Bisimulation and other behavioural equivalences for continuous-time Markov processes

Linan Chen   Florence Clerc   Prakash Panangaden

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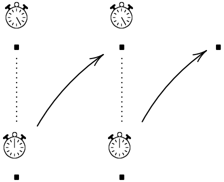
- Continuous-time Markov chains - many papers e.g. Baier et al. 2006, Desharnais and P. 2003

What do we mean by Continuous-Time?

Labelled Markov Processes

Continuous-Time Markov Chains



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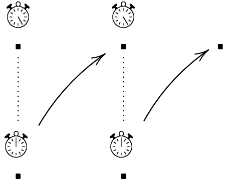

“Real” Continuous-Time: flowing rather than jumping

There is no “next step”

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- We chose Feller-Dynkin processes, perhaps we should have stuck to Lévy!

Markov kernels

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(X, Σ) a measurable space. A *Markov kernel* $k : X \times \Sigma \longrightarrow [0, 1]$ is a map of the indicated type such that:

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- We will usually think of E as a Polish space and indeed a metric space most of the time.

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- So we will go back and forth between two views: Markov kernels and function transformers.

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- So we can think of families of Markov kernels as families of such function transformers.

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- for f in $C_0(E)$, $\lim_{t\downarrow 0} \hat{P}_t f = f$; this is called *strong* continuity.

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- One can think of each $\omega \in \Omega$ as a *trajectory*.

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- We assume that if a trajectory hits ∂ it stays there.

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Definition of a filtration

Let $(\Omega, \mathcal{F}, P)$ be a probability space: a *filtration* $\mathcal{F}_t$ is an increasing family of σ-algebras $\forall t < s, \quad \mathcal{F}_t \subset \mathcal{F}_s \subset \mathcal{F}$.

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- A process is automatically adapted to its natural filtration.

Filtration associated with a process

Define Ω to be the space of trajectories: càdlàg path such that once it hits ∂, it stays at ∂.

$$\begin{aligned}\mathcal{F}_t &= \sigma(X_s \mid 0 \leq s \leq t) \\ &= \sigma(\{\omega \mid \omega(s) \in A\} \mid 0 \leq s \leq t, A \in \mathcal{E})\end{aligned}$$

The intuition is that it corresponds to the information you have about the process up to time t.

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- $P_t(x, C)$ is the probability of being in C after time t starting from x

Feller-Dynkin processes : 2

- Daniell-Kolmogorov theorem gives a space-indexed family of probability measures on Ω such that

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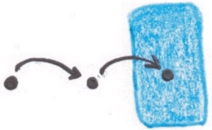
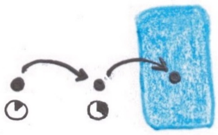
- $\mathbb{P}^x$ is a probability measure on trajectories that has support in the trajectories starting in x

A subtle difference: entry-exit points

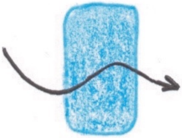
Discrete time



CTMC

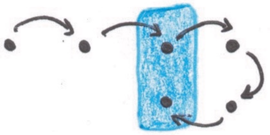


Continuous-Time



Second entry times?

Discrete time



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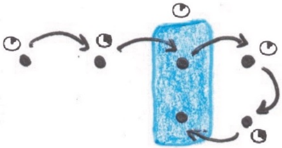

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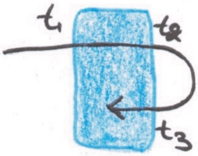
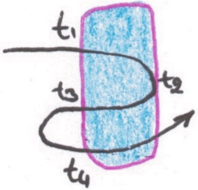


What is the 2nd entry-time of these trajectories?



The boundary is not in the blue area and at time t_2, the trajectory is on the boundary, so

$$t_2 \qquad \text{or} \qquad t_4?$$

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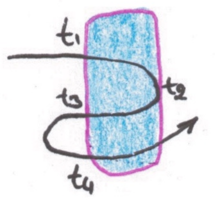

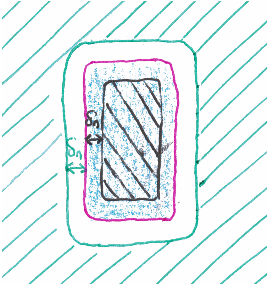

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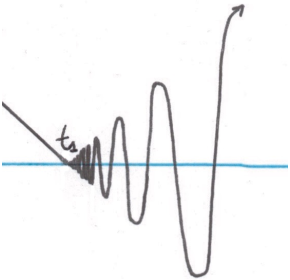

After time t_1, it behaves like $z \mapsto z \sin\left(\frac{1}{z}\right)$

Even though it looks like “we can just take limits”, we should be more careful.

Recall Discrete Time Bisimulation

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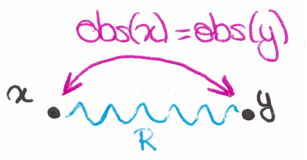
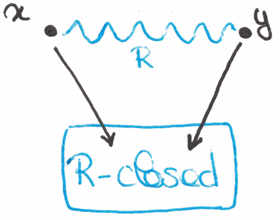
initiation condition



(co)induction condition



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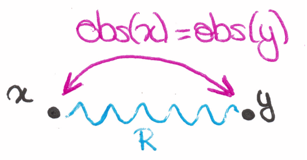

$obs(x) = obs(y)$

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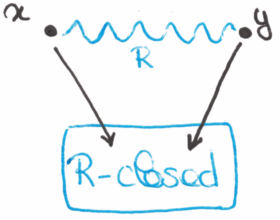

for C R-closed $\tau(x, C) = \tau(y, C)$

$(z\, R\, v) \;\Rightarrow\; (z \in C \text{ iff } v \in C)$

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What is *obs*?

- It is a function on the state space
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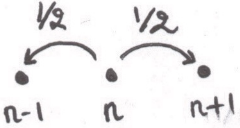
- an AP serves as an indicator and separates the state space into different areas

Example: Random Walk

State Space

Markov Kernel



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- if $|n| = |m|$, then they are bisimilar

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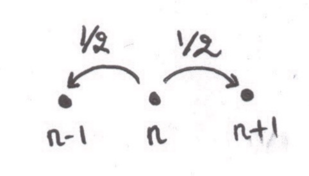
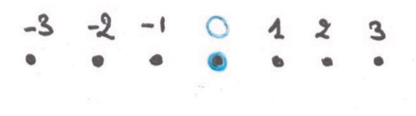
Random Walk / Brownian Motion

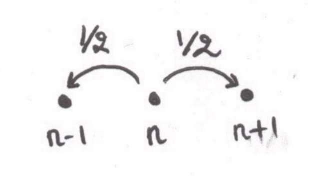
Random Walk

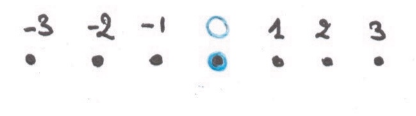




Random Walk / Brownian Motion

Random Walk





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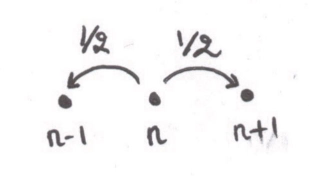
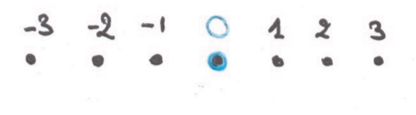
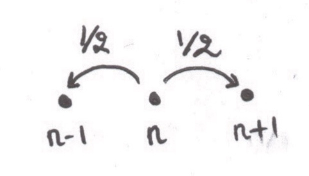


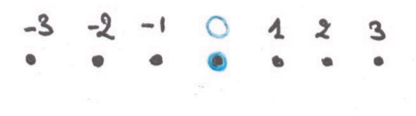


Make the time between each jump and the distance between each state increasingly small

Random Walk / Brownian Motion

Random Walk





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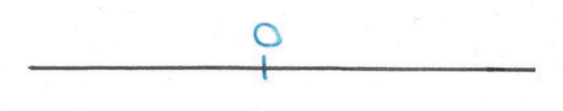
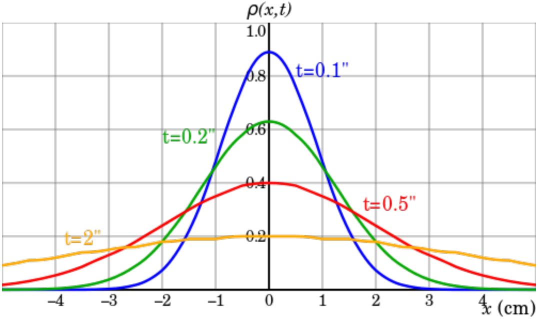
Brownian Motion



Diffusion

$$P_t(x, D) = \int_{y \in D} \rho(|y - x|, t) dy$$



A failed attempt

What happens for Brownian Motion?

When are two states x and y bisimilar?

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- Steps of time t

$$\forall z \neq 0 \; \forall t \geq 0 \;\; P_t(z, \{0\}) = 0$$

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$$\forall z \neq 0 \; \forall t \geq 0 \;\; P_t(z, \{0\}) = 0$$

- We end up with $x = y = 0$ or $x \neq 0$ and $y \neq 0$, i.e. two equivalence classes : $\{0\}$ and $\mathbb{R} \setminus \{0\}$

Going to the limit?

We cannot just replace steps by times.

Back to Random walk

What is the probability of having reached 0 **between** the $n-1$-th and the n-th steps?
What is the probability of having reached 0 **at some point** during the first n steps?

We need more than a single time-step.

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We need **trajectories**.

Where do we want to have trajectories?

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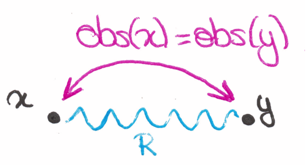
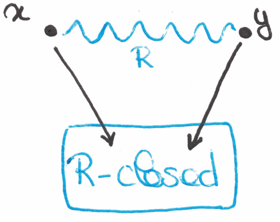
initiation condition



(co)induction condition



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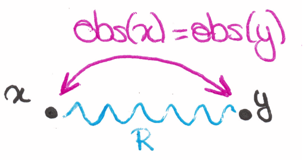
$obs(x) = obs(y)$



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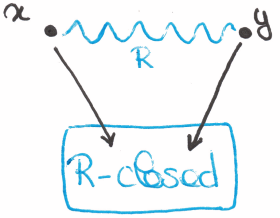

for C R-closed $\tau(x, C) = \tau(y, C)$

$(z\,R\,v) \;\Rightarrow\; (z \in C \text{ iff } v \in C)$

Extending the R-closed idea

time-R-closed

R an equivalence relation on E (extended to E_∂ by setting $\partial R \partial$). B a set of trajectories is **time-R-closed** if $\forall \omega, \omega'$, trajectories such that $\forall t \geq 0,\ \omega(t) R \omega'(t)$ we have $\omega \in B \iff \omega' \in B$.

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time-*obs*-closed

Take R to be $obs(x) = obs(y)$.

Time-R-closed sets of trajectories

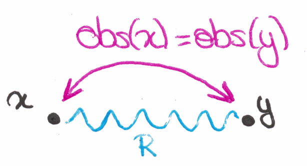
The set of measurable time-R-closed sets is a σ-algebra.
It contains the σ-algebra generated by the sets

$$\{\omega \mid \omega(t) \in C\}$$

with C R-closed and measurable.
Are these equal? No
Are they under certain conditions (which conditions?)? I don't know

Bisimulation

initiation condition



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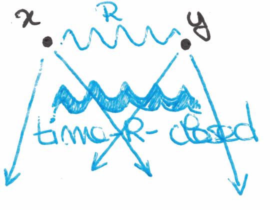

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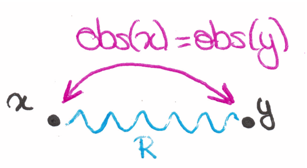

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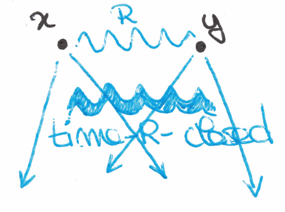


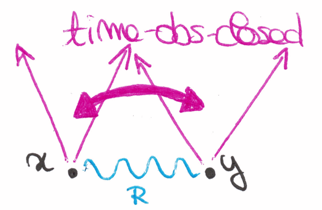
for B time-R-closed $\mathbb{P}^x(B) = \mathbb{P}^y(B)$

$\forall t \geq 0\ (\omega(t)\ R\ \omega'(t)) \ \Rightarrow\ (\omega \in B \text{ iff } \omega' \in B)$ ω, ω' trajectories

Temporal equivalence

Temporal equivalence

initiation condition (trace equivalence)



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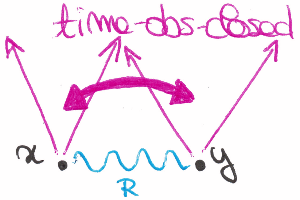

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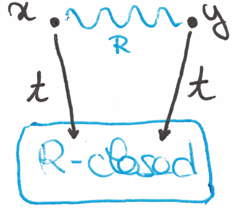


for C R-closed and $t \geq 0$,
$P_t(x, C) = P_t(y, C)$

$(z \, R \, z') \;\Rightarrow\; (z \in C$ iff $z' \in C)$

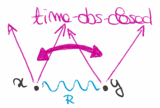
trace equivalence

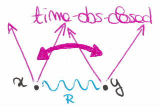
↑

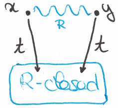
temporal equivalence

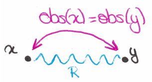
↑

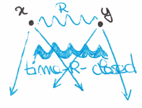
bisimulation





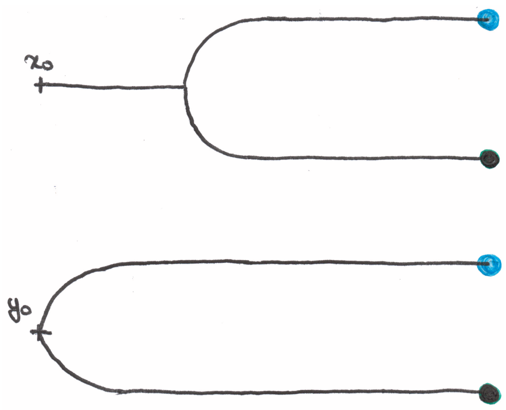






An example: the fork

deterministic drift at constant speed except at branching



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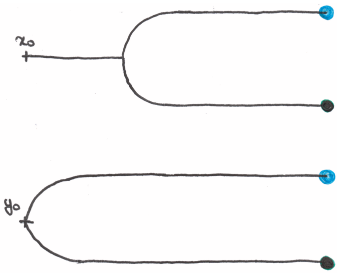

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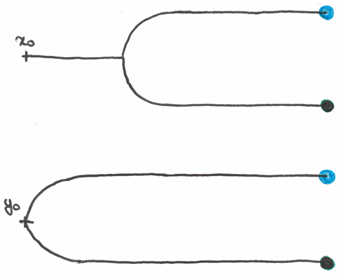

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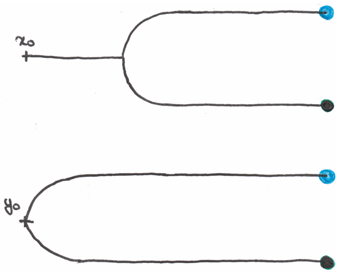

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- two trajectories from x_0: ω_x^U, ω_x^D (each prob $\frac{1}{2}$)

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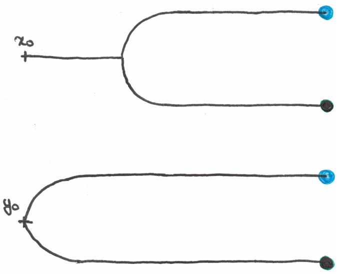

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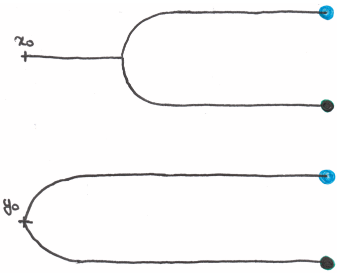

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- $obs \circ \omega_x^U = obs \circ \omega_y^U$ (similarly for the other trajectories)

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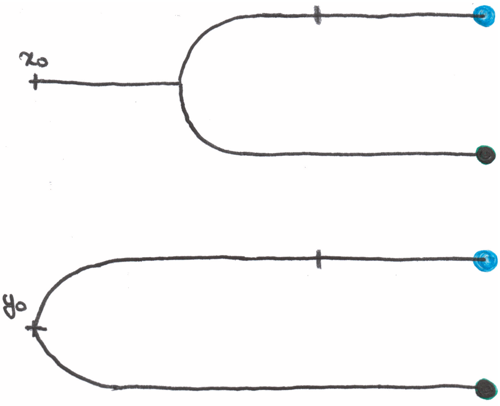

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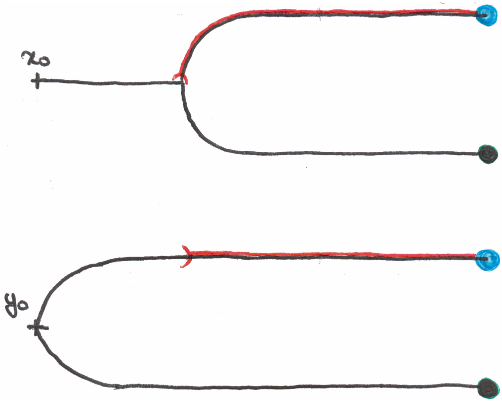
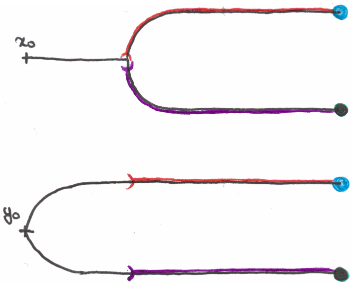
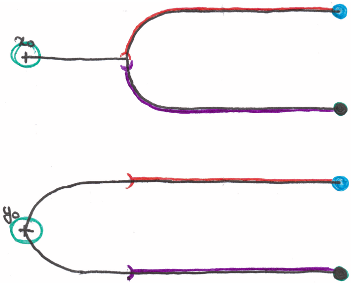


An example: the fork

Trace equivalence *strictly* includes the greatest temporal equivalence:



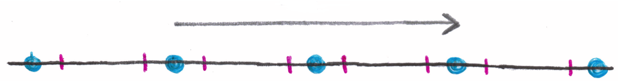


x_0 and y_0 are trace equivalent but not temporally equivalent nor bisimilar.

Another example: Brownian Motion

Other example continued

Other Example finished

translation by $k \in \mathbb{Z}$ 

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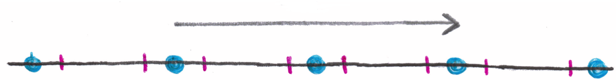
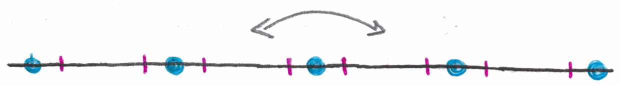


reflexion around k



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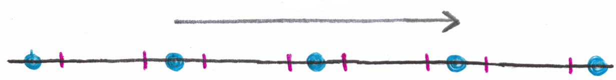

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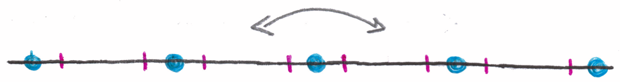
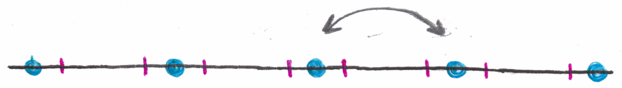


reflexion around $k + 1/2$



Common theme of many examples

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- How does one know in advance?
- By using symmetry.
- So perhaps we should promote this from a secret intuition to definition.

Group of symmetries - I

- Group of homeomorphisms on the state space

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- that commute with obs

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- Group of homeomorphisms on the state space
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- and that leave the dynamics of the system unchanged

Group of symmetries - II

Group of symmetries

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Group of symmetries - II

Group of symmetries

- for all $h \in \mathcal{H}$, $obs \circ h = obs$, and
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$$\mathbb{P}^x(B) = \mathbb{P}^{f(x)}(B).$$

Brownian motion with 0 distinguished

- The set $\{s, id\}$ (with $s(x) = -x$) is a group of symmetries

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- Hence the smallest equivalence R such that $x\ R\ -x$ is a bisimulation and a temporal equivalence
- Another consequence is that x and $-x$ are trace equivalent.

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FD-homomorphisms

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Feller-Dynkin homomorphisms

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- $obs = obs' \circ f$,
- for all $x \in E$ and for all measurable sets $B' \subset \Omega'$,

$$\mathbb{P}^{f(x)}(B') = \mathbb{P}^{x}(B)$$

  where $B := \{\omega \in \Omega \mid f \circ \omega \in B'\}$.

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Game interpretation for temporal equivalence

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- Two states x, y are temporally equivalent if and only if Duplicator has a winning strategy starting from x and y.

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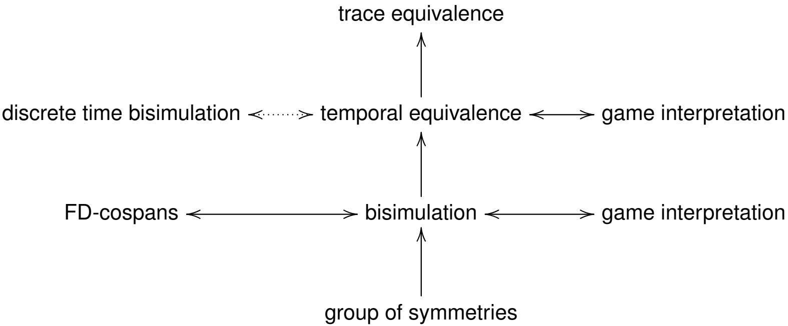
Behavioural equivalences: summary of results



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Thanks for your attention

This work is published in Mathematical Foundations of Programming Semantics 2019 and 2020.
Journal paper is under review.