Quantitative equational algebra

Radu Mardare Prakash Panangaden Gordon Plotkin

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Many examples: mostly probabilistic

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Later (1981) Giry: monad on measure spaces
and also on Polish spaces.

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quantitative equational theories will define monads on **Met**.

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Axioms:

$$(B_1) \vdash t +_1 t' = t$$
$$(B_2) \vdash t +_\epsilon t' = t$$
$$(SC) \vdash t +_\epsilon t' = t' +_{1-\epsilon} t$$
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Equations define an equivalence relation on terms.
Our quantitative equations will define a metric on terms.

Deducibility relations

(Refl) $\emptyset \vdash t =_0 t$

(Symm) $\{t =_\epsilon s\} \vdash s =_\epsilon t$.

(Triang) $\{t =_e s, s =_{e'} u\} \vdash t =_{e+e'} u$.

(Max) For $e' > 0$, $\{t =_e s\} \vdash t =_{e+e'} s$.

(Arch) For $e \geq 0$, $\{t =_{e'} s \mid e' > e\} \vdash t =_e s$.

(NExp) For $f : n \in \Omega$,
$$\{t_1 =_e s_1, \ldots, t_n =_e s_n\} \vdash f(t_1, ..t_i, ..t_n) =_e f(s_1, ..s_i, ..s_n)$$

(Subst) If $\sigma \in \Sigma(X)$, $\Gamma \vdash t =_e s$ implies $\sigma(\Gamma) \vdash \sigma(t) =_e \sigma(s)$.

(Cut) If $\Gamma \vdash \phi$ for all $\phi \in \Gamma'$ and $\Gamma' \vdash \psi$, then $\Gamma \vdash \psi$.

(Assumpt) If $\phi \in \Gamma$, then $\Gamma \vdash \phi$.

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Equational theory: $\mathcal{U} = \vdash_S \bigcap \mathcal{E}(\mathbb{T}X)$

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We write $\mathbb{K}(\mathcal{U}, \Omega)$ for the algebras satisfying $\mathcal{U}$.

Completeness

$$\forall \mathcal{A} \in \mathbb{K}(\mathcal{U}, \Omega), \mathcal{A} \text{ satisfies } \Gamma \vdash \phi \text{ iff } \Gamma \vdash \phi \in \mathcal{U}.$$

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$d^{\mathcal{U}}(s,t) = \inf\{e | \emptyset \vdash s =_e t \in \mathcal{U}\}$.

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From this we can construct $(\mathbb{T}[M], d_M) \in \mathbb{K}(\mathcal{U}, \Omega)$.

Universal property

$$
\begin{array}{cc}
\text{in } \mathbf{Met} & \mathbb{K}(\mathcal{U},\Omega) \\
\begin{tikzcd}
(M,d) \arrow[r, "\eta"] \arrow[dr, "f"'] & U(\mathbb{T}[M]) \arrow[d, "Uh"] \\
& U\mathcal{A}
\end{tikzcd}
&
\begin{tikzcd}
\mathbb{T}[M] \arrow[d, dashed, "h"] \\
\mathcal{A}
\end{tikzcd}
\end{array}
$$

$U : \mathbb{K}(\mathcal{U},\Omega) \to \mathbf{Met}$: forgetful functor

$\mathbb{T}$ defines a monad on $\mathbf{Met}$

Example 1: Semiadditive barycentric algebras

Signature: $\mathcal{B} = \{+_\epsilon | \epsilon \in [0,1]\}$ (a set of operations)

Equations:

$$(B_1) \vdash t +_1 t' =_0 t$$

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The freely generated algebra is the space of probability distributions with the *total variation* metric.

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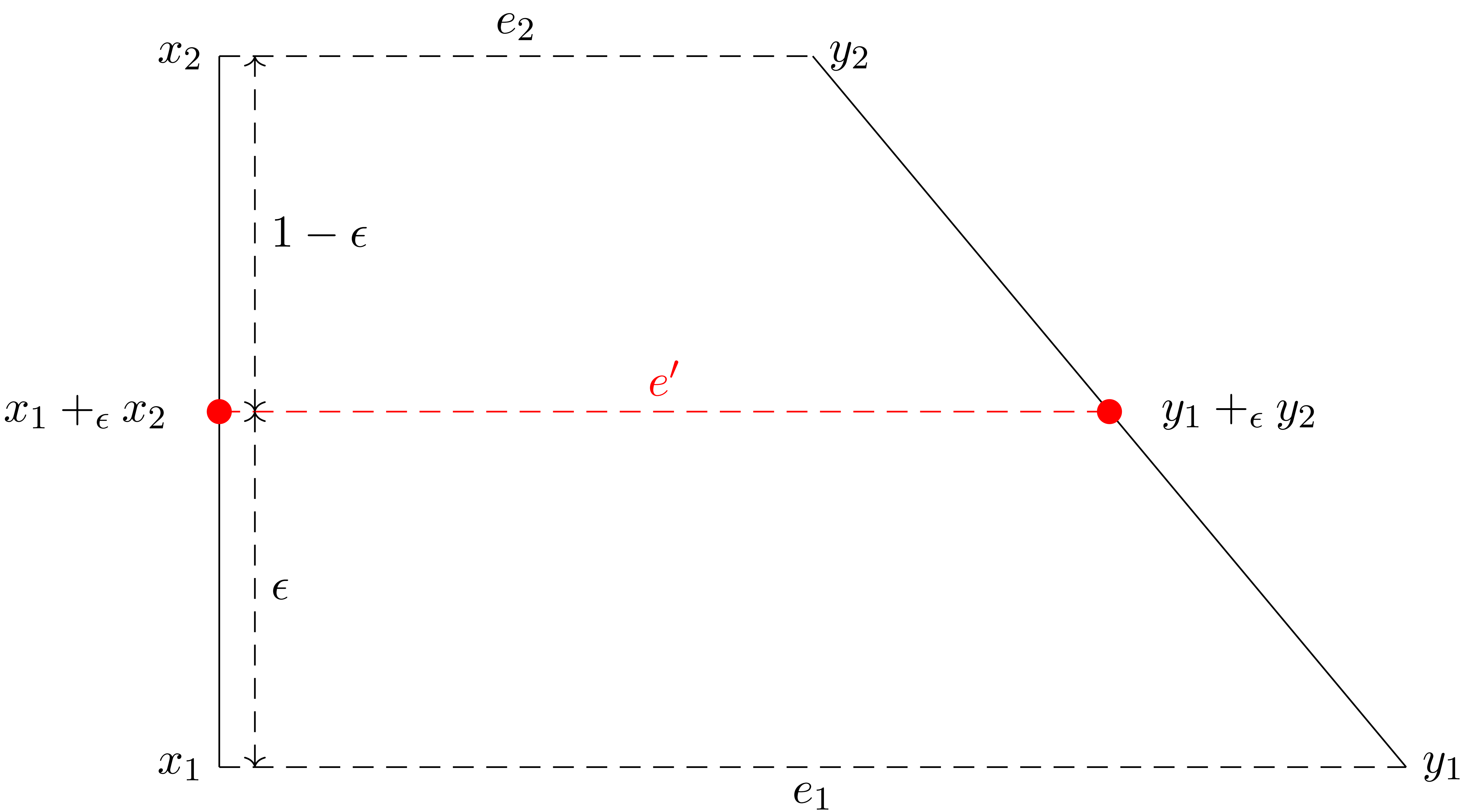
A picture of equation (K)



Finitary case

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$\mu +_\epsilon \nu = \epsilon\mu + (1-\epsilon)\nu$, convex sum.

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Prop. $(\Pi[M], W^p)$ is in $\mathbb{K}$.

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Proof ideas:
Couplings form a convex set.
Convexity properties of spaces of measures.
Linearity of integration.
Splitting lemma.
Non-expansiveness – induction on size of the support.

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Using this we can extend the barycentric algebra results to the continuous case.

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4. Exceptions, state, IO - quantitative analogues
(not in the paper)

Related work

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We can easily adapt Kantorovich to Wasserstein.

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Birkhoff variety theorem?