states would be definable.

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- Thus we can look at states in $\mathcal{M}''$ as collections of $\mathcal{M}$-formula equivalence classes.

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- The proof is by an easy induction on φ.

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- The statement above can be strengthened to show that we actually have an isomorphism of automata.
- In general, the double dual is the minimal machine with the same behaviour!
- For deterministic machines bisimulation is the same as trace equivalence so there is no question about what equivalence we have in mind.

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Brzozowski's Algorithm 1962

- Take a NFA and just reverse all the transitions and interchange initial and final states.
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- This gives the minimal DFA recognizing the same language. The intermediate step can blow up the size of the automaton exponentially before minimizing it.

Probabilistic systems

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- Usually MDPs have rewards but I will not consider them for now.

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- In POMDPs we assume actions are always accepted but with each transition some propositions are true, or some boolean observables are "on."
- Note that the observations can depend probabilistically on the action taken and the *final* state. Many variations are possible.

Formal Definition of a POMDP

- $\mathcal{M} = (S, \mathcal{A}, \mathcal{O}, \delta : S \times \mathcal{A} \times S \longrightarrow [0,1], \gamma : S \times \mathcal{A} \times \mathcal{O} \longrightarrow [0,1]),$

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- where S is the set of states, $\mathcal{O}$ is the set of observations, $\mathcal{A}$ is the set of actions, δ is the transition probability function and γ gives the observation probabilities.

Automata with State-based Observations

- A **deterministic automaton with stochastic observations** is a quintuple

$$\mathcal{E} = (S, \mathcal{A}, \mathcal{O}, \delta : S \times \mathcal{A} \longrightarrow S, \gamma : S \times \mathcal{O} \longrightarrow [0, 1]).$$

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- In AI these are called “e-tests.”

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- This machine has deterministic transitions and γ' is just the transpose of γ.

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- An easy calculation shows:

$$[\![a_1 a_2 \cdots a_k o]\!]_{\mathcal{E}''}([\![s]\!]_{\mathcal{E}'})$$

$$= [\![a_1 a_2 \cdots a_k o]\!]_{\mathcal{E}}(s).$$

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  This does not hold in the primal.
- The double dual does not conditionalize with respect to intermediate observations.

More General Tests

- Recall the definition of a POMDP

$$\mathcal{M} = (S, \mathcal{A}, \mathcal{O}, \delta_a : S \times S \longrightarrow [0,1], \gamma_a : S \times \mathcal{O} \longrightarrow [0,1]).$$

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- A **test** t is a non-empty sequence of actions followed by an observation, i.e. $t = a_1 \cdots a_n o$, with $n \geq 1$.
- An **experiment** is a non-empty sequence of tests $e = t_1 \cdots t_m$ with $m \geq 1$.

Some Notation

- We need to generalize the transition function to keep track of the final state.

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\begin{aligned}
\delta_\epsilon(s, s') &= \mathbf{1}_{s=s'} && \forall s, s' \in S \\
\delta_{a\alpha}(s, s') &= \sum_{s''} \delta_a(s, s'')\delta_\alpha(s'', s') && \forall s, s' \in S.
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- We have written $\mathbf{1}_{s=s'}$ for the indicator function.
- We define the symbol $\langle s|t|s'\rangle$ which gives the probability that the system starts in s, is subjected to the test t and ends up in the state s'; similarly $\langle s|e|s'\rangle$.

Notation continued

- We have

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Equivalence on Experiments

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- The construction of the dual proceeds as before by making equivalence classes of experiments the states of the dual machine and
- the states of the primal machine become the observations of the dual machine.

The Dual Machine

- We define the dual as $\mathcal{M}' =$

$$(S', \mathcal{A}, \mathcal{O}', \delta' : S' \times \mathcal{A} \longrightarrow S', \gamma' : S' \times \mathcal{O}' \longrightarrow [0, 1]),$$

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- $\delta'([e]_{\mathcal{M}}, a_0) = [a_0 e]_{\mathcal{M}}$ and
- $\gamma'([e]_{\mathcal{M}}, s) = \langle s|e\rangle$.
- We get a deterministic transition system with stochastic observations.

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- $\gamma''([t]_{\mathcal{M}'}, [t]_{\mathcal{M}}) = \langle [t]_{\mathcal{M}} | e \rangle = \langle s | \alpha^R t \rangle \qquad (e = \alpha s).$

The Main Theorem

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- The main result is: The probability of a state s in the primal satisfying a experiment e, i.e. $\langle s|e\rangle$ is given by $\langle [s]_{\mathcal{M}'}|[e]_{\mathcal{M}}\rangle = \gamma''([s]_{\mathcal{M}'})|[e]_{\mathcal{M}}\rangle$, where $[s]$ indicates the equivalence class of the e-test on the dual which has s as an observation and an empty sequence of actions.

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- Learning is hopeless when one has no idea what the state space is.
- There should be no such thing as absolute state! State is just a summary of past observations that can be used to make predictions.
- The double dual shows that the state can be regarded as just the summary of the outcomes of experiments.

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- We know that machines are co-algebras and logics are algebras but
- why is the dual another automaton?

Automata as Coalgebras

Our automata are coalgebras of the following functor:

$$F(S) = S^{\mathcal{A}} \times \mathbf{2}^{\mathcal{O}}, \;\; F(f : S \longrightarrow S') = \lambda(\alpha : \mathcal{A} \longrightarrow S, \; O \subset \mathcal{O}).(f \circ \alpha, \; O).$$

The category of these coalgebras is called **PODFA**.

Homomorphisms

A homomorphism for these coalgebras is a function $f : S \longrightarrow S'$ such that the following diagram commutes:

$$
\begin{tikzcd}
S \arrow[r, "f"] \arrow[d, "{(\delta,\ \gamma)}"'] & S' \arrow[d, "{(\delta',\ \gamma')}"] \\
S^{\mathcal{A}} \times \mathbf{2}^{\mathcal{O}} \arrow[r, "f^{\mathcal{A}} \times \mathrm{id}"'] & S'^{\mathcal{A}} \times \mathbf{2}^{\mathcal{O}}
\end{tikzcd}
$$

where $f^{\mathcal{A}}(\alpha) = f \circ \alpha$.

This translates to the following conditions:

$$\forall s \in S, \omega \in \mathcal{O},\ \omega \in \gamma(s) \iff \omega \in \gamma'(f(s)) \tag{1}$$

and

$$\forall s \in S, a \in \mathcal{A},\ f(\delta(s,a)) = \delta'(f(s),a). \tag{2}$$

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- the usual operations $\wedge$, $\neg$ and constants T and $\perp$ and, in addition,
- together with unary operators (a) and constants $\underline{\omega}$.
- We denote an object by $\mathcal{B} = (B, \{(a) | a \in \mathcal{A}\}, \{\underline{\omega} | \omega \in \mathcal{O}\}, \mathsf{T}, \wedge, \neg)$.

Morphisms

The morphisms are the usual boolean homomorphisms preserving, in addition, the constants and the unary operators.
The following three equations hold:

$$(a)(b_1 \wedge b_2) = (a)b_1 \wedge (a)b_2, \tag{3a}$$

$$(a)\mathsf{T} = \mathsf{T}, \tag{3b}$$

$$\neg(a)\neg b = (a)b. \tag{3c}$$

Duality Theorem

There is a dual equivalence of categories

$$\mathbf{PODFA}^{op} \cong \mathbf{FBAO}.$$

One functor $\mathcal{P}$ is just the contravariant power set functor and the other one $\mathcal{H}$ maps a boolean algebra to its set of atoms.

Minimization?

- Obviously, if we have an equivalence of categories we get the same machine back when we go back and forth.

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- Obviously, if we have an equivalence of categories we get the same machine back when we go back and forth.
- So how do we explain the minimization?

Definable Subsets

Define a logic $\mathcal{L}$ by

$$\phi ::== \mathsf{T} | \bot | \phi_1 \wedge \phi_2 | \neg\phi | (a)\phi | \underline{\omega}$$

and define the **definable subsets** $\mathcal{D}(S)$ of a machine $\mathcal{M} = (S, \delta, \gamma)$ as sets of the form $[\![\phi]\!]$.

- $\mathcal{D}(S)$ is a subobject of $\mathcal{P}(\mathcal{M})$

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- in fact it is the *smallest* possible subalgebra and
- any other subalgebra must contain $\mathcal{D}(S)$.

In Pictures

$$\mathcal{M} \dashrightarrow \mathcal{P}(\mathcal{M})$$

$$
\begin{tikzcd}
\mathcal{M} \arrow[r, dotted] & \mathcal{P}(\mathcal{M}) \\
 & D(S) \arrow[u, hook, "1"']
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$$
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 & \mathcal{M} \arrow[r, dotted] \arrow[d, two heads, "2"'] \arrow[ddl, "3"'] & \mathcal{P}(\mathcal{M}) & \\
 & \mathcal{H}(D(S)) & D(S) \arrow[l, dotted] \arrow[u, hook, "1"'] \arrow[dr, hook, "6"'] & \\
\mathcal{S}' \arrow[rrr, dotted, "4"'] & & & \mathcal{P}(\mathcal{S}') \arrow[uul, hook, "5"']
\end{tikzcd}
$$

The Secret of Minimization

$$
\begin{tikzcd}
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 & \mathcal{H}(D(S)) & D(S) \arrow[l, dotted] \arrow[u, hook, "1"'] \arrow[dr, hook, "6"'] & \\
\mathcal{S}' \arrow[ur, "7"'] \arrow[rrr, dotted, "4"'] & & & \mathcal{P}(\mathcal{S}') \arrow[uul, hook, "5"']
\end{tikzcd}
$$

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- With this logic the definable subsets $E(S)$ do not form a boolean algebra
- it is just a “set with operations”
- in other words it can be viewed as an automaton!

Deterministic vs Nondeterministic Automata

- For deterministic automata we can flatten formulas like $(a)(\omega_1 \wedge (b)\omega_2)$ to $(a)\omega_1 \wedge (a)(b)\omega_2$.

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- Thus for **deterministic** automata the boolean algebra generated by $E(S)$ is just the same as $D(S)$ so the minimization picture works with boolean algebra generated by $E(S)$.
- For nondeterministic automata the story is different.

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- These are automata where the state space is a vector space and the transitions are given by matrices.

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- exploiting the fact that the category of vector spaces is self dual.

Probabilistic Automata

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- The dual is a C* algebra with operations.
- The same picture applies.

Diagram of an automaton

$$\begin{tikzcd} 1 \arrow[dr, "i"] & & 2 \\ & S \arrow[ur, "f"] \arrow[d, "\delta"] & \\ & S^A & \end{tikzcd}$$

S is the state space
δ is the transition function
f defines the final states.

The butterfly

$$
\begin{tikzcd}
1 \arrow[d, "\varepsilon"'] \arrow[dr, "i"] & & 2 \\
A^* \arrow[r, dashed, "r"] \arrow[d, "\alpha"'] & S \arrow[ur, "f"] \arrow[r, dashed, "o"] \arrow[d, "\delta"] & 2^{A^*} \arrow[u, "\varepsilon?"'] \arrow[d, "\beta"] \\
(A^*)^A \arrow[r, dashed, "r^A"'] & S^A \arrow[r, dashed, "o^A"'] & (2^{A^*})^A
\end{tikzcd}
$$

Left: Automaton of words (initial)

$$\alpha\colon A^* \longrightarrow (A^*)^A \qquad \alpha(w)(a) = w \cdot a$$

Right: Automaton of languages (terminal)

$$\beta\colon 2^{A^*} \longrightarrow (2^{A^*})^A \qquad \beta(L)(a) = \{w \in A^* \mid a \cdot w \in L\}$$

r defines reachability; o defines observability.

Reachability and observability

A deterministic automaton (S, δ, i, f) is *reachable* if r is surjective, it is *observable* if o is injective, and it is *minimal* if it is both reachable and observable.

Contravariant power set functor

$$2^{(-)} \; : \quad \begin{array}{c} V \\ {\scriptstyle f}\big\downarrow \\ W \end{array} \quad \mapsto \quad \begin{array}{c} 2^V \\ \big\uparrow{\scriptstyle 2^f} \\ 2^W \end{array}$$

which is defined, for a set V, by $2^V = \{S \mid S \subseteq V\}$ and, for $f : V \longrightarrow W$ and $S \subseteq W$, by

$$2^f \; : \; 2^W \longrightarrow 2^V \qquad 2^f(S) = \{v \in V \mid f(v) \in S\}$$

Reversing

$$
\begin{array}{c||c|c||c}
S & A \times A & 2^{S\times A} & (2^S)^A \\
\Big\downarrow{\scriptstyle \delta} & \Big\downarrow & \Big\uparrow & \Big\uparrow{\scriptstyle 2^{\delta}} \\
S^A & S & 2^S & 2^S
\end{array}
$$

Initial becomes final

Applying the operation $2^{(-)}$ to the initial state (function) of our automaton gives

$$\begin{array}{c|c} 1 & 2 \\ \downarrow{\scriptstyle i} & \uparrow{\scriptstyle 2^i} \\ X & 2^S \end{array}$$

Reachable becomes observable - I

Apply the powerset functor:

$$\begin{tikzcd}
 & 2 \\
2^S \arrow[ur, "2^i"] \arrow[r, "2^r"] \arrow[d, "2^\delta"'] & 2^{A^*} \arrow[u, "2^\varepsilon"'] \arrow[d, "2^\alpha"] \\
(2^S)^A \arrow[r, "2^{r^A}"'] & (2^{A^*})^A
\end{tikzcd}$$

For any $L \in 2^{A^*}$, we have $2^\varepsilon(L) = \varepsilon?(L)$ and, for any $a \in A$,

$$2^\alpha(L)(a) = \{w \in A^* \mid w \cdot a \in L\}$$

Like $\beta(L)(a)$ but it uses $w \cdot a$ instead of $a \cdot w$.

Reachable becomes observable - II

By finality of $(2^{A^*}, \beta, \varepsilon?)$, there exists a unique homomorphism $rev\colon 2^{A^*} \to 2^{A^*}$

$$\begin{tikzcd}
 & & 2 \\
2^{A^*} \arrow[rru, "2^{\varepsilon}"] \arrow[rr, dashed, "rev"] \arrow[d, "2^{\alpha}"'] & & 2^{A^*} \arrow[u, "\varepsilon?"'] \arrow[d, "\beta"] \\
(2^{A^*})^A \arrow[rr, dashed, "rev^A"'] & & (2^{A^*})^A
\end{tikzcd}$$

which sends a language L to its reverse

$$rev(L) = \{w \in A^* \mid w^R \in L\}$$

where w^R is the reverse of w.

Reachable becomes observable - III

Combining diagrams yields:

$$\begin{tikzcd}
 & & 2 \\
2^S \arrow[rru, "2^i"] \arrow[r, "2^r"'] \arrow[d, "2^\delta"'] & 2^{A^*} \arrow[ru, "2^\varepsilon"] \arrow[r, "rev"'] \arrow[d, "2^\alpha"] & 2^{A^*} \arrow[u, "\varepsilon?"'] \arrow[d, "\beta"] \\
(2^S)^A \arrow[r, "2^{r^A}"'] & (2^{A^*})^A \arrow[r, "rev^A"'] & (2^{A^*})^A
\end{tikzcd}$$

Reachable becomes observable - IV

Thus the composition of rev and 2^r is the unique function that makes the following diagram commute:

$$\begin{tikzcd} & 2 \\ 2^S \arrow[ur, "2^i"] \arrow[r, dashed, "O"] \arrow[d, "2^\delta"'] & 2^{A^*} \arrow[u, "\varepsilon?"'] \arrow[d, "\beta"] \\ (2^S)^A \arrow[r, dashed, "O^A"'] & (2^{A^*})^A \end{tikzcd}$$

$$O = rev \circ 2^r$$

One can easily show that it satisfies, for any $X \subseteq S$,

$$O(X) = \{w^R \in A^* \mid i_w \in X\}$$

Final becomes initial

$$\begin{array}{c||c} 2 & 1 \\ f \big\uparrow & \big\downarrow f \\ S & 2^S \end{array}$$

Putting everything together

We have obtained the following, new deterministic automaton:

$$
\begin{tikzcd}
1 \arrow[d, "\varepsilon"'] \arrow[dr, "f"] & & 2 \\
A^* \arrow[r, dashed, "R"'] \arrow[d, "\alpha"'] & 2^X \arrow[r, dashed, "O"'] \arrow[d, "2^t"] \arrow[ur, "2^i"] & 2^{A^*} \arrow[u, "\varepsilon?"'] \arrow[d, "\beta"] \\
(A^*)^A \arrow[r, dashed, "R^A"'] & (2^X)^A \arrow[r, dashed, "O^A"'] & (2^{A^*})^A
\end{tikzcd}
$$

The main theorem

Let (S, δ, i, f) be a deterministic automaton and let $(2^S, 2^\delta, f, 2^i)$ be the reversed deterministic automaton constructed as above.

1. If (S, δ, i, f) is reachable, then $(2^S, 2^\delta, f, 2^i)$ is observable.

The main theorem

Let (S, δ, i, f) be a deterministic automaton and let $(2^S, 2^\delta, f, 2^i)$ be the reversed deterministic automaton constructed as above.

1. If (S, δ, i, f) is reachable, then $(2^S, 2^\delta, f, 2^i)$ is observable.
2. If (S, δ, i, f) accepts the language L, then $(2^S, 2^\delta, f, 2^i)$ accepts $rev(L)$.

Conclusions

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- We are experimenting with these ideas for use in *approximation* in the RL Lab at McGill; joint with Doina Precup and Joelle Pineau and their students.
- Extension to continuous observation and continuous state spaces.
- It is possible to eliminate state completely in favour of histories; when can this representation be compressed and made tractable?