

Duality in Logic and Computation

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- Controllability and observability in control theory, Kalman.
- Many examples from semantics and logic.

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- The two mathematical universes are *mirror images* of each other.
- Two completely different sets of theorems that one can use.

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- Labelled Markov processes and C^* -algebras with operators. [Mislove, Ouaknine, Pavlovic, Worrell]

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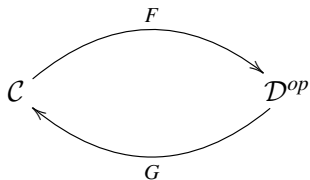
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- Compositions are *preserved or reversed*.
- This is *functoriality*.
- From this one can often conclude *invariance properties*.

Duality categorically



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Given

$$\begin{array}{c} A \in \mathcal{C} \\ \downarrow f \\ B \in \mathcal{C} \end{array}$$

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We get

$$\begin{array}{ccc} A \in \mathcal{C} & & F(A) \in \mathcal{D} \\ \downarrow f & & \\ B \in \mathcal{C} & & F(B) \in \mathcal{D} \end{array}$$

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and

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Similarly, given

$$\begin{array}{c} C \in \mathcal{D} \\ \downarrow g \\ D \in \mathcal{D} \end{array}$$

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We get

$$\begin{array}{ccc} G(C) \in \mathcal{C} & & C \in \mathcal{D} \\ & & \downarrow g \\ G(D) \in \mathcal{C} & & D \in \mathcal{D} \end{array}$$

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Isomorphisms

We have isomorphisms

$$A \simeq G(F(A)) \text{ and } C \simeq F(G(C)).$$

Duality categorically

Stone-type Duality

We have a (contravariant) adjunction between categories \mathcal{C} and \mathcal{D} , which is an *equivalence* of categories.

Often obtained by looking at maps into an object living in both categories: a schizophrenic object.

Duality for high school students I

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- V^* has the same dimension as V and a (basis-dependent) isomorphism between V and V^* .
- The double dual V^{**} is also isomorphic to V
- with a “nice” canonical isomorphism: $v \in V \mapsto \lambda\sigma \in V^*.\sigma(v)$.

Duality for high school students II

$$U \xrightarrow{\theta} V$$

$$U^* \xleftarrow{\theta^*} V^*$$

Given a linear maps θ between vector spaces U and V we get a map θ^* *in the opposite direction* between the dual spaces:

$$\theta^*(\sigma \in V^*)(u \in U) = \sigma(\theta(u)).$$

Boolean algebras

A Boolean algebra is a set A equipped with two constants, $0, 1$, a unary operation $(\cdot)'$ and two binary operations \vee, \wedge which obey the following axioms, p, q, r are arbitrary members of A :

$$\begin{aligned}
 0' &= 1 & 1' &= 0 \\
 p \wedge 0 &= 0 & p \vee 1 &= 1 \\
 p \wedge 1 &= p & p \vee 0 &= p \\
 p \wedge p' &= 0 & p \vee p' &= 1 \\
 p \wedge p &= p & p \vee p &= p
 \end{aligned}$$

Boolean algebras II

$$p'' = p$$

$$(p \wedge q)' = p' \vee q'$$

$$(p \vee q)' = p' \wedge q'$$

$$p \wedge q = q \wedge p$$

$$p \vee q = q \vee p$$

$$p \wedge (q \wedge r) = (p \wedge q) \wedge r$$

$$p \vee (q \vee r) = (p \vee q) \vee r$$

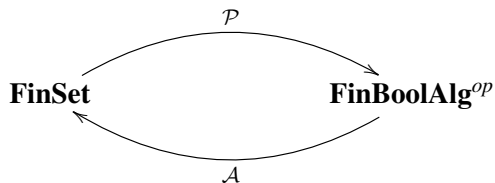
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The operation \vee is called *join*, \wedge is called *meet* and $(\cdot)'$ is called *complement*.

Maps are Boolean algebra homomorphisms.

(Toy) Stone duality



Here \mathcal{P} is power-set and \mathcal{A} takes the *atoms* of a Boolean algebra.

Stone spaces

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- *Totally disconnected*: the only connected sets are singletons.
- Many, but not all, Stone spaces are Polish.

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- Everything that can, and should be, an isomorphism is an isomorphism.

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- Elegant and (almost) compositional version: Plotkin's *structured operational semantics*.
- Denotational semantics: compositional, equivalent to operational semantics.

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- Relate predicate-transformer semantics to state-transformer semantics: Jaco De Bakker (1978).
- Duality: The category of state transformers is equivalent to the (opposite of) the category of predicate transformers: Plotkin (1979).

Duality for probabilistic programs: Kozen

Probabilistic programs and *expectation transformers*: Kozen (1981)

Logic	Probability
States s	Distributions μ
Formulas P	Random variables f
Satisfaction $s \models P$	Integration $\int f d\mu$

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- The two interpretations are Stone duals.
- Ties together semantics, logic and verification.

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- Pratt's observation: many interesting categories embed in Chu categories.
- Stone duality is “transposition” of matrices.

Many other contributors

- Bart Jacobs,
- Achim Jung and Drew Moshier
- Mai Gehrke, Jean-Eric Pin, ...
- Bezhanishvilis

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- The intermediate step can blow up the size of the automaton exponentially before minimizing it.
- But experimental results seem to indicate that it often works well in practice.

Deterministic Automata

- $\mathcal{M} = (S, \mathcal{A}, \mathcal{O}, \delta, \gamma)$: a deterministic finite (Moore) automaton. S is the set of **states**, \mathcal{A} an **input alphabet** (actions), \mathcal{O} is a set of **observations**.
- $\delta : S \times \mathcal{A} \rightarrow S$ is the **state transition function**.
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- which means that reachability makes no sense.
- We will worry about that in a minute.

A Simple Modal Logic

- View \mathcal{O} as propositions, define a simple modal logic. A *formula* φ is:

$$\varphi ::= \omega \in \mathcal{O} \mid (a)\varphi$$

where $a \in \mathcal{A}$.

- We say $s \models \omega$, if $\omega \in \gamma(s)$ (or $\gamma(s, \omega) = T$).
We say $s \models (a)\varphi$ if $\delta(s, a) \models \varphi$.
- Now we define $\llbracket \varphi \rrbracket_{\mathcal{M}} = \{s \in S \mid s \models \varphi\}$.

An Equivalence Relation on Formulas

- We write sa as shorthand for $\delta(s, a)$.
- Define $\sim_{\mathcal{M}}$ between *formulas* as $\varphi \sim_{\mathcal{M}} \psi$ if $\llbracket \varphi \rrbracket_{\mathcal{M}} = \llbracket \psi \rrbracket_{\mathcal{M}}$.

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- Equivalence class for φ same as of states $\llbracket \varphi \rrbracket_{\mathcal{M}}$ that satisfy φ .

A Dual Automaton

- Given a finite automaton $\mathcal{M} = (S, \mathcal{A}, \mathcal{O}, \delta, \gamma)$.
Let T be the set of $\sim_{\mathcal{M}}$ -equivalence classes of formulas on \mathcal{M} .
- We define $\mathcal{M}' = (S', \mathcal{A}, \mathcal{O}', \delta', \gamma')$ as follows:
 - $S' = T = \{[\varphi]_{\mathcal{M}}\}$
 - $\mathcal{O}' = S$
 - $\delta'([\varphi]_{\mathcal{M}}, a) = [(a)\varphi]_{\mathcal{M}}$
 - $\gamma'([\varphi]_{\mathcal{M}}) = [\varphi]_{\mathcal{M}}$ or $\gamma'([\varphi]_{\mathcal{A}}, s) = (s \models \varphi)$.

The intuition

Interchange states and observations.

Minimality Properties

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- but it does not really address the role of reachability properly.

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- Dual automaton from tests: probabilistic analogues of modal formulas.
- Main point: not minimization, but can learn systems from data even when the state is not directly observable
- because the double-dual serves as a substitute for the original machine.

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- There should be no such thing as absolute state!
- State is just a summary of past observations that can be used to make predictions.
- Double dual: state can be regarded as the summary of the outcomes of experiments.

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- why is the dual another automaton?

Automata as Coalgebras

- Our automata are coalgebras of the following functor:

$$F(S) = S^{\mathcal{A}} \times \mathbf{2}^{\mathcal{O}}, \quad F(f : S \rightarrow S') = \lambda \langle \alpha : \mathcal{A} \rightarrow S, \mathcal{O} \subseteq \mathcal{O} \rangle. \langle f \circ \alpha, \mathcal{O} \rangle.$$

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- with S : states, \mathcal{A} : actions, \mathcal{O} : observations, δ is a transition function and γ is an observation function.
- They are well known as Moore machines.

Homomorphisms

A homomorphism for these coalgebras is a function $f : S \rightarrow S'$ such that the following diagram commutes:

$$\begin{array}{ccc}
 S & \xrightarrow{f} & S' \\
 \langle \delta, \gamma \rangle \downarrow & & \downarrow \langle \delta', \gamma' \rangle \\
 S^{\mathcal{A}} \times \mathbf{2}^{\mathcal{O}} & \xrightarrow{f^{\mathcal{A}} \times \text{id}} & S'^{\mathcal{A}} \times \mathbf{2}^{\mathcal{O}}
 \end{array}$$

where $f^{\mathcal{A}}(\alpha) = f \circ \alpha$.

This translates to the following conditions:

$$\forall s \in S, \omega \in \mathcal{O}, \omega \in \gamma(s) \iff \omega \in \gamma'(f(s)) \quad (1)$$

and

$$\forall s \in S, a \in \mathcal{A}, f(\delta(s, a)) = \delta'(f(s), a). \quad (2)$$

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- the usual operations \wedge , \neg and constants \top and \perp and, in addition,
- together with unary operators (a) and constants $\underline{\omega}$.
- We denote an object by $\mathcal{B} = (B, \{(a) \mid a \in \mathcal{A}\}, \{\underline{\omega} \mid \omega \in \mathcal{O}\}, \top, \wedge, \neg)$.

Equations

The following three equations hold:

$$(a)(b_1 \wedge b_2) = (a)b_1 \wedge (a)b_2, \quad (3a)$$

$$(a)\top = \top, \quad (3b)$$

$$\neg(a)\neg b = (a)b. \quad (3c)$$

Morphisms

The morphisms are the usual boolean homomorphisms preserving, in addition, the constants and the unary operators.

Duality Theorem

There is a dual equivalence of categories

$$\mathbf{PODFA}^{op} \cong \mathbf{FBAO}.$$

One functor \mathcal{P} is just the contravariant power set functor and the other one \mathcal{H} maps a boolean algebra to its set of atoms.

Minimization?

- Obviously, if we have an equivalence of categories we get the same machine back when we go back and forth.

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- Obviously, if we have an equivalence of categories we get the same machine back when we go back and forth.
- So how do we explain the minimization?

Definable Subsets

Define a logic \mathcal{L} by

$$\phi ::= \top \mid \perp \mid \phi_1 \wedge \phi_2 \mid \neg \phi \mid (a)\phi \mid \underline{\omega}$$

and define the **definable subsets** $\mathcal{D}(S)$ of a machine $\mathcal{M} = (S, \delta, \gamma)$ as sets of the form $\llbracket \phi \rrbracket$.

- $\mathcal{D}(S)$ is a subobject of $\mathcal{P}(\mathcal{M})$

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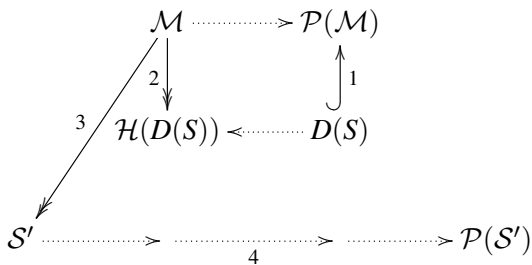
- $\mathcal{D}(S)$ is a subobject of $\mathcal{P}(\mathcal{M})$
- in fact it is the *smallest* possible subalgebra and
- any other subalgebra must contain $\mathcal{D}(S)$.

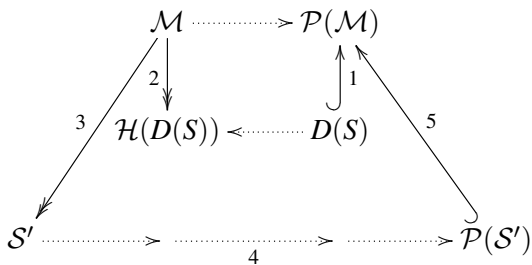
In Pictures

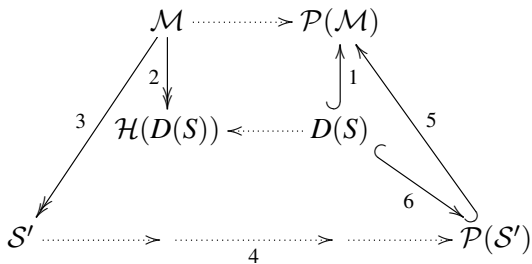
$$\mathcal{M} \dashrightarrow \mathcal{P}(\mathcal{M})$$

$$\begin{array}{ccc} \mathcal{M} & \dashrightarrow & \mathcal{P}(\mathcal{M}) \\ & & \uparrow 1 \\ & & D(S) \end{array}$$

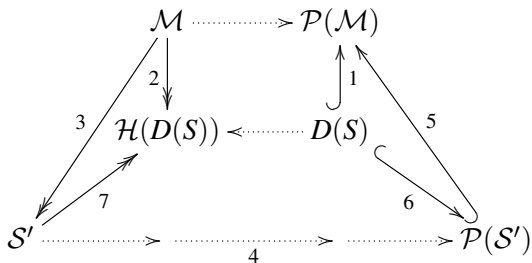
$$\begin{array}{ccc}
 \mathcal{M} & \xrightarrow{\quad \dots \quad} & \mathcal{P}(\mathcal{M}) \\
 \downarrow 2 & & \uparrow 1 \\
 \mathcal{H}(D(S)) & \xleftarrow{\quad \dots \quad} & D(S)
 \end{array}$$







The Secret of Minimization



A Simpler Logic

- Why did the minimization work with just the logic

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$$\phi ::= \underline{\omega}(a)\phi?$$

- With this logic the definable subsets $E(S)$ do not form a boolean algebra,
- it is just a “set with operations”
- in other words, it can be viewed as an automaton!

Why the simpler logic works

- For deterministic automata we can flatten formulas like $(a)(\omega_1 \wedge (b)\omega_2)$ to $(a)\omega_1 \wedge (a)(b)\omega_2$.

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- Thus for **deterministic** automata the boolean algebra generated by $E(S)$ is just the same as $D(S)$ so the minimization picture works with boolean algebra generated by $E(S)$.
- For nondeterministic automata the story is different.

Minimality = fewest states?

- The minimal machines defined above are really the “most quotiented versions” of a system.

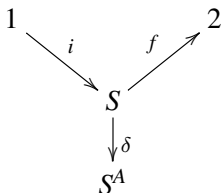
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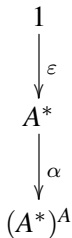
- The minimal machines defined above are really the “most quotiented versions” of a system.
- To really get the system with the fewest states one needs to deal with reachability.
- The following discussion is a rapid version of what Jan Rutten discussed in his beautiful MFPS talk on Monday.

An automaton in diagrams



- Here S is the state space, A is the set of actions, 1 is the one-element set and 2 is a two-element set.
- The map i defines an initial state and f defines a set of final states. I will write i for the map and for the initial state itself.
- the transition function $\delta : S \times A \rightarrow S$ has been written as $\delta : S \rightarrow S^A$.
- There is a natural extension $\delta^* : S \rightarrow S^{A^*}$.

A very special (infinite) automaton



- This automaton has all words as its state space.
- The initial state is the empty word ε .
- The transition function α acts by $\alpha(w) = \lambda a : A.w \cdot a$.
- We do not bother to define “final” states in this machine.

Reachability

$$\begin{array}{ccc}
 \mathbf{1} & & \\
 \downarrow \varepsilon & \searrow i & \\
 A^* & \overset{r}{\dashrightarrow} & S \\
 \downarrow \alpha & & \downarrow \delta \\
 (A^*)^A & \overset{r^A}{\dashrightarrow} & S^A
 \end{array}$$

- Given any function between sets $f : V \rightarrow W$, we have a map $f^A : V^A \rightarrow W^A$, given by $f^A(\phi) = \lambda a : A. f(\phi(a)) = f \circ \phi$.

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- Note, final states play no role.

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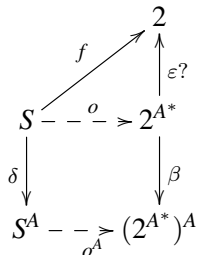
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- Here o is the map that takes a state to the language recognized starting from that state.
- It is the unique map making the upper triangle and the lower square commute.
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- A machine is *observable* exactly when distinct states recognize different languages, i.e. when o is an injection.

The butterfly

$$\begin{array}{ccccc}
 1 & & & & 2 \\
 \varepsilon \downarrow & \searrow i & & \nearrow f & \\
 A^* & \overset{r}{\dashrightarrow} & S & \overset{o}{\dashrightarrow} & 2^{A^*} \\
 \alpha \downarrow & & \downarrow \delta & & \downarrow \beta \\
 (A^*)^A & \overset{r^A}{\dashrightarrow} & S^A & \overset{o^A}{\dashrightarrow} & (2^{A^*})^A
 \end{array}$$

A deterministic automaton (S, δ, i, f) is minimal if it is both reachable and observable.

The power-set functor

Given sets U, V and a function $f : U \rightarrow V$ we define

$$\mathcal{P}(f) : \mathcal{P}(V) \rightarrow \mathcal{P}(U)$$

by

$$\mathcal{P}(f)(P \subseteq V) = f^{-1}(P).$$

Reverse functorially

$$\begin{array}{c}
 S \\
 \delta \downarrow \\
 S^A
 \end{array}
 \parallel \parallel
 \begin{array}{c}
 S \times A \\
 \downarrow \\
 S
 \end{array}
 \mid
 \begin{array}{c}
 2^{S \times A} \\
 \uparrow \\
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- Initial becomes final under powerset. The final state $S \rightarrow \mathbf{2}$ becomes the new initial state by observing that such a function is the same thing as a subset.
- It makes reachable into observable, *but not vice versa*.

Why Brzozowski's algorithm works

Theorem

If (S, δ, i, f) is a reachable deterministic automaton accepting L , then $(2^S, 2^\delta, f, 2^i)$ is an observable deterministic automaton accepting $rev(L)$.

If, we take its reachable part again and reverse it again we again get an observable automaton this time recognizing L . If we take the reachable part we get a minimal automaton recognizing L .

Abstract nonsense?

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- “Surely, this is categorical mumbo-jumbo for something that can be explained simply!”

No, it is generalized abstract nonsense!

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- Moore automata work by changing the functor slightly.
- Kleene algebra with tests.
- Weighted automata (i.e. automata over vector spaces) can be minimized by using the same idea with the self duality of vector spaces.
- Belief automata can be minimized using Gelfand duality.

What is Gelfand duality?

- Problem in undergrad algebra courses: Given the ring of continuous real-valued functions defined on a compact Hausdorff space X , call this $C(X)$, show that for every maximal ideal M there is an $x \in X$ such that

$$M = \{f \in C(X) \mid f(x) = 0\}.$$

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- $C(X)$ is more than a ring it is a *commutative, unital C^* -algebra*.
- Incidentally, this works just as well with complex-valued functions.

C^* -algebras

- A (complex) C^* -algebra is a (complex) vector space with
- an associative multiplication (satisfying obvious laws)
- and a norm $\|\cdot\|$ with respect to which it is complete (hence a Banach space).
- The norm satisfies: $\|a \cdot b\| \leq \|a\| \cdot \|b\|$.
- There is also an involution $*$ satisfying $(ab)^* = b^*a^*$ and $(\alpha a)^* = \bar{\alpha}a^*$.
- The crucial property is:
 - $\|a^*a\| = \|a\|^2$.
 - Morphisms are homomorphisms preserving the $*$.
 - We say that A is *unital* if there is a unit element for the multiplication.

Gelfand duality

The category of commutative unital C^* -algebras is dually equivalent to the category of compact Hausdorff spaces.

It does not matter if the C^* algebras are complex (Gelfand) or real (Stone); though the proofs are very different.

Belief automata

- A **probabilistic automaton with observations** is

$$\mathcal{F} = (S, \mathcal{A}, \mathcal{O}, \delta : S \times \mathcal{A} \times S \rightarrow [0, 1], \gamma : S \times \mathcal{O} \rightarrow [0, 1]).$$

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- Minimization via Stone duality \longrightarrow minimization via Gelfand duality.

Conclusions

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- Completeness theorems, which typically work by constructing transition systems from consistent sets of formulas embody a key aspect of duality results *but*,
- the arrow part of the duality is crucial for proving our minimization results.

Ongoing and Future Work

- Unify the BKP picture with the BBHPRS picture: BBBHKKPRS unified picture in progress.

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- Metric analogue of Stone duality: Mardare and Kozen.
- Pressing research topic of great interest in quantum information theory: what is the duality theory for *non-commutative* C^* -algebras?: Tobias Fritz.

Thank you!